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ABSTRACT

The complex-ray-tracing technique, together with the uniform approximation at complex caustics, is used to determine the field diffracted by a smooth convex reflecting object. In this way one gets some improvements with respect to the preceding diffraction theories: firstly, the treatment results simpler; secondly, the number of assumptions is considerably reduced; lastly, a method is found for calculating more precisely the decay exponents and the diffraction coefficients relative to the "modes" of excitation of creeping waves.

1. - INTRODUCTION

Certain physical theories result to be short-wave limits of more general theories. In particular, it is well-known that geometrical optics corresponds to the leading term of an asymptotic expansion of the solution of a boundary problem for the reduced wave-equation. Similarly classical mechanics can be viewed as the limit to which wave mechanics tends when the Planck constant is reduced to zero. The condition of validity for these asymptotic theories is that the refractive index - or, equivalently, the potential - should vary slowly with respect to the displacement of one wave-length.

On the contrary, if the refractive index presents surfaces of discontinuity or regions of sudden variation, we are faced with diffraction effects, for which one has to take into account higher-order terms in the asymptotic expansion. Levy and Keller[1,2] elaborated a theory for evaluating systematically these terms, as regards diffraction produced by discontinuities in the refractive index. Such a theory is not completely deductive, in the sense that one has to make some assumptions, based partly on physical intuition, partly on the study of asymptotic expansions of exact solutions of the so-called "canonical problems". These non-deductive parts include the concepts of surface rays, decay exponents and diffraction coefficients. Lewis et al.[3] improve Levy and Keller's theory by reducing the non-deductive parts; however they are faced with rather involved and complicated equations. In a rather different approach, Hong[4] uses an asymptotic trial solution modeled on those which arise in canonical problems, obtaining an integral equation, from which he deduces the first two asymptotic terms. Lastly Ursell[5] establishes rigorously the asymptotic nature of the ray optics as the short-wave limit of the wave optics.

However, to the author's knowledge, an asymptotic theory of diffraction mathematically rigorous and physically intuitive has not get been built; in this sense it is worth mentioning the method suggested by Ludwig[6] in the bidimensional case. The main purpose of this paper is to give some contributions in such a direction.

Before illustrating our method, let us examine briefly some other short-wave problems in which higher-order terms of the asymptotic expansion have been calculated. Firstly we recall the method of Knoll and Schaeffer[7,8], who have shown that complex solutions of the classical equations of motion arise from considering the effects of the rapid variation of the nuclear potential at the boundary; see also references quoted in [8]. Similarly Balian and Bloch[9] and Crowley[10] suggest that non-classical effects should be described in terms of complex trajectories. As regards asymptotic problems arising in electromagnetism, the complex-ray-tracing (CRT) technique has been employed by Felsen et al.[11] in the description of "modal" propagation in curved open waveguides (see also [12]). This method results particularly flexible in the calculation of non-geometrical effects; moreover it seems to be preferable to the evanescent-wave method[13], since it is more adaptable to different situations.

In this paper we present a method, based on CRT, for determining an asymptotic solution of the problem of diffraction by a smooth convex reflecting object. The determination of the approximate solution is developed in two steps:

i) As a first step, we interpret the asymptotic solution of a canonical problem in terms of complex rays and complex caustics in order to obtain a hint for the trial solution in the general case.

ii) Secondly, we construct our trial solution starting from complex ray-systems with given (unknown) caustic and using the uniform approximation by Ludwig[14] near to these caustics, which we determine by imposing the proper boundary condition on the surface of the object[(*)]. The complex caustics are related in a simple way to the decay exponents of the "modes" of excitation of the creeping waves (see Sects. 4 and 6).

The method presents the advantage of a great simplicity, in the sense that it avoids the involved equations that arise from the use of real surface rays; moreover it allows us to significantly reduce the number of assumptions which have been made in previous treatments[2, 3]; lastly, it gives rise to a more precise calculation of the decay exponents and diffraction coefficients, such that Levy and Keller's results are obtained as a first-order approximation of an iterative procedure. This method could be applied, with suitable modifications, to subnuclear elastic scattering at very high energies, where the interaction region is pictured as an opaque body of revolution[15].

