ISTITUTO NAZIONALE DI FISICA NUCLEARE

Sezione di Catania

INFN/AE-85/2
25 Marzo 1985

E. Recami and W.A.Rodrigues:

A MODEL-THEORY FOR TACHYONS IN TWO DIMENSIONS

A MODEL-THEORY FOR TACHYONS IN TWO DIMENSIONS (°)

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Abstract - The subject of *Tachyons*, even if still speculative, may deserve some attention for reasons that can be divided into a few categories, two of which we want preliminarily to mention right now: (i) the larger scheme that one tries to build up in order to incorporate space-like objects in the relativistic theories can allow a better understanding of many aspects of the *ordinary* relativistic physics, even if Tachyons would not exist in our cosmos as "asymptotically free" objects; (ii) Superluminal classical objects can have a role in elementary particle interactions (and perhaps even in astrophysics); and it might be tempting to verify how far one can go in reproducing the quantum-like behaviour at a classical level just by taking account of the possible existence of faster-than-light classical particles.

This article is divided in two parts, the first one having nothing to do with tachyons. In fact, to prepare the ground, in Part I (Sect.2) we shall merely show that Special Relativity —even *without* tachyons— can be given a form such to describe both particles and anti-particles. The plan of Part II is confined only to a "model-theory" of Tachyons in two dimensions, for the reasons stated in Sect.3.

(°) Supported in part by MPI, CNR, and by IBM-do-Brazil.

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1. INTRODUCTION

The subject of *Tachyons*, even if still speculative, may deserve some attention for reasons that can be divided into a few categories, two of which we want preliminarly to mention right now: (i) the larger scheme that one tries to build up in order to incorporate space-like objects in the relativistic theories can allow a better understanding of many aspects of the *ordinary* relativistic physics, even if Tachyons would not exist in our cosmos as "asymptotically free" objects; (ii) Superluminal classical objects can have a role in elementary particle interactions (and perhaps even in astrophysics); and it might be tempting to verify how far one can go in reproducing the quantum-like behaviour at a classical level just by taking account of the possible existence of faster-than-light classical particles.

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PART I : PARTICLES AND ANTIPARTICLES IN SPECIAL RELATIVITY (SR)

2. SPECIAL RELATIVITY WITH ORTHO- AND ANTI-CHRONOUS LORENTZ TRANSFORMATIONS.

In this Part I we shall *forget* about Tachyons.

From the ordinary postulates of Special Relativity (SR) it follows that in such a theory — which refers to the class of Mechanical and Electromagnetic Phenomena — the class of reference-frames equivalent to a given inertial frame is obtained by means of transformations L (Lorentz Transformations, LT) which satisfy the following sufficient requirements: (i) to be linear;

$$x'^{\mu} = L^{\mu}_{\nu}\, x^{\nu} \;; \tag{1}$$

(ii) to preserve space-isotropy (with respect to electromagnetic and mechanical phenomena); (iii) to form a group; (iv) to leave the quadratic form invariant:

$$\eta_{\mu\nu}\, dx^{\mu} dx^{\nu} = \eta'_{\alpha\beta}\, dx'^{\alpha} dx'^{\beta} \;. \tag{2}$$

From condition (i), if we confine ourselves to sub-luminal speeds, it follows that in eq.(2):

$$\eta'_{\alpha\beta} = \mathrm{diag}(+1,\, -1,\, -1,\, -1) = \eta_{\mu\nu}. \tag{3}$$

Eqs. (1)-(3) imply that $\det L^2 = 1$; $(L^o_{\ o})^2 \geq 1$. The set of all sub-luminal (Lorentz) transformations satisfying all our conditions consists — as is well-known — of four pieces, which form a noncompact, non-connected group (the Full Lorentz Group). Wishing to confine ourselves to space-time "rotations" only, i.e. to the case $\det L = +1$, we are left with the two pieces:

$$\begin{cases} \{L^{\uparrow}_{+}\} : L^{o}_{o} \geq +1 \quad ; \quad \det L = +1 \quad ; & \text{(4a)} \\ \{L^{\downarrow}_{+}\} : L^{o}_{o} \leq -1 \quad ; \quad \det L = -1 \quad , & \text{(4b)} \end{cases}$$

which give origin to the group of the proper (orthochronous *and* anti-chronous) transformations

$$\mathcal{L}_{+} \equiv \mathcal{L}^{\uparrow}_{+} \cup \mathcal{L}^{\downarrow}_{+} \equiv \{L^{\uparrow}_{+}\} \cup \{L^{\downarrow}_{+}\} \tag{5}$$

and to the subgroup of the (ordinary) proper ortochronous transformations

$$\mathcal{L}^{\uparrow}_{+} \equiv \{L^{\uparrow}_{+}\} \quad , \tag{6}$$

both of which being, incidentally, invariant subgroups of the Full Lorentz Group. For reasons to be seen later on, let us rewrite $\mathcal{L}_+$ as follows

$$\mathcal{L}_+ = \mathcal{L}_+^{\uparrow} \otimes \mathbf{Z}(2) \; ; \quad \mathbf{Z}(2) \equiv \{ \sqrt[2]{+1} \} \equiv \{+1, -1\}. \tag{5'}$$

We shall skip in the following, for simplicity's sake, the subscript + in the transformations $L_+^{\uparrow}$, $L_+^{\downarrow}$. Given a transformation $\bar{L}^{\downarrow}$, another transformation $\bar{L}^{\uparrow} \in \mathcal{L}_+^{\uparrow}$ always exists such that

$$\bar{L}^{\downarrow} = (-\mathbb{1}) \cdot \bar{L}^{\uparrow}, \quad \forall \bar{L}^{\downarrow} \in \mathcal{L}_+^{\downarrow} \; , \tag{7}$$

and vice-versa. Such a one-to-one correspondence allows us to write formally

$$\mathcal{L}_+^{\downarrow} = - \mathcal{L}_+^{\uparrow} \; . \tag{7'}$$

It follows in particular that the central elements of $\mathcal{L}_+$ are: $\mathbb{C} \equiv$ $\equiv (+\mathbb{1}, -\mathbb{1})$.

Usually, even the piece (4b) is discarded. Our present aim is to show — on the contrary — that a physical meaning can be attributed also to the transformations (4b). Confining ourselves here to the *active* point of view (cf. Recami and Rodrigues[1] and references therein), we wish precisely to show that the theory of SR, once based on the *whole* proper Lorentz group (5) and not only on its orthochronous part, will describe a Minkowski space-time populated by both matter and antimatter.

2.1. The Stückelberg-Feynman "switching principle" in SR.

Besides the usual postulates of SR (Principle of Relativity, and Light-Speed Invariance), let us assume — as commonly admitted, e.g. for the reasons in Garuccio *et al.*[9], Mignani and Recami[3] — the following:

Assumption – ⟨⟨ negative-energy objects travelling forward in time do *not* exist⟩⟩ . We shall give this Assumption, later on, the status of a fundamental postulate.

Let us therefore start from a positive-energy particle P travelling forward in time. As wellknown, any orthochronous LT (4a) transforms it into another particle still endowed with positive energy and motion forward in time. On the contrary, any antichronous (=non-orthochronous) LT (4b) will change sign — among the others — to the time-components of *all the four-vectors* associated with P. Any $L^{\downarrow}$ will transform P into a particle P' endowed in particular with negative energy *and* motion backwards in time. (Fig. 1).

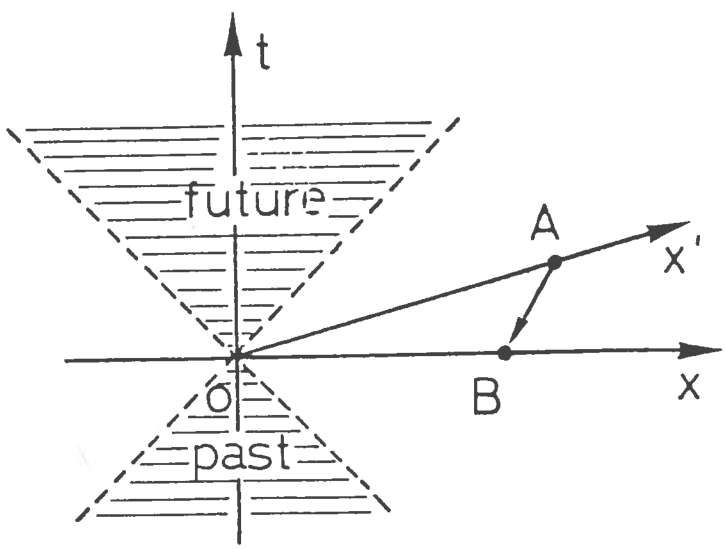


Fig. 1

In other words, SR together with the natural Assumption above *implies* that a particle going backwards in time (Gödel[4]) (Fig. 1) corresponds in the four-momentum space, Fig. 2, to a particle carrying negative energy; and, vice-versa, that changing the energy sign in one space corresponds to changing the sign of time in the dual space. It is then easy to see that these two paradoxical occurrences ("negative energy" and "motion backwards in time") give rise to a phenomenon that

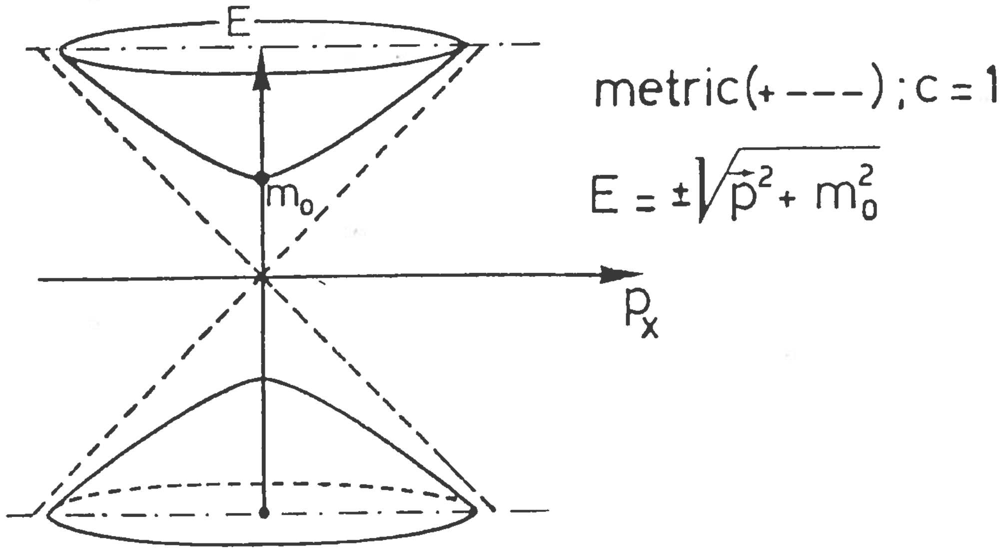


Fig. 2

any observer will describe in a quite *orthodox* way, when they are — as they actually are — simultaneous (Recami[5,6] and refs. therein).

Notice, namely, that: (i) every observer (a macro-object) explores space-time, Fig. 1, in the positive t-direction, so that we shall meet B as the first and A as the last event; (ii) emission of positive quantity is equivalent to absorption of negative quantity, as (-) · (-) = = (+) · (+); and so on.

Let us now suppose (Fig. 3) that a particle P' with negative energy (and e.g. charge $-e$) moving backwards in time is emitted by A at time t_1 and absorbed by B at time $t_2 < t_1$. Then, it follows that at time t_1 the object A "looses" negative energy and charge, i. e. *gains* positive energy and charge. And that at time $t_2 < t_1$ the object B "gains" negative energy and charge, i.e. *looses* positive energy and charge. The physical phenomenon here described is nothing but the exchange *from* B *to* A of a particle Q with *positive* energy, charge $+e$, and going *forward* in time. Notice that Q has, however, charges *opposite* to P'; this means that in a sense the present "switching procedure" (previously called "RIP") effects a "charge conjugation" C, among

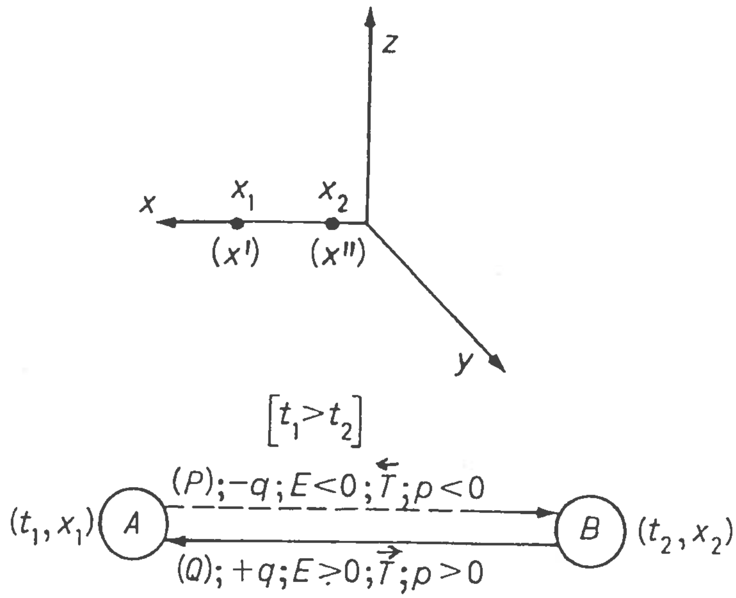


Fig. 3(a)

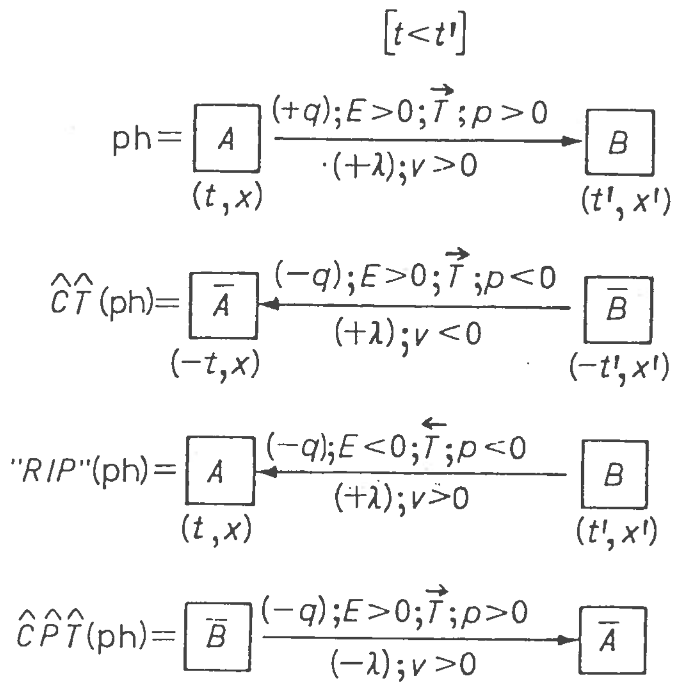


Fig.3(b)

the others. Notice also that "charge", here and in the following, means *any* additive charge; so that our definitions of charge conjugation, etc., are more general than the ordinary ones (Recami and Mignani [(7)], hereafter called Review I; Recami [(8)]). Incidentally, such a switching procedure has been shown to be equivalent to applying the chirality operation γ_5 (Recami and Ziino[9]).