The paper is organized as follows. In Sect. 2 we illustrate the assumption on which our trial solution relies and we introduce the complex caustics to be determined (see eq. (18)). In Sect. 3 we trace the complex-ray system relative to such caustics and determine the field amplitude along the complex rays. In Sect. 4 we apply the uniform approximation near to complex caustics, which we determine by imposing the proper boundary conditions. In Sect. 5 we study the field near to real focal points. Sect. 6 is devoted to the determination of the diffraction coefficients relative to the various "modes" and to the discussion of particular cases. Lastly in Sect. 7 we draw some brief conclusions.

2. - MATHEMATICAL FORMULATION OF THE PROBLEM. ASSUMPTIONS

Our aim is to determine the Green function for the reduced wave equation, with the condition that it should vanish on a smooth convex surface S and fulfill the radiation condition at infinity. In particular we assume this surface to be represented by parametric equations of the type

$$\vec{r} = \vec{r}_s(\sigma_1, \sigma_2) , \qquad (1)$$

where σ_1 and σ_2 are real parameters and all the functions involved are analytic in their arguments. Let P_o be the location of the source and P the observation point, which we denote, respectively, $\vec{r}_o$ and $\vec{r}$. Then our problem amounts to determining

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(*) - This condition cannot be imposed if one tries to apply the uniform approximation to the real diffracted rays; see Sect. 4 and also ref. (3).

a function $G(r; r_o)$ such that

i) it vanishes on S;

ii) $r(\frac{\partial G}{\partial r} - ikG) \rightarrow 0$ as $r = |\vec{r}| \rightarrow \infty$; (2)

iii) G is defined and non-singular everywhere outside S, except at $\vec{r}_o$, and satisfies the Helmholtz equation, i. e.

$$(\Delta + k^2) G(\vec{r}; \vec{r}_o) = 0 \tag{3}$$

outside S, except at r_o;

iv) as $r \rightarrow r_o$ along any path,

$$G(\vec{r}; \vec{r}_o) \sim \frac{1}{4\pi R}, \qquad R = |\vec{r} - \vec{r}_o| . \tag{4}$$

As usual, we distinguish among shadow and lit region, which are separated from each other by the "shadow" cone generated by the straight lines tangent to S from P. We shall determine the field in the shadow region, where the situation is simpler, since only diffractive effects are involved.

As a first step we consider the exact Green function for a so-called canonical problem and examine the asymptotic behaviour for large values of k, interpreting the asymptotic behaviour in terms of complex rays; this will suggest an assumption for writing the trial solution in the general case.

Let us consider the problem of a perfectly reflecting circular indefinite cylinder which scatters a stationary wave emitted by a linear source parallel to the axis of the cylinder. This amounts to determining the Green function in two dimensions, such that it should vanish on a given circumference.

The problem has been solved exactly by Levy and Keller(2), who also have determined the asymptotic behaviour of the solution in the limit of $k \rightarrow \infty$. Here we examine this result. To this end we define a reference frame such that the z-axis coincidences with the axis of the cylinder and the x-axis intersects the source. Due to the symmetry of the problem, the solution does not depend on z; moreover it is useful to introduce plane polar coordinates in the (x,y)-plane, as illustrated in Fig. 1. If a is the radius of the cylinder, we have the following asymptotic

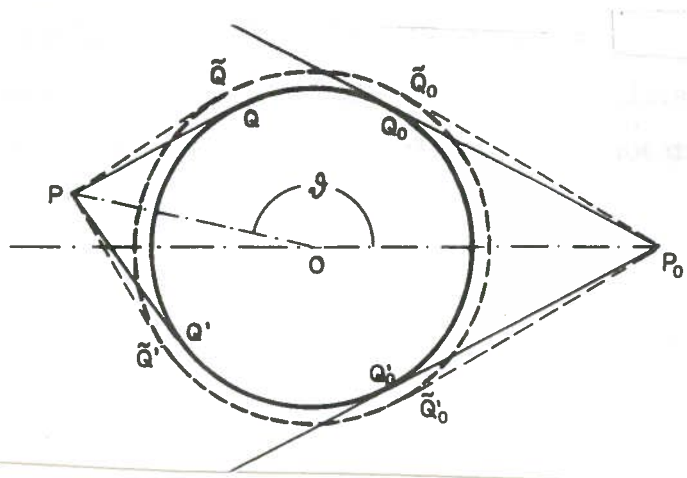

FIG. 1 - Diffraction by a cylinder. The dashed lines represents the complex rays.