2.2. Matter and Antimatter from SR

A close inspection shows the application of any antichronous transformation $L^{\downarrow}$, together with the switching procedure, to transform P into an object

$$Q \equiv \overline{P} \tag{8}$$

which is indeed the *antiparticle* of P. We are saying that *the concept of antimatter is a purely relativistic one*, and that, on the basis of the double sign in [c = 1]

$$E = \pm\sqrt{\vec{p}^{\,2} + m_o^2} \;, \tag{9}$$

the existence of antiparticles could have been predicted from 1905, exactly with the properties they actually exibited when later discovered, provided that recourse to the "switching procedure" had been made. We therefore maintain that the points of the lower hyperboloid sheet in Fig. 2 — since they corresponds not only to negative energy but also to motion backwards in time — represent the kinematical states of the *antiparticle* $\overline{P}$ (*of* the particle P represented by the upper hyperboloid sheet). Let us explicitly observe that the switching procedure exchanges the roles of *sources* and *detector*, so that (Fig. 1) any observer will describe B to be the source and A the detector of the antiparticle $\overline{P}$.

Let us stress that the switching procedure not only can, but *must* be performed, since any observer can do nothing but explore space-time along the positive time-direction. That procedure is merely the translation into a purely relativistic language of the Stückelberg-Feynman[10] "Switching principle". Together with our Assumption above, it can take the form of a "Third Postulate": ⟨⟨ Negative - energy objects travelling forward in time do not exist; any negative-energy object P travelling backwards in time can and must be described as its anti-object $\overline{P}$ going the opposite way in space (but endowed with positive energy and motion forward in time)⟩⟩ . Cf. e.g. Caldirola and Recami[11], Recami[6] and references therein.

2.3. Further remarks

a) Let us go back to Fig. 1. In SR, when based only on the two ordinary postulates, nothing prevents a priori the event A from influencing the event B. Just to forbid such a possibility we introduced our Assumption together with the Stückelberg-Feynman "Switching procedure". As a consequence, not only we eliminate any particle-motion backwards in time, but we also "predict" and naturally explain within SR the existence of antimatter.

b) The Third Postulate, moreover, helps solving the paradoxes connected with the fact that all relativistic equations admit, besides standard "retarded" solutions, also "advanced" solutions: The latter will simply represent antiparticles travelling the opposite way (Mignani and Recami[12]). For instance, if Maxwell equations admit solutions in terms of *outgoing* (polarized) photons of helicity $\lambda=+1$, then they will admit also solutions in terms of incoming (polarized) photons of helicity $\lambda=-1$; the actual intervention of one or the other solution in a physical problem depending only on the initial conditions.

c) Eqs. (7), (8) tell us that, in the case considered, any $\overline{L}^{\downarrow}$ has the same kinematical effect than its "dual" transformation $\overline{L}^{\uparrow}$ just defined through eq. (7), except for the fact that it moreover transforms P into its antiparticle $\overline{P}$. Eqs. (7), (7') then lead (Mignani and Recami[13,15]) to write

$$-\mathbb{1} \equiv \overline{P}\,\overline{T} = CPT \ , \qquad (10)$$

where the symmetry operations $\overline{P}$, $\overline{T}$ are to be understood in the "strong sense": For instance, $\overline{T}$ = reversal of the time-components of all fourvectors associated with the considered phenomenon (namely, inversion of the time *and* energy axes). We shall come back to this point. The discrete operations P,T have the ordinary meaning. When the particle considered in the beginning can be regarded as an extended object, Pavšič and Recami(16) have shown the "strong" operations $\overline{P}$, $\overline{T}$ to be equivalent to the space, time reflections acting on the space-time *both* external *and* internal to the particle world-tube.

Once accepted eq. (10), then eq.(7') can be written

$$\mathcal{L}_+^{\downarrow} = (\overline{P}\,\overline{T})\,\mathcal{L}_+^{\uparrow} \equiv (\mathrm{CPT})\mathcal{L}_+^{\uparrow} \quad . \qquad (7'')$$

In particular, the total-inversion $\overline{L}^{\downarrow} = -\mathbb{1}$ transforms the process $a + b \rightarrow c + d$ into the process $\overline{d} + \overline{c} \rightarrow \overline{b} + \overline{a}$ without any change in the velocities.

d) All the ordinary relativistic laws (of Mechanics and Electromagnetism) are actually already covariant under the *whole* proper group $\mathcal{L}_+$, eq.(5), since they are CPT-symmetric besides being covariant under $\mathcal{L}_+^{\uparrow}$.

e) A few quantities that happened (cf. Sect. 5.17 in the following) to be Lorentz-invariant under the transformations $L^{\uparrow} \in \mathcal{L}_+^{\uparrow}$, are no more invariant under the transformations $L \in \mathcal{L}_+$. We have already seen this to be true for the *sign* of the additive charges, e.g. for the sign of the electric charge $\underline{e}$ of a particle P. The ordinary derivation of the electric-charge invariance is obtained by evaluating the integral flux of a current through a surface which, under $L^{\uparrow}$, does move, changing the angle formed with the current. Under $L^{\downarrow} \in \mathcal{L}_+^{\downarrow}$ the surface "rotates" so much with respect to the current (cf. also Figs. 6,12 in the following) that the current enters is through the opposite face; as a consequence, the integrated flux (i.e. the charge) changes sign.

PART II : BRADYONS AND TACHYONS IN SR

3. PRELIMINARIES ABOUT TACHYONS

Let us now take on the issue of Tachyons (T).

Tachyons, or space-like particles, are already known to exist as *internal, intermediate states* or exchanged objects. Can they also exist as "asymptotically free" objects?

We shall see that the particular — and unreplaceable — role in SR of the light-speed c in vacuum is due to its *invariance* (namely, to the experimental fact that c does not depend on the velocity of the source), and *not* to its being or not the maximal speed (Recami and Modica[17]).

Since a priori we know nothing about Ts, the safest way to build up a theory for them is trying to generalize the ordinary theories (starting with the classical relativistic one, only later on passing to the quantum field theory) through "minimal extensions", i.e. by performing modifications as small as possible. Only afer possessing a theoretical model we shall be able to start experiments: Let us remember that, not only good experiments are required before getting sensible ideas (Galilei[18,19]), but also a good theoretical background is required before sensible experiments can be performed.

The first step consists therefore in facing the problem of extending SR to Tachyons. In so doing, some authors limited themselves to consider objects both subluminal and Superluminal, always referred however to subluminal observers ("weak approach"). Other authors attempted on the contrary to generalize SR by introducing both subluminal observers (s) and Superluminal observers (S), and then by extending the Principle of Relativity ("strong approach"). This second approach is theoretically more worth of consideration (tachyons, e.g., get *real* proper-masses), but is meets of course the greatest obstacles. In fact, the extension of the Relativity Principle to Superluminal inertial frames seems to be straighforward only in the pseudo-Euclidean space-times M(n,n) having the same number n of space-axes and of time-axes.

For instance, when facing the problem of generalizing the Lorentz transformations to Superluminal frames in four dimensions one meets no-go theorems as Gorini's *et al.* (Gorini[20] and refs. therein), stating *no* such extensions exist which satisfy *all* the following properties: (i) to refer to the four-dimensional Minskowski space-time $M_4 \equiv M(1,3)$; (ii) to be real; (iii) to be linear; (iv) to preserve the space isotropy; (v) to preserve the light-speed invariance; (vi) to possess the prescribed group-theoretical properties.

We shall therefore start by sketching the simple, instructive and very promising "model-theory" in two dimensions (n=1).

Let us first revisit, however, the postulates of the ordinary SR.

4. THE POSTULATES OF SR REVISITED

Let us adhere to the ordinary postulates of SR. A suitable choice of Postulates is the following one (Review I; Maccarrone and Recami[21,22] and refs. therein):

1) First Postulate - Principle of Relativity: ⟨⟨The physical laws of Electromagnetism and Mechanics are covariant (= invariant in form) when going from an inertial frame $\underline{f}$ to another frame moving with constant velocity $\vec{u}$ relative to $\underline{f}$⟩⟩.

2) Second Postulate - "Space and time are homogeneous and space is isotropic". For future convenience, let us give this Postulate the form: ⟨⟨The space-time accessible to any inertial observer is four-dimensional. To each inertial observer the 3-dimensional Space appears as *homogeneous and isotropic*, and the 1-dimensional Time appears as *homogeneous*⟩⟩.

3) Third Postulate - Principle of Retarded Causality: ⟨⟨Positive-energy objects travelling backwards in time *do not exist*; and any negative-energy particle P travelling backwards in time can and must be described as its antiparticle $\overline{P}$, endowed with positive energy and motion forward in time (but going the opposite way in space)⟩⟩. See Sects. 2.1, 2.2.

The *First Postulate* is inspired to the consideration that all inertial frames should be *equivalent* (for a careful definition of "equivalence" see e.g. Recami[6]); notice that this Postulate does *not* impose any constraint on the relative speed $u \equiv |\vec{u}|$ of the two inertial observers, so that a priori $-\infty < u < +\infty$. The *Second Postulate* is justified by the fact that from it the conservation laws of energy, momentum and angular-momentum follow, which are well verified by experience (at least in our "local" space-time region); let us add the following comments: (i) The words homogeneous, isotropic refer to space-time properties assumed — as always — with respect to the electromagnetic and mechanical phenomena; (ii) Such properties of space-time are supposed by this Postulate to be covariant within the class of the inertial frames; this means that SR assumes the vacuum (i.e. space) to be "at rest" with respect to *every* inertial frame. The *Third Postulate* is inspired to the requirement that *for each observer* the "causes" chronologically preced their own "effects" (for the definition of causes and effects see e.g. Caldirola and Recami[11]). Let us recall that in Sect. 2 the initial statement of the Third Postulate has been shown to be equivalent — as it follows from Postulates 1) and 2) — to the more natural Assumption that ⟨⟨ negative-energy objects travelling forward in time do *not* exist⟩⟩ .

4.1. Existence of an invariant speed

Let us initially *skip* the Third Postulate.

Since 1910 it has been shown (Ignatowski[23], Frank and Rothe[24], Hahn[25], Lalan[26], Severi[27], Agodi[28], Di Jorio[29]) that the postulate of the light-speed invariance is not strictly necessary, in the sense that ourPostulates 1) and 2) *imply* the existence of an invariant speed (*not* of a maximal speed, however). In fact, from the first two Postulates it follows (Rindler[30], Berzi and Gorini[31], Gorini and Zecca[32] and refs. therein, Lugiato and Gorini[33]) that one *and only one* quantity w^2 — having the physical dimensions of the square of a speed — must exist, which has the same value according to all inertial frames:

$$w^2 = \text{invariant.} \tag{13}$$

If one assumes $w = \infty$, as done in Galilean Relativity, then one would get Galilei-Newton physics; in such a case the invariant speed is the infinite one: $\infty \oplus v = \infty$, where we symbolically indicated by $\oplus$ the operation of speed composition.

If one assumes the invariant speed to be finite and real, then one gets immediately Einstein's Relativity and physics. Experience has actually shown us the speed c of light in vacuum to be the (finite) invariant speed: $c \oplus v = c$. In this case, of course, the infinite speed is no more invariant: $\infty \oplus v = V \neq \infty$. It means that in SR the operation $\oplus$ is not the operation + of arithmetics.

Let us notice once more that the unique role in SR of the light-speed c in vacuum rests on its being invariant and not the maximal one (see e.g. Shankara(34), Recami and Modica(17)); if tachyons — in particular infinite-speed tachyons — exist, they could *not* take over the role of light in SR (i.e. they could not be used by different observers to compare the sizes of their space and time units, etc.), just in the same way as bradyons cannot replace photons. The speed c turns out to be a limiting speed; but any limit can possess *a priori* two sides (Fig. 4).

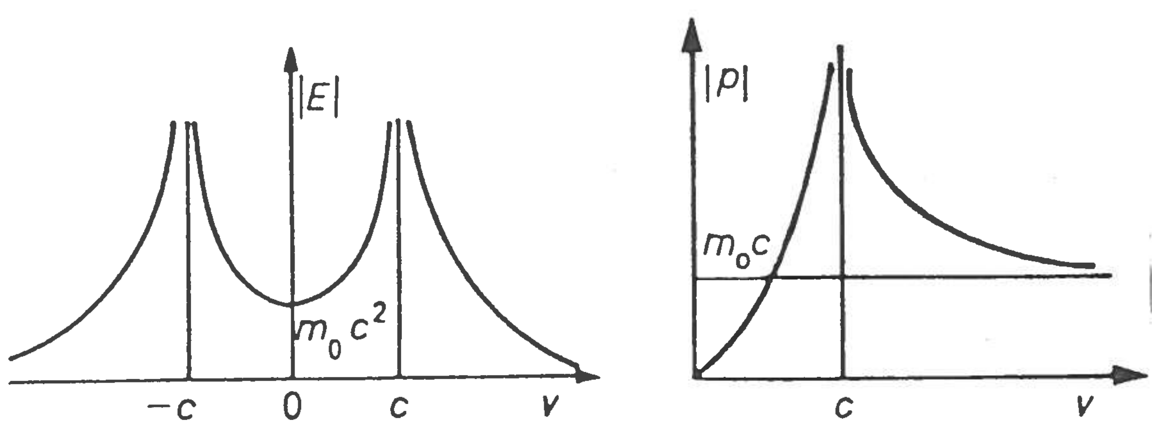


Fig.4(a) Fig.4(b)

4.2. The Problem of Lorentz Transformations.

Of course one can substitute the light-speed invariance Postulate for the assumption of space-time homogeneity and space isotropy (see the Second Postulate).

In any case, from the first two Postulates it follows that the transformations connecting two generic inertial frames f, f', a priori with $-\infty < |u| < +\infty$,

$$dx'_{\mu} = G^{\nu}_{\mu} dx_{\nu} \tag{14}$$

must (cf. Sect. 2):

(i) transform inertial motion into inertial motion;
(ii) form a group **G**;
(iii) preserve space isotropy;
(iv) leave the quadratic form invariant, *except for its sign* (Rindler[35], Landau and Lifshitz[36]):

$$dx'_{\mu} dx'^{\mu} = \pm dx_{\mu} dx^{\mu} . \tag{15}$$

Notice that eq.(15) imposes — among the others — the light-speed to be invariant (Jammer[37]). Eq.(15) holds for any quantity dx_{μ} (position, momentum, velocity, acceleration, current, etc.) that be a G-*fourvector*, i.e., that behaves as a fourvector under the transformations belonging to **G**. If we explicitly confine ourselves to slower-than-light relative speeds, $u^2 < c^2$, then we have to skip in eq.(15) the sign minus, and we are left with eq.(2) of Sect.2. In this case, in fact, one can start from the identity transformation $G = \mathbb{1}$, which requires the sign plus, and then retain such a sign for continuity reasons.