solution in the shadow region:

$$G(r,\vartheta;r_o) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\left[\psi^{+}_{mn}(r,\vartheta)+\psi^{-}_{mn}(r,\vartheta)\right] , \tag{5}$$

with

$$\psi^{\pm}_{mn}(r,\vartheta) = -\frac{i}{2}A_n \frac{e^{i\sqrt{k^2r_o^2-\lambda_n^2}}}{\sqrt[4]{k^2r_o^2-\lambda_n^2}}\; e^{i\lambda_n(\varphi_n^{\pm}+2m\pi)}\; \frac{e^{i\sqrt{k^2r^2-\lambda_n^2}}}{\sqrt[4]{k^2r^2-\lambda_n^2}} , \tag{6}$$

and

$$\varphi_n^{+} = \vartheta - \left(\arccos\frac{\lambda_n}{kr} + \arccos\frac{\lambda_n}{kr_o}\right) , \tag{7}$$

$$\varphi_n^{-} = 2\pi - \vartheta - \left(\arccos\frac{\lambda_n}{kr} + \arccos\frac{\lambda_n}{kr_o}\right), \tag{8}$$

$$A_n = -\frac{e^{-i\frac{\pi}{6}}}{2\pi}\left(\frac{ka}{2}\right)^{\frac{1}{3}}\left[A_i'(-x_n)\right]^{-2} , \tag{9}$$

$$\lambda_n = ka + e^{i\frac{\pi}{3}}x_n(ka)^{\frac{1}{3}} . \tag{10}$$

In (9) and (10) $A_i(\zeta)$ is the Airy function,

$$A_i(\zeta) = \frac{3^{1/3}}{\pi}\int_0^{\infty}\cos\left(t^3+3^{1/3}\zeta t\right)dt , \tag{11}$$

and $-x_n$ is the n-th zero of the Airy function. Expression (6) holds true as long as

$$\vartheta > \arccos\frac{a}{r_o} + \arccos\frac{a}{r} , \tag{12}$$

which delimitates the shadow region. We take the principal value of the function arccos z which appears in eqs. (7) and (8). At this point Levy and Keller approximate λ_n by ka in the square roots that appear in (6) and in the expressions (7) and (8) of $\varphi_n^{\pm}$, and show that (6) is interpretable as the amplitude of a ray which propagates from the source to the cylinder, then describes an arc of circumference (along which it decays) and lastly leaves the surface tangentially, arriving at the point P of co-ordinates (r,ϑ).

We give here a slightly different interpretation of (6), in terms of complex rays. To this and we extend the real (xy)-plane in such a way that the coordinates assume complex values. We denote the complex space so obtained by $\mathbb{C}^2$ and define new co-ordinates X and Y. The real (x,y)-plane is re-defined as a subspace of $\mathbb{C}^2$, such that X and Y are restricted to real values. In this subspace we denote the point of intersection of the real (x,y)-plane with the source by P_o and by P the point of observation. At this point let us define a circumference of equation

$$X^2 + Y^2 = \frac{\lambda_n^2}{k^2} , \tag{13}$$

where λ_n is defined by (10): we consider a fixed value of n. Let us remark that this circumference - like any curve in $\mathbb{C}^2$ - is a twofold variety with respect to the real numbers. Now we trace the straight lines tangent to that circumference through P and P_o. We denote by $\tilde{Q}_o$, $\tilde{Q}'_o$, $\tilde{Q}$ and $\tilde{Q}'$ the points of tangency: as $k \to \infty$, these points tend to the real points which we have denoted in Fig. 1 by the same letters, without tilde: in that limit, in fact, the radius of the complex circumference tends to a, as can be easily checked from (10) and (13).