On the contrary, the sign minus will play an important role when we are ready to go beyond the light-cone discontinuity. In such a perspective, let us preliminary clarify — on a formal ground — what follows (Maccarrone and Recami[21,22]).

4.3. Orthogonal and Antiorthogonal Transformations: Digression

4.2.1. Let us consider a space having, in a certain initial base, the metric $g^{\mu\nu}$, so that for vectors dx^{α} and tensors $M^{\alpha\beta}$ it is

$$dx^{\alpha} = g^{\alpha\beta}\, dx_{\beta} \quad ; \qquad M^{\alpha\beta} = g^{\alpha\gamma} g^{\beta\delta} M_{\gamma\delta} \; .$$

When passing to another base, one writes

$$dx'^{\mu} = g'^{\mu\nu} dx'_{\nu} \quad ; \qquad M'^{\alpha}{}_{\beta} = g'^{\alpha\gamma} M_{\gamma\beta} \; .$$

In the two bases, the scalar products are defined

$$dx_{\alpha} dx^{\alpha} \equiv dx_{\alpha} g^{\alpha\beta} dx_{\beta} \; ; \quad dx'_{\mu} dx'^{\mu} \equiv dx'_{\mu} g'^{\mu\nu} dx'_{\nu} \quad ,$$

respectively.

Let us call A the transformation from the first to the second base, in the sense that

$$dx'^{\mu} = A^{\mu}{}_{\rho}\, dx^{\rho} \, ,$$

that is to say

$$dx^{\mu} = (A^{-1})^{\mu}{}_{\rho}\, dx'^{\rho} \; .$$

Now, if we *impose* that

$$dx_{\alpha} dx^{\alpha} = + dx'_{\mu} dx'^{\mu}, \text{ (assumption)} \tag{16}$$

we get

$$g_{\alpha\beta} = g'_{\mu\nu} A^{\mu}{}_{\alpha} A^{\nu}{}_{\beta} \quad ; \tag{17}$$

however, if we *impose* that

$$dx_\alpha dx^\alpha = -dx'_\mu dx'^\mu \text{ , (assumption)} \tag{16'}$$

we get that

$$g_{\alpha\beta} = -g'_{\mu\nu} A^\mu{}_\alpha A^\nu{}_\beta . \tag{17'}$$

4.3.2. Let us consider the case (16)-(17), i.e.

$$dx_\alpha dx^\alpha = dx'_\mu dx'^\mu, \text{ (assumption)} \tag{16}$$

and let us look for the properties of transformations A which yield

$$g'_{\mu\nu} = g_{\mu\nu} . \qquad \text{(assumption)} \tag{18}$$

It must be

$$g_{\alpha\beta} = g_{\mu\nu} A^\mu{}_\alpha A^\nu{}_\beta ; \qquad \text{i.e.} \qquad g_{\alpha\beta} = A^\mu{}_\alpha A_{\mu\beta} , \tag{17}$$

wherefrom

$$g^{\gamma\alpha} g_{\alpha\beta} = g^{\gamma\alpha} A^\mu{}_\alpha A_{\mu\beta} = A^{\mu\gamma} A_{\mu\beta} = (A^T)^{\gamma\mu} A_{\mu\beta} . \tag{19}$$

At this point, if we impose that in the initial base

$$g_{\mu\nu} \equiv \eta_{\mu\nu} , \qquad \text{(assumption)} \tag{20}$$

then eq.(19) yields

$$\delta^\gamma{}_\beta = (A^T)^{\gamma\mu} A_{\mu\beta} = g^{\mu k} (A^T)^\gamma{}_k g_{\mu\sigma} A^\sigma{}_\beta = \delta^k{}_\sigma (A^T)^\gamma{}_k A^\sigma{}_\beta = (A^T)^\gamma{}_k A^k{}_\beta ,$$

that is to say

$$(A^T)(A) = \mathbf{1} . \tag{21}$$

4.3.3. Now, in the case (16')-(17'), i.e.

$$dx_\alpha dx^\alpha = -dx'_\mu dx'^\mu \quad , \text{ (assumption)} \tag{16'}$$

when

$$g_{\alpha\beta} = -g'_{\mu\nu} A^\mu{}_\alpha A^\nu{}_\beta , \tag{17'}$$

let us investigate which are the properties of transformations A that yield

$$g'_{\mu\nu} = - g_{\mu\nu} . \quad \text{(assumption)} \tag{18'}$$

In the particular case, again, when

$$g_{\mu\nu} \equiv \eta_{\mu\nu} \quad , \quad \text{(assumption)} \tag{20}$$

it must be

$$g_{\alpha\beta} = - (-g_{\mu\nu}) A^\mu{}_\alpha A^\nu{}_\beta ,$$

i.e. transformations A must still be orthogonal:

$$(A^T)(A) = \mathbb{1} . \tag{21}$$

In conclusion, transformations A when *orthogonal* operate in such a way that

either: (i) $dx_\alpha dx^\alpha = + dx'_\mu dx'^\mu$ and $g'_{\mu\nu} = + \eta_{\mu\nu}$, (22<u>a</u>)

or: (ii) $dx_\alpha dx^\alpha = -dx'_\mu dx'^\mu$ and $g'_{\mu\nu} = - \eta_{\mu\nu}$. (22<u>b</u>)

4.3.4. On the contrary, let us now require that

$$dx_\alpha dx^\alpha = -dx'_\mu dx'^\mu \quad \text{(assumption)} \tag{16'}$$

when

$$g_{\alpha\beta} = -g'_{\mu\nu}\, A^{\mu}{}_{\alpha} A^{\nu}{}_{\beta} , \qquad (17')$$

and simultaneously let us look for the transformations A such that

$$g'_{\mu\nu} = + g_{\mu\nu} \ . \ \text{(assumption)} \qquad (18)$$

In this case, when in particular assumption (20) holds, $g_{\mu\nu} \equiv \eta_{\mu\nu}$, we get that transformations A must be *anti-orthogonal*:

$$(A^T)(A) = -\mathbb{1}. \qquad (23)$$

4.3.5. The same result (23) is easily obtained when assumptions (16) and (18') hold, together with condition (20).

In conclusion, transformations A when *anti-orthogonal* operate in such a way that

$$\text{either: (i)} \quad dx_{\alpha}dx^{\alpha} = -dx'_{\mu}dx'^{\mu} \quad \text{and} \quad g'_{\mu\nu} = +\eta_{\mu\nu} \ , \qquad (24\underline{a})$$

$$\text{or:} \quad \text{(ii)} \quad dx_{\alpha}dx^{\alpha} = +dx'_{\mu}dx'^{\mu} \quad \text{and} \quad g'_{\mu\nu} = -\eta_{\mu\nu} \ . \qquad (24\underline{b})$$

4.3.6. For passing from sub- to Super-luminal frames we shall have(see the following) to adopt *antiorthogonal* transformations. Then, our conclusions (22) and (24) show that we will have to impose a sign-change either in the quadratic form (20'), or in the metric (22'), but *not* — of course — in both: otherwise one would get, as known, an ordinary and not a Superluminal transformation (cf. e.g. Mignani and Recami[38]). We expounded here such considerations, even if elementary, since they arose some misunderstandings (e.g., in Kowalczyński[39]). We choose to assume always (unless differently stated in explicit way):

$$g'_{\mu\nu} = + g_{\mu\nu} \ . \qquad (25)$$

Let us add the following comments. One could remember the theorems of Riemannian geometry (theorems so often used in General Relativity), which state the quadratic form to be positive-definite and the $g_{\mu\nu}$-signature to be invariant, and therefore wonder how it can be possible for our anti-orthogonal transformations to act in a different way. The fact is that the pseudo-Euclidean (Minkowski) space -time is *not* a particular Riemannian manifold, but rather a particular Lorentzian (i.e., pseudo-Riemannian) manifold. The space-time itself of General Relativity (GR) is *pseudo-Riemannian* and not Riemannian (only space is Riemannian in GR): see e.g. Sachs and Wu[(40)]). In other words, the antiorthogonal transformations do *not* belong to the ordinary group of the so-called "arbitrary" coordinate-transformations *usually* adopted in GR, as outlined e.g. by Møller[(41)]. This has been overlooked, by authors as e.g. Kowalczyński[(39)]. However, by introducing suitable scale-invariant coordinates (e.g. dilation-covariant "light-cone coordinates"), both sub- and Super-luminal "Lorentz transformations" can be formally written (Maccarrone *et al.*[(42)] in such a way to preserve the quadratic form, its sign included (see Sect. 5.8).

Throughout this paper we shall adopt (when convenient) natural units [c = 1].

5. A MODEL-THEORY FOR TACHYONS: AN "EXTENDED RELATIVITY" (ER) IN TWO DIMENSIONS.

Till now we have not taken account of tachyons. Let us finally take them into consideration, starting from a *model-theory*, i.e. from "Extended Relativity" (ER) (Maccarrone and Recami[(21,22)], Maccarrone *et al.*[(42)], Barut *et al.*[(43)], Review I) in two dimensions.

5.1. A Duality Principle.

We got from experience that the invariant speed is $w = c$. Once an inertial frame s_o is chosen, the invariant character of the light-speed allows an exhaustive partition of the set {f}, of all inertial frames f (cf. Sec. 4), into the two disjoint, complementary subsets {s}, {S} of the frames having speeds $|u| < c$ and $|U| > c$ relative to s_o, respectively. In the following, for simplicity, we shall consider ourselves as "the observer s_o". At the present time we neglect the *luminal* frames ($u \equiv U = 0$) as "unphysical". The First Postulate requires frames s and S to be equivalent (for such an extension of the criterion of "equivalence" see Caldirola and Recami[11], Recami[6]), and in particular observers S — if they exist — to have at their disposal the same physical objects (rods, clocks, nucleons, electrons, mesons,...) than observers s. Using the language of multidimensional space-times for future convenience, we can say the first two Postulates to require that even observers S must be able to fill their space (as seen by themselves) with a "lattice-work" of meter-sticks and synchronized clocks (Taylor and Wheeler[44]). It follows that objects must exist which are at rest relatively to S and faster-than-light relatively to frames s; this, together with the fact that luxons ℓ show the same speed to any observers s or S, implies that the objects which are braydons B(S) with respect to a frame S must appear as tachyons T(s) with respect to any frame s, and vice-versa:

$$B(S) = T(S) \;;\;\; T(S) = B(s) \;;\; \ell(S) = \ell(s). \qquad (26)$$

The statement that the terms B,T,s,S do not have an absolute, but only a *relative* meaning, and eq.(26), constitute the so-called *duality principle* (Olkhovsky and Recami[45], Recami and Mignani[46,47], Mignani *et al.*[48], Antippa[49], Mignani and Recami[50]).

This means that the relative speed of two frames s_1, s_2 (or S_1, S_2) will always be smaller than c; and the relative speed between two frames s, S will be always larger than c. Moreover, the above exhaustive partition is invariant when s_o is made to vary inside {s} (or inside {S}), whilst the subsets {s}, {S} get on the contrary interchanged when we pass from $s_o \in \{s\}$ to a frame $S_o \in \{S\}$.

The main problem is finding out *how* objects that are subluminal w.r.t. (= with respect to) observers S appear to observers s (i.e. to us). It is, therefore, finding out the (Superluminal) Lorentz transformations — if they exist — connecting the observations by S with the observations by s.

5.2. Sub - and Super-Luminal Lorentz Transformations: Preliminaries.

We neglect space-time translations, i.e. consider only *restricted* Lorentz transformations. All frames are supposed to have the same event a their origin. Let us also recall that in the chronotopical space Bs are characterized by time-like, ℓs by light-like, and Ts by space-like world-lines.

The ordinary, subluminal Lorentz transformations (LT) from s_1 to s_2, or from S_1 to S_2, are known to preserve the four-vector type. After Sect. 5.1, on the contrary, it is clear that the "Superluminal Lorentz transformations" (SLT) from s to S, or from S to s, must transform time-like into space-like quantities, and vice-versa. With the assumption (25) it follows that in eq.(15) the *plus* sign has to hold for LT's and the *minus* sign for SLTs:

$$ds'^2 = \pm\, ds^2 \quad ; \quad [\underline{u}^2 \lessgtr 1] \qquad (15)$$

therefore, in "Extended Relativity" (ER), with the assumption (25), the quadratic form

$$ds^2 \equiv dx_\mu dx^\mu$$

is a *scalar* under LTs, but is a *pseudo-scalar* under SLTs. In the present case, we shall write that LTs are such that

$$dt'^2 - dx'^2 = +\,(dt^2 - dx^2) \ ; \quad [u^2 < 1] \qquad (27\underline{a})$$

while for SLTs it must be

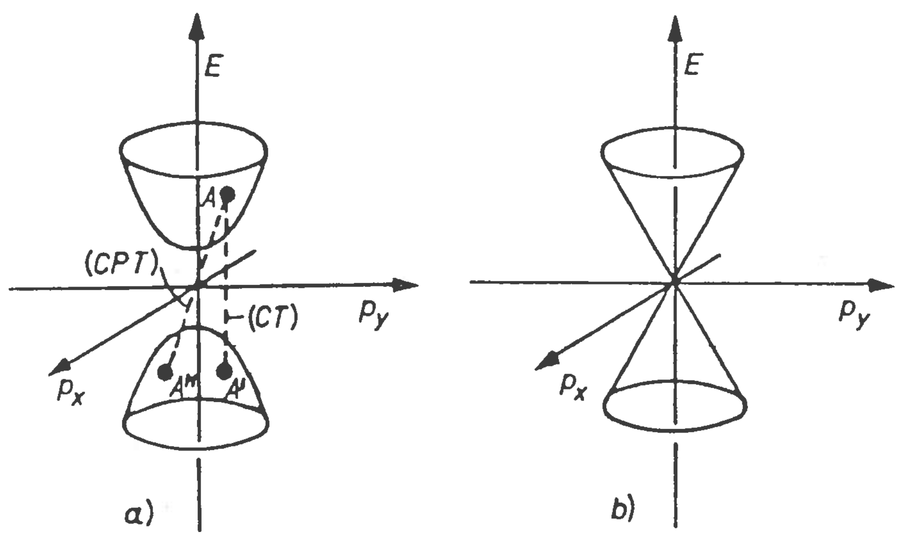
$$dt'^2 - dx'^2 = -\,(dt^2 - dx^2)\ . \quad [u^2 > 1] \qquad (27\underline{b})$$

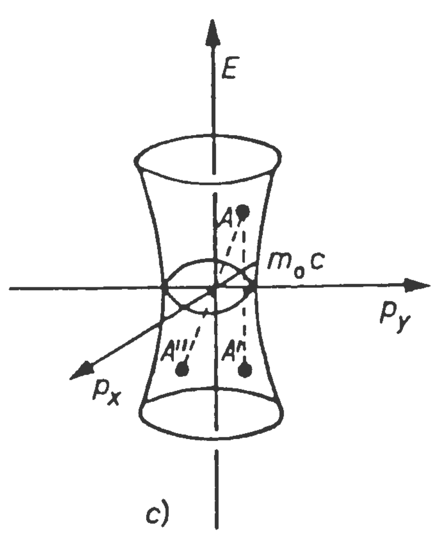
5.3. Energy-Momentum Space.