Now it is straightforward to see that the (complex) lengths of the segments of tangency to the circumference from P_o and P are,respectively, $k^{-1}\sqrt{k^2r_o^2 - \lambda_n^2}$ and $k^{-1}\sqrt{k^2r^2 - \lambda_n^2}$. Moreover, as we are showing in a moment, $(\varphi_n^{\pm} + 2m\pi)\lambda_n k^{-1}$ are interpretable as arclengths from $\tilde{Q}_o$ to $\tilde{Q}$ and from $\tilde{Q}'_o$ to $\tilde{Q}'$. To see that, we define the polar angle in the complex space $\mathbb{C}^2$ as

$$\Theta = \arcsin \frac{Y}{\sqrt{X^2 + Y^2}} . \tag{14}$$

Θ is a multivaluted function, even if we restrict curselves to real values of the co-ordinates X and Y: in this case we have

$$\Theta = \vartheta + 2m_o\pi , \tag{15}$$

where m_o is any integer (to be distinguished from m), while ϑ is the usual plane polar angle, $0 \leq \vartheta \leq 2\pi$. The complex arclengths from $\tilde{Q}_o$ to $\tilde{Q}$ and from $\tilde{Q}'_o$ to $\tilde{Q}'$ along the circumference are, respectively

$$s_{\pm} = k^{-1}\lambda_n(\pm\Theta - \arccos\frac{\lambda_n}{kr} - \arccos\frac{\lambda_n}{kr_o}) , \tag{16}$$

where Θ is given by (15). Moreover, in order to give an appropriate "ordering" to the points on a ray, we choose to define the curvilinear abscissa along the ray in such a way that its real part be positive (see also ref. (16)). Therefore we have to restrict ourselves to non-negative values of m_o for s_+ and the negative values of m_o for s_-. As a result we get

$$s_{\pm} = k^{-1}\lambda_n(\varphi_n^{\pm} + 2m\pi) , \tag{17}$$

as we wished to show. At this point, by applying the eikonal and the transport equation to each factor of (6), it can be checked that $\psi^{\pm}_{mn}$ represents the amplitude of a complex ray that propagates from P_o to $\tilde{Q}_o$ (or to $\tilde{Q}'_o$) along a straight line, then describes an arc of circumference from $\tilde{Q}_o$ to $\tilde{Q}$ (or from $\tilde{Q}'_o$ to $\tilde{Q}'$) and lastly arrives at P along a straight line.

We observe that the approximation (6) does not hold true when r approaches the value λ_n/k, since the circumference (13) is a caustic of the complex rays. In Section 4 we shall define a function which describes the amplitude uniformly near and far from the complex caustic.

Now, returning to the general case of a smooth convex object, we are ready to formulate our assumption for the trial solution to be put into equation (3). To this end, primarily we define a complex space $\mathbb{C}^3$. Moreover we re-define the real space as a subspace of $\mathbb{C}^3$, such that the co-ordinates X, Y and Z are restricted to real values. To this subspace belong, obviously, both the point P_o and the surface S of the reflecting body, defined by eq. (1): we denote the locations of these points again by $\vec{r}_o$ and by $\vec{r}_s\ (\sigma_1,\ \sigma_2)$ respectively.

Now, in analogy to the case of the circular cylinder, we assume that the asymptotic solution can be expressed as a sum over amplitudes along surface rays. More precisely we consider rays which start at $\vec{r}_o$ and are tangent to a suitable complex surface S' (to be defined below) and lastly, after describing a curve on this surface, leave it tangentially. The surface S' is defined by the following parametric vector equation:

$$\vec{R}_{s'}(\Sigma_1,\ \Sigma_2) = \vec{r}_s(\Sigma_1,\ \Sigma_2) + W_o(\Sigma_1,\ \Sigma_2)\vec{N}(\Sigma_1,\ \Sigma_2)\ , \tag{18}$$

where Σ_1 and Σ_2 are complex extensions of the real parameters σ_1 and σ_2 introduced in (1): $\vec{N}$ is a "complex normal versor", which shall be defined in a moment; $W_o(\Sigma_1, \Sigma_2)$ is an unknown function to be determined in such a way that the solution should vanish on the real surface S. Now, since we assume S to be analytic, we can define the "complex normal versor" $\vec{N}$ as the complex extension of the normal versor $\vec{n}$ of the real surface, i. e. of

$$\vec{n}(\sigma_1,\ \sigma_2) = \mathrm{vers}\left(\frac{\partial \vec{r}_s}{\partial \sigma_1} \wedge \frac{\partial \vec{r}_s}{\partial \sigma_2}\right). \tag{19}$$

We assume W_o to be analytic in its arguments and to vanish in the limit as $k \to \infty$.