Since tachyons are just usual particles w.r.t. their own rest frames f, where the fs are Superluminal w.r.t. us, they will possess *real* rest-masses m_o (Recami and Mignani[46], Leiter[5], Parker[52]). From eq.(27b) applied to the energy-momentum vector p^μ, one derives for free tachyons the ralation

$$E^2 - p_x^2 = - m_o^2 < 0 \quad , \quad [m_o \ \text{real}] \tag{28}$$

provided that p^μ is so defined to be a $\mathbb{G}$-vector (see the following); so that one has (cf. Figs. 5):





Figs. 5

$$p_\mu p^\mu = \begin{cases} + m_o^2 > 0 & \text{for braydons (time-like case)} \quad (29\underline{a}) \\ 0 & \text{for luxons (light-like case)} \quad (29\underline{b}) \\ - m_o^2 < 0 & \text{for tachyons (space-like case)} \quad (29\underline{c}) \end{cases}$$

Eqs. (27) - (29) tell us that the roles of space and time and of energy and momentum get interchanged when passing from bradyons to tachyons (See Sect. 5.6). Notice that in the present case (eqs. (29)) it is $\mu = 0,1$. Notice also that tachyons slow down when their energy increases and accelerate when their energy decreases. In particular, divergent energies are needed to slow down the tachyons' speed towards its (lower) limit c. On the contrary, when the tachyons' speed tends to infinity, their energy tends to zero; in ER, therefore, energy can be transmitted only at *finite* velocity. From Figs. 5a,c it is apparent that a braydon may have zero momentum (and minimal energy $m_o c^2$), and a tachyon may have zero energy (and minimal momentum $m_o c$); however Bs cannot exist at zero energy, and tachyons cannot exist at zero momentum (w.r.t. the observers to whom they appear as tachyons!). Incidentally, since *transcedent* (=infinite--speed) tachyons do *not* transport energy but do transport momentum ($m_o c$), they allow getting thr *rigid body* behaviour even in SR (Bilaniuk and Sudarshan[53], Review I, Castorina and Recami[54]). In particular, in elementary particle physics, they might a priori be useful for interpreting in the suitable reference frames the diffractive scatterings, elastic scatterings, etc. (Maccarrone and Recami[55] and refs. therein).

5.4. Generalized Lorentz Transformations (GLT): Preliminaries

Eqs. (27a,b), together with requirements (i)-(iii) of Sect. 4.2, finally imply the LTs to be *orthogonal* and the SLTs to be *anti-orthogonal* (Maccarrone *et al.*[42] and refs. therein):

$$G^T G = + \mathbb{1} \qquad \text{(subluminal case:} \quad u^2 < 1); \qquad (30\underline{a})$$

$$G^T G = - \mathbb{1} \qquad \text{(Superluminal case:} \; u^2 > 1), \qquad (30\underline{b})$$

as anticipated at the end of Sect. 4.3. Both sub- and Super-luminal Lorentz transformations (let us call them "Generalized Lorentz transformations", GLT) result to be unimodular. In the two-dimensional case, however, the SLTs can a priori be special or not; to given them a form coherent with the four-dimensional case (cf. also Sects.5.5, 5.6), one is led to adopt SLTs with negative trace: det $SLT_2 = -1$. In four dimensions, however, all the GLTs will result to be unimodular and special:

$$\det G = + 1 \,, \qquad \forall G \in \mathbb{G} \,. \qquad (31)$$

5.5. The Fundamental Theorem of (Bidimensional) ER.

We have now to write down the SLTs, satisfying the conditions (i)-(iv) of Sect. 4.2 with the sign *minus* in eq.(15), still however with $g'_{\mu\nu} = g_{\mu\nu}$ (cf. Sect.4.3, and Maccarrone and Recami(56)), and show that the GLTs actually from a (new) group $\mathbb{G}$. Let us remind explicitly that an essential ingredient of the present procedure is the assumption that the space-time interval dx^μ is a (chronotopical) vector even with respect to $\mathbb{G}$: see eq. (14).

Any SLT from a sub- to a Super-luminal frame, $s \to S'$, will be identical with a suitable (ordinary) LT — let us call it the "dual" transformation — *except for the fact* that it must change time-like into space-like vectors, and vice-versa, according to eqs.(27b) and (25).

Alternatively, one could say that a SLT is *identical* with its dual subluminal LT, provided that we impose the primed observer S' to use the opposite metric-*signature* $g'_{\mu\nu} = g_{\mu\nu}$, however *without changing* the signs into the definitions of time-like and space-like quantities! (Mignani and Recami(38), Shah(57)).

It follows that a generic SLT, corresponding to a Superluminal velocity U, will be *formally* expressed by the *product* of the dual LT corresponding to the subluminal velocity $u \equiv 1/U$, *by* the matrix $S \equiv S_2 \equiv \mathbb{1}$, where $\mathbb{1}$ is the two-dimensional identity:

$$\begin{cases} SLT(U) = \pm S \cdot LT(u) & (32) \\ \qquad\qquad [u \equiv \frac{1}{U} ;\ u^2 < c^2 ;\ U^2 > c^2] \\ S \equiv i\mathbb{1} . & (33) \end{cases}$$

Transformation $S \equiv S_2 \in \mathbf{G}$ plays the role of the "transcendent SLT" since for $u \to 0$ one gets $SLT(U \to \infty) = \pm\, i\mathbb{1}$. The double sign in eq. (32) is required by condition (ii) of Sect. 4.2; in fact, given a particular subluminal Lorentz transformation L(u) and the SLT = + iL(u), one gets

$$[iL(u)]\,[iL^{-1}(u)] \equiv [iL(u)]\,[iL(-u)] \equiv -\mathbb{1} . \qquad (34a)$$

However

$$[iL(u)]\,[-iL^{-1}(u)] \equiv [iL(u)]\,[-iL(-u)] \equiv +\mathbb{1} . \qquad (34b)$$

Eqs. (34) show that

$$[iL(u)]^{-1} = -iL^{-1}(u) \equiv -iL(-u).$$

5.6. Explicit Form of the Superluminal Lorentz Transformations (SLT) in Two Dimensions.

In conclusion, the Superluminal Lorentz transformations $\pm\, iL(u)$ form a group $\mathbb{G}$ together with *both* the orthochronous *and* the antichronous subluminal LTs of Sect. 2: see Fig. 6. Namely, if $\mathbf{Z}(n)$ is the discrete group of the n-th roots of unity, then the new groups $\mathbb{G}$ of GLTs can be *formally* written down as

$$\begin{cases} \mathbb{G} = \mathbf{Z}(4) \otimes \mathcal{L}_+^{\uparrow} ; & (35) \\ \mathbf{Z}(4) \equiv \{\sqrt[4]{1}\} \equiv \{+1, -1, +i, -i\} , & (36) \end{cases}$$

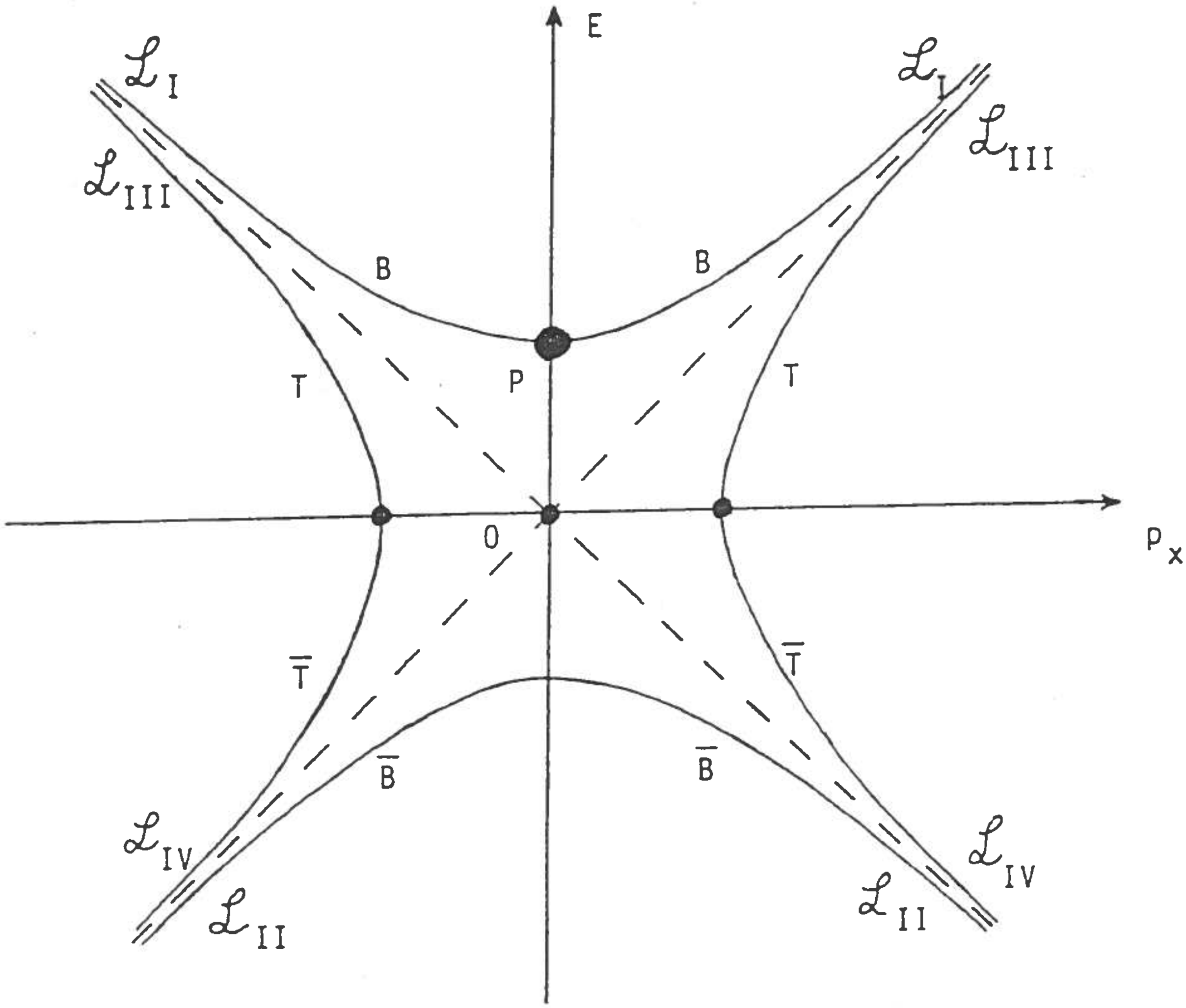


Fig. 6

where $\mathcal{L}_+^\uparrow$ represents *here* the *bidimensional* proper orthochronous Lorentz group. Eq. (35) should be compared with eq.(5'). It is

$$\begin{cases} G \in \mathbb{G} \Longrightarrow -G \in \mathbb{G} \,, \quad \forall G \in \mathbb{G} \,; \\ G \in \mathbb{G} \Longrightarrow \mathcal{S}G \in \mathbb{G} \,, \quad \forall G \in \mathbb{G} \,. \end{cases} \tag{37}$$

The appearance of imaginary units into eqs.(33)-(36) is *only* formal, as it can be guessed from the fact that the transcendent operation $\mathcal{S}_2 \equiv \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$ goes into $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ through a "congruence" (Maccarrone *et al.*[42]):

$$\begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix} = M \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} M^T . \qquad (38)$$

Actually, the GLTs given by eq.(32)-(33), or (35)-(36), simply represent (Review I, p. 232-233) *all* the space-time pseudo-rotations for $0 \leq \alpha \leq 360^o$: see Fig. 7. To show this, let us write down explicitly the SLTs in the following formal way

$$\begin{cases} dt' = \pm i \dfrac{dt - udx}{\sqrt{1-u^2}} ; \\ dx' = \pm i \dfrac{dx - udt}{\sqrt{1-u^2}} . \end{cases} \qquad \left[\begin{matrix}\text{Superluminal case} \\ \underline{u}^2 < 1\end{matrix}\right] \qquad (39)$$

Notice that a priori $\sqrt{\beta^2 - 1} = \pm i\sqrt{1-\beta^2}$, since $(\pm\, i)^2 = -1$. Moreover, we shall always understand that $\sqrt{1-\beta^2}$ for $\beta^2 > 1$ represents the upper half-plane solution.