Moreover we assume the complex caustic S' to behave, in a certain sense, like a "convex surface", i. e. that for a point P outside S' there are exactly two rays tangent to S' which pass through P; this condition is expressed mathematically by inequality (41)). We shall see in Sects. 4 and 6 that, for sufficiently high values of k, there are a set of complex caustics which fulfill the conditions exposed above.

In the following Section we shall describe the "complex-ray tracing" relative to the complex rays that we have mentioned. We shall also determine the amplitude along each complex ray, according to the geometrical approximation. In other words, we shall solve the eikonal and transport equation for the complex rays which are tangent to S'.

3. - THE EIKONAL AND TRANSPORT EQUATION

As is well-known, the homogeneous Helmholtz equation (3) can be solved, approximately in the limit of large values of k, by setting a trial solution of the type

$$\psi = A e^{ik\Phi} , \tag{20}$$

where Φ and A fulfill the eikonal and transport equation, i. e.

$$(\vec{\nabla}\Phi)^2 = 1 , \tag{21}$$

$$2\vec{\nabla}\Phi \cdot \vec{\nabla}A + A\Delta\Phi = 0 . \tag{22}$$

Approximation (20) is not valid in the whole complex space, as we shall see. In this section we study the solution of the system (21-22), imposing proper initial conditions; we also determine the "caustic", i. e. the set of singular points where approximation (20) fails.

In particular, as regards the eikonal equation (21), we want to determine Φ in such a way that

a) it should have the surface S' as a "caustic", i. e. as an envelope of the rays, which are the lines determined by $\vec{\nabla}\Phi$;

b) it should satisfy the following initial condition: given a curve C, which is the intersection of the incident rays with S', we require that

$$\Phi\left[\vec{R}(V)\right] = T_0(V) , \tag{23}$$

where $\vec{R}(V)$ is the parametrization of C (V being a complex parameter) and T_0 is the phase of the incident ray on S'.

We solve this problem in two stages:

i) we determine Φ on the surface S', such that it should satisfy (23);

ii) we determine Φ at points lying outside S', in such a way that it should coincide, on S', with the function determined in (i).

3.1. - The surface eikonal equation

Firstly we are faced with the stage i) of the eikonal-equation problem that we have posed above. If S' is to be a caustic, $\vec{\nabla}\Phi$ must be tangent to S', then the eikonal equation (21) can be rewritten as a surface eikonal equation, i. e.

$$G^{ij} \frac{\partial \Phi}{\partial \Sigma_i} \frac{\partial \Phi}{\partial \Sigma_j} = 1 , \tag{24}$$

where G^{ij} is inverse to the "complex metric tensor"

$$G_{ij} = \frac{\partial \vec{R}_{s'}}{\partial \Sigma_i} \frac{\partial \vec{R}_{s'}}{\partial \Sigma_j} \tag{25}$$

and $\vec{R}_{s'} = \vec{R}_{s'}(\Sigma_1, \Sigma_2)$ has been defined in (19).

As shown in Appendix A, equation (24) results to be equivalent to the following "characteristic system":

$$\frac{d\Sigma_i}{dU} = G^{ij}\Pi_j , \tag{26}$$

$$\frac{d\Pi_i}{dU} = -\frac{1}{2} \frac{\partial G^{jk}}{\partial \Sigma_i} \Pi_j \Pi_k , \tag{26'}$$

where

$$\Pi_i = \frac{\partial \Phi}{\partial \Sigma_i} \tag{26''}$$

and

$$dU = (G_{ij}\, d\Sigma_i\, d\Sigma_j)^{1/2} \tag{26'''}$$

is the "complex curvilinear abscissa" along a characteristic line; this function is defined in such a way that

$$\mathrm{Re}(dU) > 0 . \tag{26''''}$$

We note, incidentally, that the vanishing of W_o in the limit of $k \to \infty$ (as we have assumed in the preceeding Section, see eq. (18), implies that G_{ij} and dU reduce, respectively, to the usual metric tensor and curvilinear abscissa on the surface S. In this connection we observe that the parameter V can be conveniently defined as the curvilinear abscissa along the curve C, by means of (26''').