The two-dimensional space-time $M(1,1) \equiv (t,x)$ can be regarded as a complex-plane; so that the imaginary unit

$$i \equiv \exp[\tfrac{1}{2} i \pi] \qquad (40)$$

operates there as a 90^o pseudo-rotation. The same can be said, of course, for the operation S_2; in accord with eq.(38). Moreover, with regard the *axes* $\underline{x}'$, $\underline{t}'$, $\underline{x}$, $\underline{t}$, both observers s_o, S' will agree in the case of a SLT that: $\underline{t}' \equiv \underline{x}$; $\underline{x}' \equiv \underline{t}$. It follows that eqs.(39) can be immediately rewritten:

$$\left\{ \begin{aligned} dt' &= \pm \frac{dx - udt}{\sqrt{1-u^2}} \, ; \\ dx' &= \pm \frac{dt - udx}{\sqrt{1-u^2}} \, , \end{aligned} \right. \qquad \begin{bmatrix} \text{Superluminal case} \\ u^2 < 1 \end{bmatrix} \tag{39'}$$

where the roles of the space and the time coordinates appear interchanged, but the imaginary units *disappeared.*

Let us now take advantage of a very important symmetry property of the ordinary Lorentz boosts, expresse by the *identities*

$$\left\{ \begin{aligned} \frac{dx - udt}{\sqrt{1-u^2}} &\equiv - \frac{dt - Udx}{\sqrt{U^2 - 1}} \, ; \\ \frac{dt - udx}{\sqrt{1-u^2}} &\equiv - \frac{dx - Udt}{\sqrt{U^2 - 1}} \, . \end{aligned} \right. \qquad [U \equiv 1/u] \tag{41}$$

Eqs.(39') eventually write

$$\left\{ \begin{aligned} dt' &= \mp \frac{dt - Udx}{\sqrt{U^2 - 1}} \, ; \\ dx' &= \mp \frac{dx - Udt}{\sqrt{U^2 - 1}} \, , \end{aligned} \right. \qquad \begin{bmatrix} \text{Superluminal case} \\ U^2 > 1 \end{bmatrix} \tag{39''}$$

which can be assumed as *the canonical form* of the SLTs in two dimensions. Let us observe that eqs.(39') or (39") yield for the speed of s_o w.r.t. S':

$$x \equiv 0 \implies \frac{dx'}{dt'} = \pm\left(-\frac{1}{u}\right) = \mp U\,, \quad \left[\begin{array}{c} u^2 < 1\,; \quad U^2 > 1 \\ \\ U \equiv 1/u \end{array}\right] \tag{42}$$

where u, U are the speeds of the two *dual* frames s, S'. This confirms that eqs.(39), (39") do actually refer to Superluminal relative motion. Even for eqs.(39) one could have derived that the **G**-vectorial velocity $u^\mu \equiv dx^\mu/d\tau_o$ (see the following) changes under trabsformation (39) in such a way that $u'_\mu u'^\mu = - u_\mu u^\mu$; so that from $u_\mu u^\mu = +1$ it follows $u'_\mu u'^\mu = -1$ (that is to say, bradyonic speeds are transformed into tachyonic speeds).

The group **G** of the GLTs in two dimensions can be finally written (Fig. 6):

$$G = \{+L^\uparrow\} \cup \{-L^\uparrow\} \cup \{-SL^\uparrow\} \cup \{+SL^\uparrow\}\ ; \tag{35'}$$

$$S \equiv S_2 \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{36'}$$

Notice that the transcendent SLT, S, when applied to the motion of a particle, just interchanges the values of energy and impulse, as well as of time and space: Cf. also Sects.5.2, 5.3 (Review I; see also Vysin(58)).

5.7. Explicity Form of GLTs

The LTs and SLTs together, i.e. the GLTs, can be written of course in a form covariant under the whole group **G**; namely, in "G-covariant" form, they can be written (Figs.7):

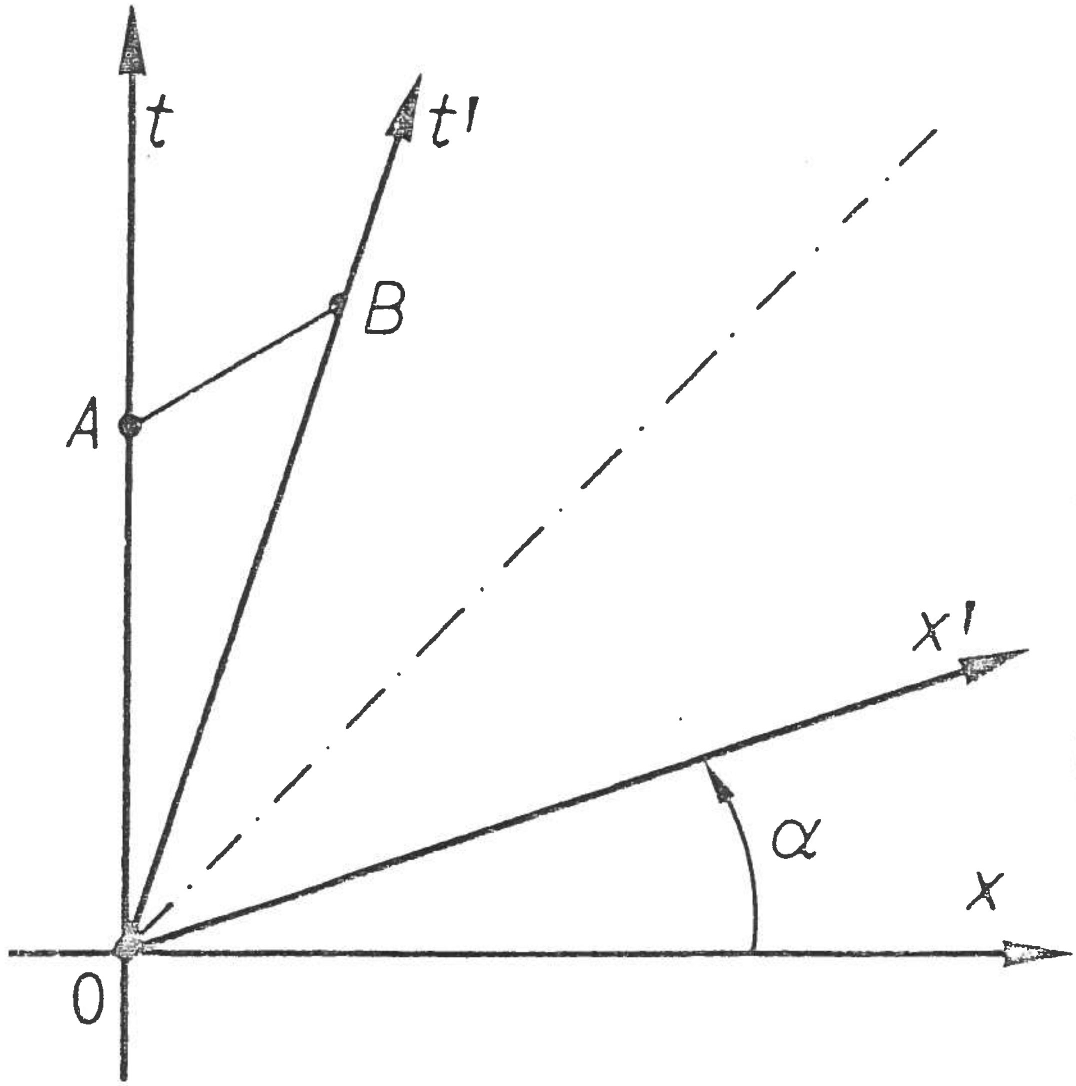


Fig.7(a)

$$\begin{cases} dt' = \pm \dfrac{dt - udx}{\sqrt{|1-u^2|}} \; ; \\ \\ dx' = \pm \dfrac{dx - udt}{\sqrt{|1-u^2|}} \end{cases} \qquad \left[\begin{array}{c}\text{Generalized case} \\ -\infty < u < +\infty\end{array}\right] \tag{43}$$

or rather (Recami and Mignani[47]), in terms of the continuous parameter $\sigma \in [0,2\pi]$,

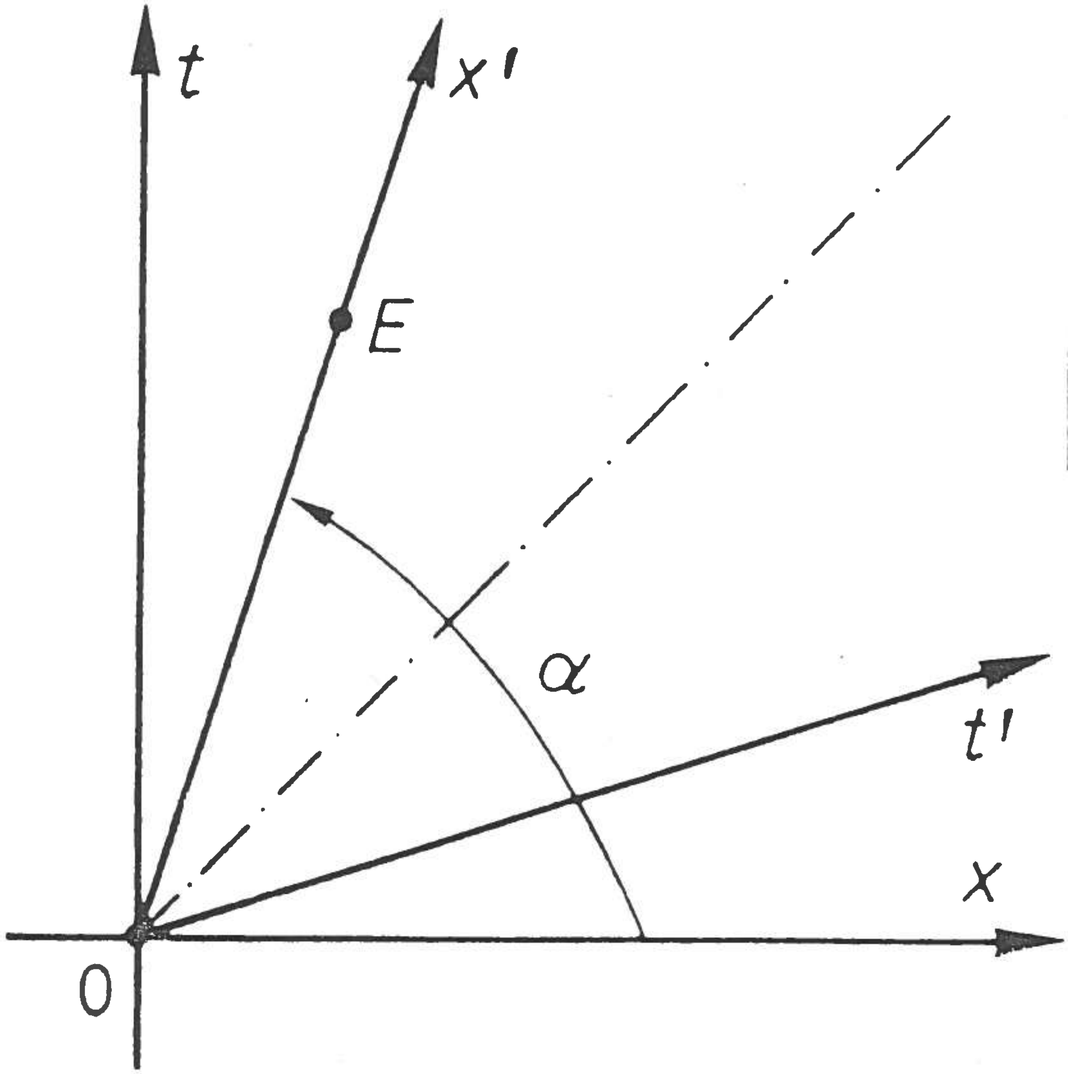


Fig.7(b)

$$\begin{cases} dt' = \Omega\gamma_o(dt - dx\ tg\,\sigma)\ ; \\ dx' = \Omega\gamma_o(dx - dt\ tg\,\sigma)\ , \end{cases} \qquad \begin{bmatrix} -\infty < u < +\infty \\ 0 \leq \sigma \leq 2\pi \end{bmatrix} \tag{43a}$$

with

$$\begin{cases} u \equiv tg\,\sigma\ ;\ \Omega \equiv \Omega\,(\sigma) \equiv \dfrac{\cos\sigma}{|\cos\sigma|}\ \delta^2\ ;\ \delta \equiv +\sqrt{\dfrac{1 - tg^2\sigma}{|1 - tg^2\sigma|}}\ ; \\ \gamma_o = +\,(|1 - tg^2\sigma|)^{-1/2}\ , \end{cases} \tag{43b}$$

where the form (43a) of the GLTs explicitly shows *how* the signs in front of t', x' *succeed* one another as functions of u, or rather of σ (see also the figs. 2-4 and 6 in Review I).

Apart from Somigliana's early paper, only recently rediscovered (Caldirola *et al.*[59]), the eqs. (39"),(43) first appeared in Olkhovsky and Recami[60], Recami and Mignani[46], Mignani *et al.*[48], and then — independently — in a number od subsequent papers: see e.g. Antippa[49] and Ramanujam and Namasivayam[61]. Eqs. (39'), (39") have been shown by Recami and Mignani[46] to be equivalent to the pioneering — even if more complicated — equations by Parker[52]. Only in Mignani *et al.*[48], however, it was first realized that eqs. (39)-(43) need their double sign, necessary in order that *any* GLT admits an inverse transformation (see also Mignani and Recami[50]).

5.8. The GLTs by Discrete Scale Transformations.

If you want, you can regard eqs. (39')-(39") as entailing a "reinterpretation" of eqs. (39), — such a reinterpretation having nothing to do, of course, with the Stückelberg-Feynman "switching procedure", also known as "reinterpretation principle" ("RIP").— Our interpretation procedure, however, not only is straightforward (cf. eqs. (38), (40)), but has been also rendered *automatic* in terms of new, scale-invariant "light-cone coordinates" (Maccarrone *et al.*[42]).

Let us first rewrite the GLTs in a more compact form, by the language of the discrete (real or imaginary) scale transformations (Pavšič and Recami[62], Pavšič[63]):

$$ds'^2 = \rho^2 ds^2 \quad ; \quad \rho^2 = \pm 1 \quad ; \qquad (15')$$

notice that, in eq. (36), $\mathbb{Z}(4)$ is nothing but the discrete group of the dilations $D: x'_\mu = \rho x_\mu$ with $\rho^2 = \pm 1$. Namely, let us introduce the new (discrete) dilation-invariant coordinates (Kastrup[64])

$$\eta^\mu \equiv k x^\mu \, , \quad [k = \pm 1 \, , \pm i] \qquad (44)$$

k being the *intrinsic* scale-factor of the considered object: and let us observe that, under a dilation D, it is $\eta'_\mu = \eta_\mu$ with $\eta'_\mu \equiv k' x'_\mu$, while $k' = \rho^{-1} k$. Braydons (antibraydons) correspond to $k = +1$ ($k=-1$), whilst tachyons and antitachyons correspond to $k = \pm i$. It is interesting that in the present formalism the quadratic form $d\sigma^2 \equiv d\eta_\mu d\eta^\mu$ is invariant, its sign included, under *all* the GLTs:

$$d\sigma'^2 = + d\sigma^2 \;, \quad \forall G \in \mathbb{G}. \qquad (15'')$$

Moreover, under an orthochronous Lorentz transformation $L \in \mathcal{L}_+^\uparrow$, it holds that $\eta'_\mu = L^\mu_{\ \nu} \eta^\nu$; $k' = k$.

It follows — when going back to eq.(14), i.e. to the coordinates x^μ, k — that the generic GLT = G can be written in two dimensions

$$\begin{cases} G = k'^{-1} L k \\ \\ L \in \mathcal{L}_+^\uparrow . \end{cases} \qquad \begin{bmatrix} \rho^2, k^2 = \pm 1 \\ \\ k' = \rho^{-1} k \end{bmatrix} \qquad (45)$$

5.9. The GLTs in the "Light-Cone Coordinates" *Automatic* Interpretation.

It is known (Bjorken *et al.*(65)) that the ordinary subluminal (proper, orthochrounous) boosts along x can be written in the generic form:

$$\begin{cases} d\xi' = \alpha d\xi \; ; \; d\zeta' = \alpha^{-1} d\zeta \; ; \; dy' = \dfrac{\alpha}{|\alpha|} dy \; ; \; dz' = \dfrac{\alpha}{|\alpha|} dz \; ; \; [0 < \alpha < +\infty] \\ \\ \dfrac{\alpha - \alpha^{-1}}{\alpha + \alpha^{-1}} = u \; ; \quad u^2 < 1 \, , \end{cases}$$

in terms of the light-cone coordinates (Fig. 8):

$$\xi \equiv t - x \; ; \quad \zeta \equiv t + x' \; ; \quad y \; ; \; z. \qquad (46)$$

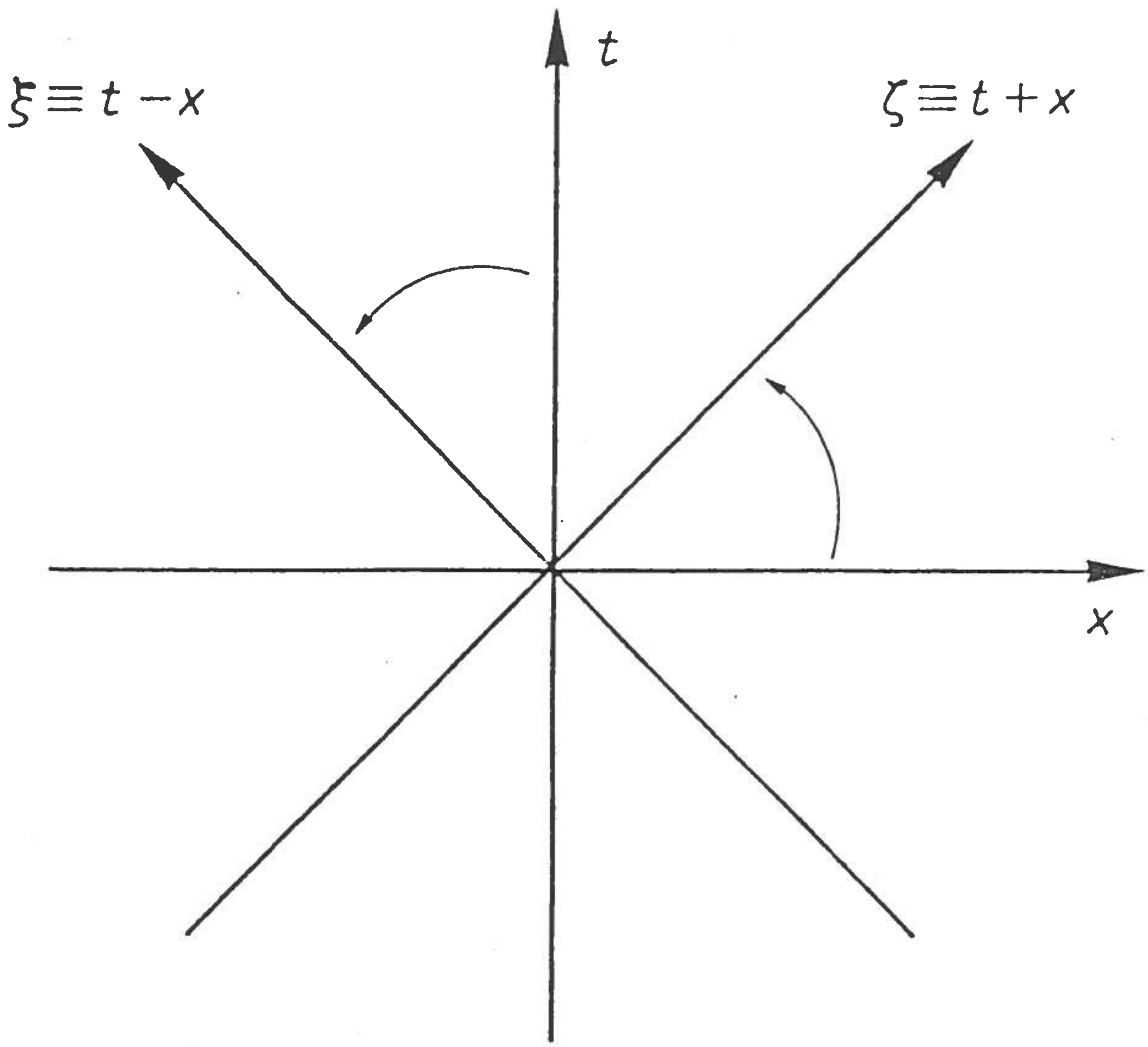


Fig. 8

It is interesting that the orthochronous Lorentz boosts along x just correspond to a dilation of the coordinates ξ,ζ (by the factors α and α^{-1}, respectively, with α any *positive real* number). In particular for $\alpha \to +\infty$ we have $u \to c^-$ and for $\alpha \to 0^+$ we have $u \to -(c^-)$. It is apparent that $\alpha = e^R$, where R is the "rapidity".

The proper *antichronous* Lorentz boosts correspond to the negative real α values (which still yield $u^2 < 1$).

Recalling definitions (44), let us eventually introduce the following *scale-invariant* "light-cone coordinates":

$$\varphi \equiv \eta^{(0)} - \eta^{(1)} \;;\quad \psi \equiv \eta^{(0)} + \eta^{(1)} \;;\; \eta^{(2)} \;;\; \eta^{(3)} \,. \tag{47}$$

In terms of coordinates (47), *all* the two-dimensional GLTs (both sub- and Superluminal) can be expressed in the synthetic form (Maccarrone *et al.*[42]):

$$\begin{cases} d\varphi' = \alpha d\varphi \;;\quad d\psi' = \alpha^{-1}\, d\psi \;;\quad [\,|u| \lessgtr 1\,] \\ k' = \rho^{-1} k \;;\quad \alpha \equiv \rho a \;; a \in (0,+\infty) \;;\; \rho^2 = \pm 1 \,, \end{cases} \tag{48}$$

and all of them preserve the quadratic form, its sign included: $\varphi'\psi' = \varphi\psi$.

The point to be emphasized is that eqs.(48) in the Superluminal case yield directly eq.(39"), i.e. they *automatically* include the "re-interpretation" of eqs.(39). Moreover, eqs.(48) yield

$$\begin{cases} u = \dfrac{\alpha - \alpha^{-1}}{\alpha + \alpha^{-1}} \;; & \begin{bmatrix} u^2 \lessgtr 1 \;; \\ 0 < a < +\infty \end{bmatrix} \\ \alpha = \rho a \;;\quad \rho^2 = \pm 1 \;; \end{cases} \tag{49}$$

i.e. also in the Superluminal case they forward the correct (faster--than-light) relative speed *without any need of "reinterpretation"*.

5.10. An Application

As an application of eqs.(39"), (43), let us consider a tachyon having (real) proper-mass m_o and moving with speed V relatively to us: then we shall observe the relativistic mass

$$m = \frac{m_o}{(|1 - V^2|)^{1/2}} \equiv \frac{-im_o}{(1 - V^2)^{1/2}} \equiv \frac{m_o}{(V^2 - 1)^{1/2}} \,,\; [V^2 > 1 \;;\; m_o \;\text{ real}]$$

and, more in general (in G-covariant form):

$$m = \pm \frac{m_o}{(|1 - v^2|)^{1/2}} \quad , \qquad [-\infty < v < +\infty] \tag{50}$$

so as anticipated in Fig. 4a.

For other applications, see e.g. Review I; for instance: (i) for the generalized "velocity composition law" in two dimensions see eq. (33) and Table I in Review I; (ii) for the generalization of the phenomenon of Lorentz contraction/dilation see Fig. 8 of Review I.

5.11. Dual Frames (Or Objects)

Eqs.(32) and follows. show that a one-to-one correspondence

$$v \longleftrightarrow \frac{c^2}{v} \tag{51}$$

can be set between subluminal frames (or objects) with speed $v < c$ and Superluminal frames (or objects) with speed $V \equiv c^2/v > c$. In such a particular conformal mapping (*inversion*) the speed c is the "united" one, and the speeds zero, infinite correspond to each other. Cf. also Fig. 9, which illustrates the important equation (32). In fact (Review I) the *relative* speed of two "dual" frames s,S (frames dual one to the other are characterized in Fig. 9 by AB being orthogonal to the u-axis) is *infinite*; the figure geometrically depicts, therefore, the circumstance that $(s_o \to S) = (s_o \to s)\cdot(s \to S)$, i.e. the fundamental theorem of the (bidimensional) "Extended Relativity":⟨⟨ The SLT : $s_o \to S(U)$ is the product of the LT : $s_o \to s(u)$, where $u \equiv 1/U$, by the *transcendent* SLT ⟩⟩ : Cf. Sect. 5.5, eq.(32). (Mignani and Recami[50]).

Even in more dimensions, we shall call "dual" two objects (or frames) moving along the same line with speeds satisfying eq.(51):

$$vV = c^2 \quad , \tag{51'}$$

i.e. with infinite *relative* speed. Let us notice that, if p^μ and P^μ are the energy-momentum vectors of the two objects, then the condition of *infinite* relative speed writes in G-invariant way as

$$p_\mu P^\mu = 0 \qquad (51'')$$

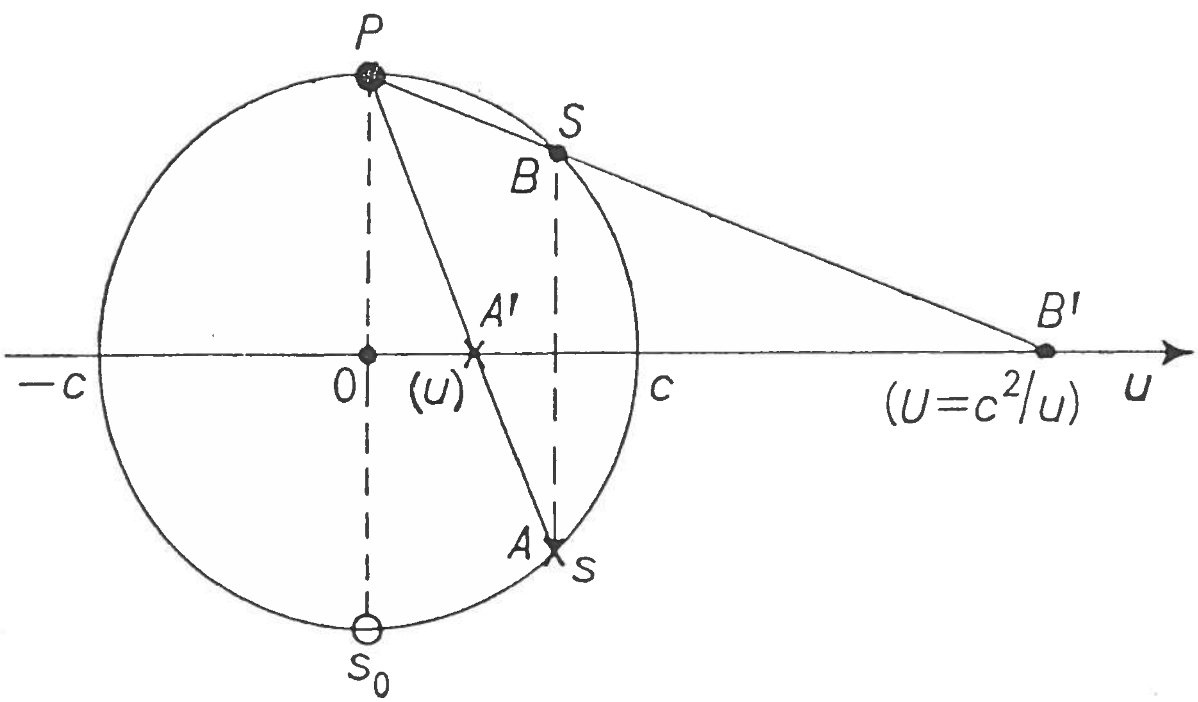


Fig. 9

5.12. The "Switching Principle" for Tachyons.

The problem of the double sign in eq.(50) has been already taken care of in Sect. 2 for the case of braydons (eq.(9)).

Inspection of Fig. 5c shows that, in the case of tachyons, it is enough a (suitable) ordinary subluminal orthochronous Lorentz transformation $L^\uparrow$ to transform a positive-energy tachyon T into a negative-energy tachyon T'. For simplicity let us here confine ourselves, therefore, to transformations $L \equiv L^{\uparrow} \in \mathcal{L}_{+}^{\uparrow}$, acting on free tachyons.

On the other hand, it is wellknown in SR that the chronological order along a space-like path is *not* $\mathcal{L}_{+}^{\uparrow}$-invariant.

However, in the case of Ts it is even clearer than in the bradyon case that the same transformation L which inverts the energy--sign will also reverse the motion-direction in time (Review I, Recami.[66], Caldirola and Recami.[67], see also Garuccio *et al.*[2]). In fact, from Fig. 10 we can see that for going from a positive - energy state $\underline{T}_i$ to a negative-energy state $\underline{T}'_f$ it is necessary to bypass the "transcendent" state $\underline{T}_\infty$ (with $V = \infty$). From Fig. 11a we see moreover that, given in the initial frames s_o a tachyon T travelling e.g. along the positive x-axis with speed V_o, the "critical observer" (i. e. the ordinary subluminal observer $s_c \equiv (t_c, x_c)$ seeing T with *infinite* speed) is simply the one whose space-axis x is superimposed to the world-line OT; its speed u_c w.r.t. s_o, along the positive x-axis, is evidently

$$u_c = c^2/V_o \ ; \quad u_c V_o = c^2 \ , \quad [\text{"critical frame"}]$$

dual to the tachyon speed V_o. Finally, from Fig. 10 and Fig. 11b we conclude that any "trans-critical" observer $s' \equiv (t', x')$ such that $u'V_o > c^2$ will see the tachyon T not only endowed with negative energy, but also travelling backwards in time. Notice, incidentally, that nothing of this kind happens when $u V_o < 0$, i.e. when the final frame moves in the direction opposite to the tachyon's.