Now we have to impose initial conditions on the system (26-26'), i. e. we have to define an "initial strip" $\left[\bar{\Sigma}_i(V), \bar{\Pi}_i(V)\right]$, such that

$$\Sigma_i(0,V) = \bar{\Sigma}_i(V) , \qquad (27)$$

$$\frac{d\Sigma_i}{dU}(0,V) = G^{ij}\bar{\Pi}_j(V) , \qquad (27')$$

Now $\bar{\Sigma}_i(V)$ and $\bar{\Pi}_i(V)$ are determined in such a way that

- $\vec{R}\left[\bar{\Sigma}_1(V), \bar{\Sigma}_2(V)\right]$ describes the points of the curve C;
- $\vec{p}(V) = G^{ij}(\bar{\Sigma}_1, \bar{\Sigma}_2)\bar{\Pi}_i(V) \dfrac{\partial \vec{R}_{S'}}{\partial \Sigma_i}$ is the versor in the direction of the incident rays.

The conditions (27-27') uniquely determine the solution of the system (26-26'). However we observe that these conditions are stronger than the initial condition (23), owing to which the incident rays that arrive at C may be oriented toward the point-like source P_o or in the opposite direction; but the requirement of a solution which is outgoing from P_o uniquely determines the orientation of $\vec{p}(V)$.

Once we have determined the solution of (26-26'), with the initial conditions (27-27'), we have a set of "complex geodesics" on S', i.e. a set of complex surface rays. We define the function Φ as

$$\Phi = U, \qquad (28)$$

where U is the arclength along the surface ray, as from a fixed surface wave-front, which, from now on, will be called C_o. The surface-ray tracing now described can be regarded as a mapping

$$(U,V) \longrightarrow (\Sigma_1, \Sigma_2) \qquad (29)$$

between the pair of complex parameters U, V and the points of S'. If we consider a (real) closed body, or a body having a finite cross-section (as is the case of the cylinder), this mapping is not one-to-one, since a surface ray winds infinitely many times (either counterclockwise) round the surface. Moreover, as we shall see, the Jacobian of this mapping may vanish for some points of S'. From now on we shall consider separately each branch of the inverse to mapping (29) and we shall consider a limited portion of the real surface S (and henceforward of S'), in such a way that singular points of the mapping (29) be excluded and the correspondence be biunivocal.

3.2. - Eikonal equation

In this subsection we solve the stage (ii) of the solution of the eikonal equation, i.e. we determine Φ in points outside S'. To this end let us define, for a general point P outside S', a set of coordinates U, V, W, in such a way that U, V be the surface co-ordinates of the projection of P on S', while W is the distance of P from S' (The normal versor $\vec{N}'$, of the surface S' is defined in such a way to coincide with the versor $\vec{n}$ normal to S in the limit of $k \to \infty$, i.e. in the limit of $W_o \to 0$). It is easy to see that this set of coordinates U, V, W is mutually orthogonal, so that, in a neighborhood of S', the eikonal equation reads as

$$\frac{1}{H_U^2}\left(\frac{\partial\Phi}{\partial U}\right)^2 + \frac{1}{H_V^2}\left(\frac{\partial\Phi}{\partial V}\right)^2 + \frac{1}{H_W^2}\left(\frac{\partial\Phi}{\partial W}\right)^2 = 1 , \tag{30}$$

where

$$H_W^2 = \left(\frac{\partial\vec{R}}{\partial W}\right)^2 = 1 , \tag{31}$$

$$H_U^2 = \left(\frac{\partial\vec{R}}{\partial U}\right)^2 = \left(\vec{p} + W\frac{\partial\vec{N}'}{\partial U}\right)^2 , \tag{31'}$$

$$H_V^2 = \left(\frac{\partial\vec{R}}{\partial V}\right)^2 = \left(\vec{q} + W\frac{\partial\vec{N}'}{\partial V}\right)^2 , \tag{31"}$$

and

$$\vec{R} \equiv (X, Y, Z) , \tag{32}$$

$$\vec{p} = \frac{\partial\vec{R}_{S'}}{\partial U} , \tag{32'}$$

$$\vec{q} = \frac{\partial\vec{R}_{S'}}{\partial V} , \tag{32"}$$

$\vec{R}_{S'}$ being a point of the surface S'. In particular H_U can be re-written in a more convenient way, thanks to the Frenet equation,

$$\frac{\partial\vec{N}'}{\partial U} = \kappa\vec{p} - \kappa_\tau\vec{b} \tag{33}$$

where $\vec{p}$ is given by (32') and $\vec{b} = \vec{N}' \wedge \vec{p}$; as a result we get

$$H_U^2 = 1 + 2\kappa W + (\kappa^2 + \kappa_\tau^2)W^2 . \tag{34}$$

It is worth observing that, in the limit of $k \to \infty$, i.e. when the surface S' tends to S, κ and κ_τ reduce, respectively, to the usual curvature and torsional curvature of a geodetic line of S; in particular, the convexity of S implies that, in this limit,

$$\kappa > 0 . \tag{35}$$

Now, in a way similar to the case of a real convex caustic[14], we look for a solution of (30) of the form

$$\Phi = U + \Gamma , \tag{36}$$

with Γ proportional to $W^{3/2}$.

Introducing (36) in (30), we get

$$\frac{\partial \Gamma}{\partial t} = \pm 2 t^2 \left\{ \frac{1}{H_U^2} \left[2\kappa + (\kappa^2 + \kappa_\tau^2) t^2 \right] - \frac{1}{H_U^2} \left[\frac{2}{t^2} \frac{\partial \Gamma}{\partial U} + t^2 \left(\frac{1}{t^2} \frac{\partial \Gamma}{\partial U} \right)^2 \right] - \frac{t^2}{H_V^2} \left(\frac{1}{t^2} \frac{\partial \Gamma}{\partial V} \right)^2 \right\}^{1/2} \tag{37}$$

where

$$t = W^{1/2} . \tag{38}$$

Now from (31'), (31") and (38) it follows that

$$H_U = H_U(U, V, t^2) , \tag{39}$$

$$H_V = H_V(U, V, t^2) , \tag{39'}$$

so that equation (37) is of the type

$$\frac{\partial \Gamma}{\partial t} = \pm t^2 F \left(\frac{1}{t^2} \frac{\partial \Gamma}{\partial U} , \frac{1}{t^2} \frac{\partial \Gamma}{\partial V} , t^2, U, V \right) . \tag{40}$$

Eq. (39) is consistent with our assumption that Γ be proportional to $W^{3/2}$, if

$$\kappa \neq 0 , \tag{41}$$

as we can easily see from (37).

We prove in Appendix B that (41) is a necessary and sufficient condition for (37) to have one and only one solution for each value of the sign. This condition, which is surely satisfied for $W_o = 0$ (i.e. in the limit of $k \to \infty$, see (35)), ensures us that there are exactly two rays which pass through a given point external to the caustic, as we assumed (see Sect. 2).

Now, if we call $\Gamma^{\pm}$ each of the two solutions of (38), we have

$$\Gamma^-(t, U, V) = \Gamma^+(-t, U, V) ; \tag{42}$$

putting (42) into (33), for each point $\vec{R}$ outside S' we have two solutions of (30), which are of the type

$$\Phi^{\pm}(\vec{R}) = U + \Gamma^{+}(\pm t, U, V) ; \qquad (43)$$

these two solutions reduce both to U in the limit of $t = 0$, i. e. on S', as we have required. The solution of (37) can be found by substitution of a power series for Γ; for small values of t one has

$$\Gamma^{\pm} \simeq \pm \frac{2}{3}(2\kappa)^{1/2} W^{3/2} . \qquad (44)$$

Now let us examine the situation in the limit of $k \to \infty$: both κ and W become real non negative quantities (κ is strictly positive, cfr. (35), so that substituting (44) in (43) we realize that Φ^{+} refers to a ray which is "outgoing" with respect to S', while Φ^{-} refers to an "ingoing" ray. Since we have fixed the orientation of $\vec{\nabla}\Phi$ on the initial curve C, this automatically rules out the "ingoing" solution; however, as we shall see in the following, it is useful to consider also the "ingoing" ray, which, from now on, we shall call "fictitious".