Therefore Ts display negative energies in the same frames in which they would appear as "going backwards in time", and vice-versa. As a consequence, we can — and must — apply also to tachyons the Stückelberg-Feynman "switching procedure" exploited in Sects. 2.1-2.3. As a result, point $\underline{A}'$ (Fig. 5c) or point $\underline{T}'_f$ (Fig. 10) do *not* refer to a "negative-energy tachyon moving backwards in time", but rather to an *antitachyon* $\overline{T}$ moving the opposite way (in space), forward in time,

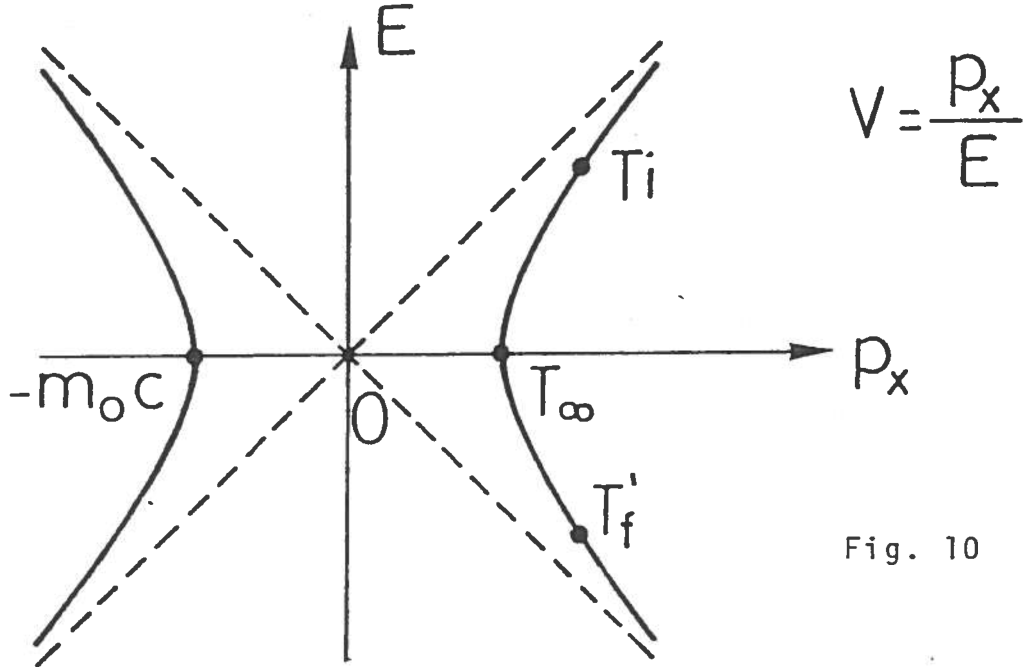
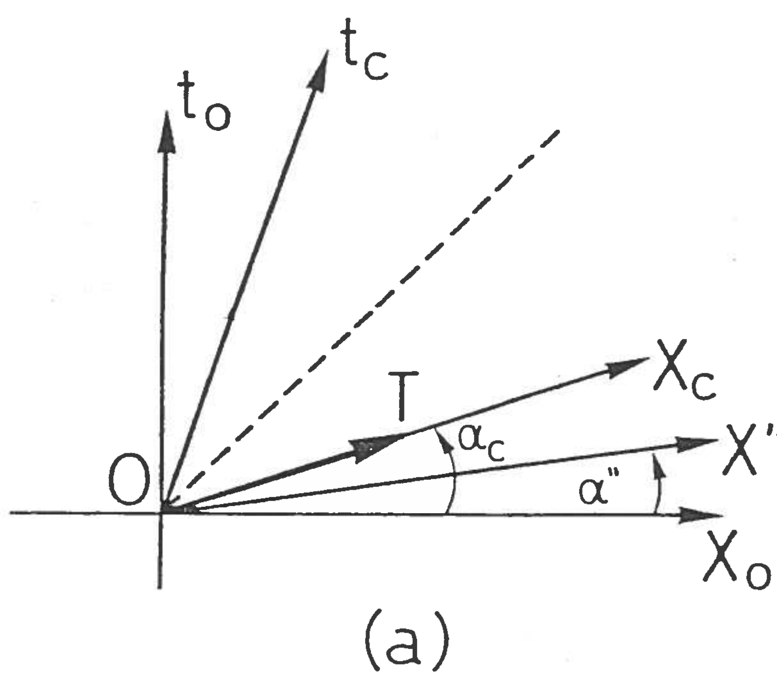


Fig. 10

Fig.11(a)



and with positive energy. Let us repeat that the "switching" never comes into the play when the sign of u is opposite to the sign of V_o. (Review I, Recami[68], Caldirola and Recami[11]).

The "Switching Principle" has been first applied to tachyons by Sudarshan and coworkers (Bilaniuk *et al.*[69]; see also Gregory[70]).

Recently Schwartz[71] gave the switching procedure an interesting formalization, in which — in a sense — it becomes "automatic".

5.13. Sources and Detectors. Causality.

After the consideration in the previous Sect.5.12, i.e. when we apply our Third Postulate (Sect.4) also to tachyons, we are left with *no* negative energies (Recami and Mignani[72]) and with *no* motions backwards in time (Maccarrone and Recami[55,73] and refs. therein).

Let us remind, however, that a tachyon T can be transformed into an antitachyon $\overline{T}$ "going the opposite way in space" even by (suitable) ordinary subluminal Lorentz transformation $L \in \mathcal{L}_+^\uparrow$. It is always essential, therefore, when dealing with a tachyon T, to take into proper consideration also its *source* and *detector*, or at least to refer T to an "interaction-region". Precisely, when a tachyon overcomes the divergent speed, it passes from appearing e.g. as a tachyon T *entering* (leaving) a certain interaction-region to appearing as the antitachyon $\overline{T}$ *leaving* (entering) that interaction-region (Arons and Sudarshan[74], Dhar and Sudarshan[74], Glück[75], Baldo *et al.*[76], Camenzind[77]). More in general, the "trans-critical" transformations $\overline{L} \in \mathcal{L}_+^\uparrow$ (cf. the caption of Fig. 11<u>b</u>) lead from a T emitted by A and absorbed by B to its $\overline{T}$ emitted by B and absorbed by A (see Figs.1 and 3<u>b</u>, and Review I).

The already mentioned fact (Sect.2.2) that the Stückelberg - Feynman-Sudarshan "switching" exchanges the roles of source and detector (or, if you want, of "cause" and "effect") led to a series of apparent "causal paradoxes" which — even if easily solvable, at least in microphysics — gave rise to much perplexity in the literature.

We shall deal with the causal problem elsewhere. Let us here anticipate that, — even if in ER the judgement about which is the "cause" and which is the "effect", and even more about the very existence of a "causal connection", is *relative* to the observer —, nevertheless in microphysics the law of "retarded causality" (see our Third Postulate)

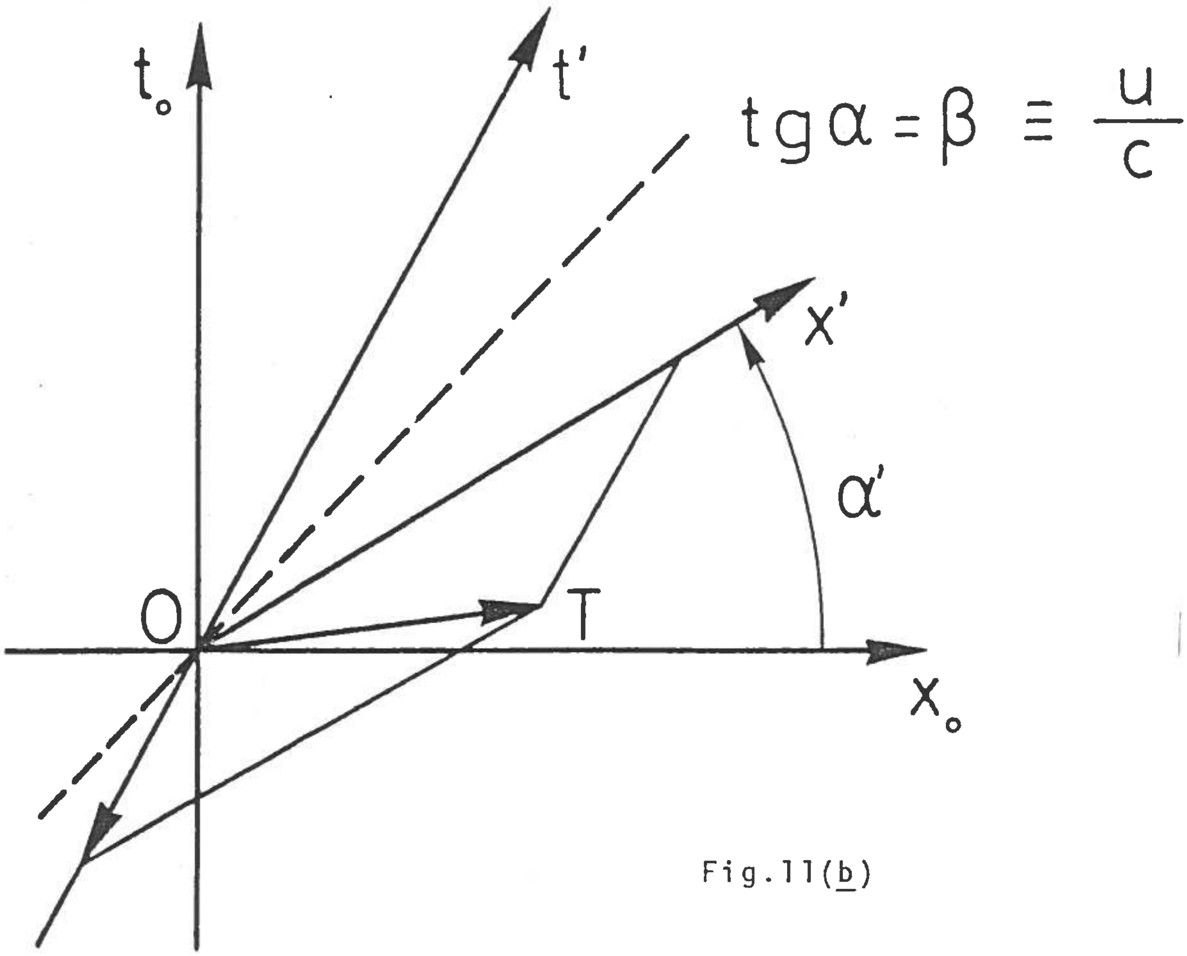


Fig.11(b)

remains covariant, since any observers will always see the cause to precede its effect.

Actually, a sensible procedure to introduce Ts in Relativity is postulating *both*: (a) tachyon existence *and* (b) retarded causality, and then trying to build up an ER in which the validity of both postulates is enforced. Till now we have seen that such an attitude —which extends the procedure in Sect.2 to the case of tachyons— has already produced, among the others, the description within Relativity of both matter and antimatter (Ts and $\overline{\text{T}}$s, and Bs and $\overline{\text{B}}$s).

5.14. Braydons and Tachyons. Particles and Antiparticles.

Fig.6 shows, in the energy-momentum space, the existence of *two* different "symmetries", which have nothing to do one with the other.

The symmetry particle/antiparticle is the mirror symmetry w.r.t. the axis $E = 0$ (or, in more dimensions, to the hyperplane $E = 0$).

The symmetry braydon/tachyon is the mirror symmetry w.r.t. the bisectors, i.e. to the two dimensional "light-cone".

In particular, when we confine ourselves to the proper orthochronous subluminal transformations $L^{\uparrow} \in \mathcal{L}_{+}^{\uparrow}$, the "matter" or "antimatter" character is invariant for braydons (but not for tachyons).

We want at this point to put forth explicitly the following simple but important argumentation. Let us consider the two "most-typical" generalized frames: the frame at rest, $s_o \equiv (t,x)$, and its *dual* Superluminal frame (cf.eq.(51) and Fig.9), i.e. the frame $S'_\infty \equiv (t',x')$ endowed with infinite speed w.r.t. s_o. The world-line of S'_∞ will be of course superimposed to the x-axis. With reference to Fig.7b, observer S'_∞ will consider as time-axis t' our x-axis and as space-axis x' our t-axis; and vice-versa for s_o w.r.t. S'. Due to the "extended principle of relativity" (Sect.4), observers s_o, S'_∞ have moreover to be equivalent.

In space-time (Fig.1) we shall have braydons and tachyons going both forward and backwards in time (even if for each observer — e.g. for s_o — the particles travelling into the past have to bear negative energy, as required by our Third Postulate). The observer s_o will of course intepret all —sub- and Superluminal— particles moving *backwards* in *his* time t as antiparticles; and he will be left only with objects going forward in time.

Just the same will be done, in his own frame, by observer S'_∞ since to him all —sub- or Superluminal— particles travelling *backwards* in *his* time t'(i.e. moving along the negative x-direction, according to us) will appear endowed with negative energy. To see this, it is enough to remember that the transcendent transformation S does exchange the values of energy and momentum (cf. eq.(38), the final part of Sect.5.6, and Review I). The same set of braydons and tachyons will be therefore described by S'_∞ in terms of particles and antiparticles all moving along its positive time-axis t'.

But, even if axes t' and x coincide, the observer s_o will see bradyons and tachyons moving (of course) along both the positive and the negative x-axis! In other words, we have seen the following: The fact S' sees only particles and atiparticles moving along its positive t'-axis *does not mean* at all that s_o sees only braydons and tachyons travelling along his *positive* x-axis! This erroneous belief entered, in connection with tachyons, in the (otherwise interesting) two-dimensional approach by Antippa[49], and later on contibuted to lead Antippa and Everett[78] to violate space-isotropy by conceiving that even in four dimensions tachyons had to move just along a unique, privileged direction — or *"tachyon corridor"*— .

5.15. Totally Inverted Frames.

We have seen that, when a tachyon T appears to overcome the infinite speed (Figs.11a,b), we must apply our Third Postulate, i.e. the "switching procedure". The same holds of course when the considered "object" is a reference frame.

More in general, we can regard the GLTs expressed by eqs. (35')-(36') from the *passive*, and no more from the active, point of view (Recami and Rodrigues[1]). Instead of Fig.6, we get then what depicted in Fig.12. For future convenience, let us use the language of multi-dimensional space-times. It is apparent that the four subsets of GLTs in eq.(35') describe the transitions from the initial frame s_o (e.g. with *right-handed* space-axes) not only to all frames $\underline{f}^R$ moving along x with *all* possible speeds $u = (-\infty,+\infty)$, but also to the "totally inverted" frames $\underline{\tilde{f}}^L = (-\mathbb{1})\underline{f}^R = (\overline{P}\,\overline{T})\,\underline{f}^R$, moving as well along x with *all* possible speeds u (cf. Figs.2-6 and 11 in Review I). With reference to Fig. 9, we can say loosely speaking that, if an ideal frame $\underline{f}$ could undergo a whole trip along the axis *(circle)* of the speeds, then —after having overtaken $\underline{f}(\infty) \equiv \underline{f}(U = \infty)$ — it would come back to rest with a *left-handed* set of space-axes and with particles transformed into antiparticles. For further details, see Recami and Rodrigues[1] and refs. therein.

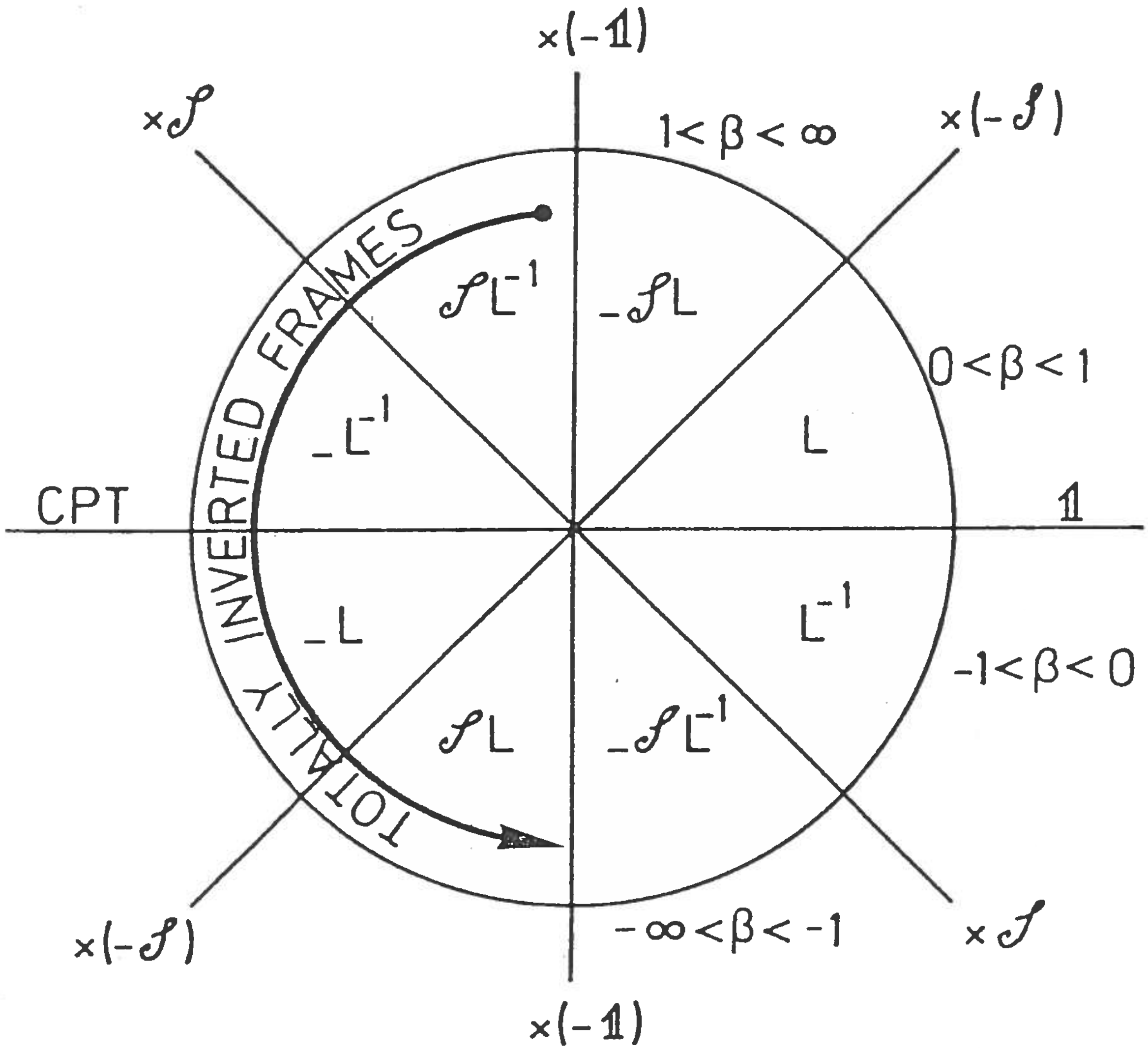


Fig. 12

5.16. About CPT

Let us first remind (sect.5.5) that the product of two SLTs (which is always a *subluminal* LT) can yield a transformation both orthochronous, $L^{\uparrow} \in \mathcal{L}_{+}^{\uparrow}$, and antichronous, $(-\mathbb{1}) \cdot L^{\uparrow} \equiv (\overline{P}\,\overline{T})\, L^{\uparrow} = L^{\downarrow} \in \mathcal{L}_{+}^{\downarrow}$ (cf.Sect. 2.3). We can then give eq.(10) the following meaning within ER.

Let us consider in particular (cf.Figs. 7a , b) the antichronous GLT$(\theta = 180^{\circ}) = -\mathbf{1} \equiv \overline{P}\,\overline{T}$. In order to reach the value $\theta = 180^{\circ}$ starting from $\theta = 0$ we must bypass the case $\theta = 90^{\circ}$ (see Fig.12), where the switching procedure has to be applied (Third Postulate). Therefore:

$$GLT(\theta = 180^o) = -\mathbb{1} \equiv \overline{P}\,\overline{T} = CPT \ . \qquad (53)$$

The "total inversion" $-\mathbb{1} \equiv \overline{P}\,\overline{T} = CPT$ is nothing but a particular "rotation" in space-time, and we saw the GLTs to consist in *all* the space--time "rotations" (Sect.5.6). In other words, we can write: $CPT \in \mathbb{G}$, and the CPT-theorem may be regarded as a particular, explicit requirement of SR (as formulated in Sect.2), and *a fortiori* of ER (Mignani and Recami[14,15], and refs. therein, Recami and Ziino[9], Pavšič and Recami[79]). Notice that, in our formalization, the operator CPT is linear and unitary.

5.17. Laws and Descriptions. Interactions and Objects.

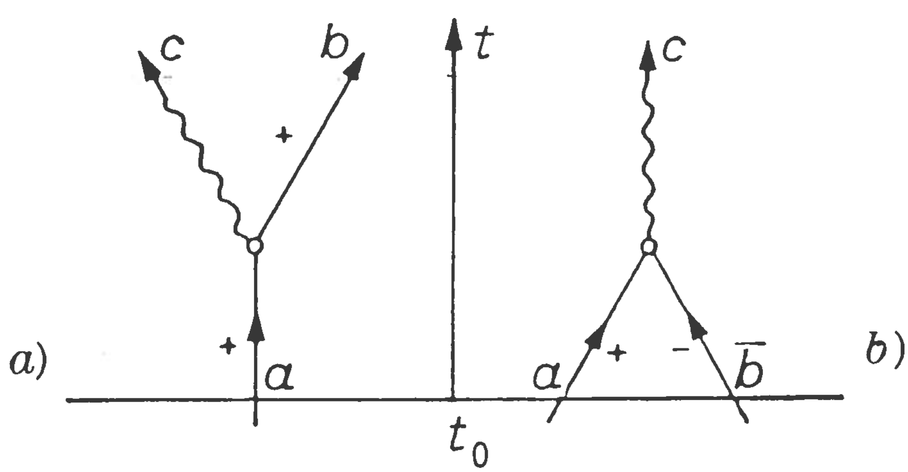
Given a certain phenomenon *ph*, the principle of relativity (First Postulate) requires two diferent inertial observers O_1, O_2 to find that *ph* is ruled by the same physical laws, but it does *not* require at all O_1, O_2 to give the same description of *ph* (cf. e.g. Review I; p. 555 in Recami[6]; p. 715, Appendix, in Recami and Rodrigues[1]).

We have already seen in ER that, whilst the "Retarded Causality" is a *law* (corollary of our Third Postulate), the assigment of the "cause" and "effect" labels is *relative* to the observer (Camenzind[77]); and is to be considered a *description-detail* (so as, for instance, the observed colour of an object).

In ER one has to become acquainted with the fact that many description-details, which by chance were Lorentz-invariant in ordinary SR, are no more invariant under the GLTs. For example, what already said (see Sect.2.3, point e)) with regard to the possible non-invariance of the sign of the additive charges under the transformations $L \in \mathcal{L}_+$ holds *a fortiori* under the GLTs, i.e., in ER. Nevertheless, the total charge of an isolated system will have of course to be constant during the time-evolution of the system — i.e. to be conserved — as seen by *any* observer.

Let us refer to the explicit in (Fig.13) (Feinberg[80], Baldo *et al.*[76]), where the pictures (a), (b) are the different descriptions of the same interaction given by two different (generalized) observers . For instance, (a) and (b) can be regarded as the description, from two *ordinary* subluminal frames O_1, O_2, of one and the same process involving the tachyons a, b (c can be a photon, e.g.). It is apparent that, before the interaction, O_1 sees one tachyons while O_2 see two tachyons. Therefore, the very number of particles —e.g. of tachyons, if we consider only subluminal frames and LTs— observed at a certain time-instant is not Lorentz-invariant. However, the *total* number of particles partecipating in the reaction either in the initial or in the final state *is* Lorentz-invariant (due to our initial three Postulates). In a sense, ER prompts us to deal in physics with interactions rather than with objects (in quantum-mechanical language, with "amplitudes" rather with "states"); (cf. e.g. Gluck[(75)], Baldo and Recami[(81)]).

Long ago Baldo *et al.*[(76,81)] introduced however a vector-space $H = \mathcal{H} \otimes \overline{\mathcal{H}}$ direct product of two vector-spaces $\mathcal{H}$ and $\overline{\mathcal{H}}$, in such a way that any Lorentz transformation was unitary in the H-space even in presence of tachyons. The spaces $\mathcal{H}$ ($\overline{\mathcal{H}}$) were defined as the vector-spaces spanned by the states representing particles and anti-particles only in the initial (final) state.



Figs. 13

5.18. SR with Tachyons in two Dimensions.

Further developments of the classical theory for tachyons in two dimensions, after what precedes, can be easily extracted for example from: Review I and refs. therein; Recami[6,82], Corben[83], Caldirola and Recami[11], Maccarrone and Recami[21,22], Maccarrone *et al.*[42] .

We merely refer here to those papers, and references therein.

Here we shall only make the following (simple, but important) remark. Let us consider two (brayonic) bodies A, B that —owing to mutual attraction— for instance *accelerate while approaching* each other. The situation in Fig.14, where A is chosen as the reference-frame $s \equiv (t,x)$ and, for simplicity, only a discrete change of velocity is depicted. From a Superluminal frame they will be described either as two (anti)tachyons that *accelerate while receding* one from the other [frame $S' \equiv (t',x')$], or as two tachyons that *decelerate while approaching* each other [$S'' \equiv (t'',x'')$]. Therefore, we expect that two tachyons *from the kinematical point of view* will *seem* to suffer a repulsion, if they attract each other in their own rest-frames (and in other frames in which they are subluminal); we shall however see that such a behaviour of tachyons *may* be still considered —from the dynamical, energetical point of view— as due to an *attraction.*

To conclude, let us explicitly remark that the results of the model-theory in two dimensions strongly prompt us to attempt building up a similar theory (based as far as possible on the same Postulates) also in more dimensions.

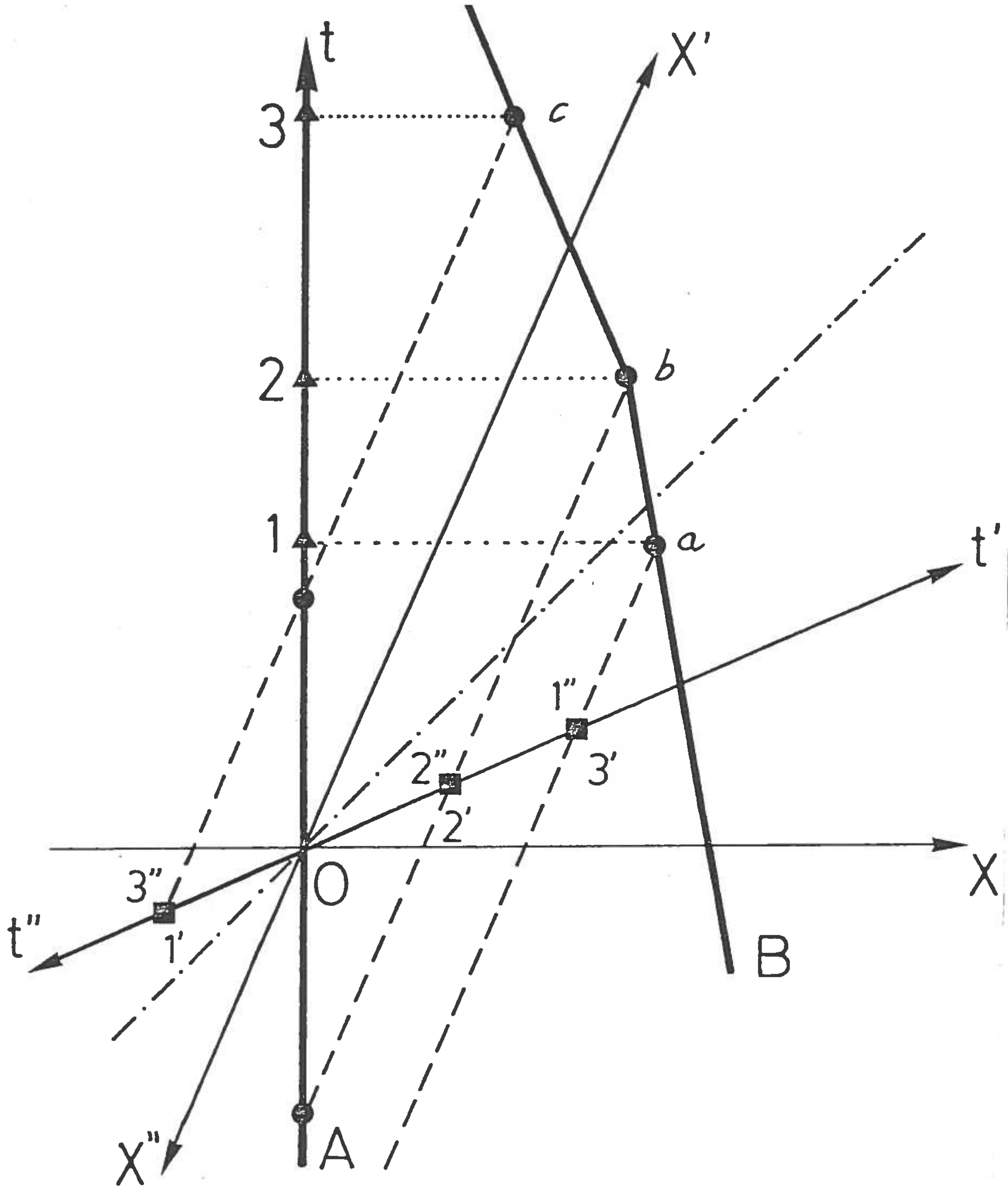


Fig. 14

6. ACKNOWLEDGEMENTS.

The authors are very grateful, for continuous discussions, to Dr. G. D. Maccarrone and Prof. R. Mignani.

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