Lastly it may be useful to write Φ in the form

$$\Phi = U' + T , \qquad (45)$$

where U' is the curvilinear abscissa relative to the point of tengency of the outgoing ray and T the segment of tangency along the outgoing ray. As shown by Lewis et al.[3] (Appendix A.2) the quantities T and U' can be found as functions of U, V, W. In particular, for small values of W one has[3]

$$W \simeq \frac{1}{2} \kappa T^2 . \qquad (46)$$

As in the case of surface rays, we adopt the convention that a ray propagates in the direction of increasing Re T; as a consequence, Re T is positive along the outgoing ray, whereas it is negative along the incoming (fictitious) ray.

3. 3. - The transport equation

Now let us turn our attention to equation (22). It is a solution of the form

$$A^2 = \frac{D(U', V')}{J} , \qquad (47)$$

where U', V' are the coordinates of the point of tangency, while D(U', V') is a function to be determined in the following; J is the Jacobian of the mapping

$$(T, U', V') \longrightarrow (X, Y, Z) . \qquad (48)$$

This Jacobian results to be

$$J = \frac{\partial(X,Y,Z)}{\partial(T,U',V')} = T(\alpha + \beta T) , \quad (49)$$

where

$$\alpha = k\vec{N}'\cdot\vec{p}\wedge\vec{q} , \quad (49')$$

and

$$\beta = \vec{p}\ \frac{\partial\vec{p}}{\partial U'} \wedge \frac{\partial\vec{p}}{\partial V'} . \quad (49'')$$

The sign of α and β - and therefore of J - results to be undetermined, until we do not choose the orientation of the parameter V along the initial curve C; this sign will be uniquely determined in Section 5.

The Jacobian (47) vanishes either for

$$T = 0 , \quad (50)$$

or for points such that

$$\alpha + \beta T = 0 . \quad (50')$$

The set of points which satisfy (50) or (50') constitute the caustic of the complex rays. One sheet of the caustic, which is described by eq. (50), coincides with S', as expected; the other sheet (which is present only if $\beta \neq 0$) is described by eq. (50') and will be referred to as S", or as the second sheet of the caustic. Approximation (20) fails in a neighborhood of the caustic; in the next two Sections we shall consider the approximation to be used near to these singular points. In particular, in Section 4 we shall treat the uniform approximation for points near to and far from S'; instead, in Section 5 we shall give an outline of the problems which arise in calculating the solution near to points of S"; we shall also tackle the question as to which branch of (47) has to be taken.

4. - UNIFORM APPROXIMATION

In this section we draw an asymptotic solution of the stationary homogeneous wave equation, with appropriate boundary conditions at infinity and on the body surface. In order to motivate the trial solution that we shall use, we introduce the uniform approximation by Ludwig[14], adapting some of his considerations to a complex caustic.

The eikonal approximation cannot be used near to a caustic; however we assume that in every point the asymptotic solution of the reduced wave equation be of the type

$$\psi(\vec{r}) \simeq \int e^{ik\Phi(\vec{r},\beta)} A(\vec{r},\beta)\, d\beta \quad , \tag{51}$$

where β is a real variable and $\Phi(\vec{r},\beta)$ and $A(\vec{r},\beta)$ are real functions which satisfy the eikonal and transport equation identically in β. Such asymptotic solution result from some scattering prolems, like, e. g. , the scattering of a stationary wave by a cylinder(2), or by a sphere(17), or more generally by a central potential(8) : the exact solution, expressed as a partial wave sum, is transformed into a series of integrals by means of Poisson's sum formula and the integrands are replaced by their asymptotic (WKB) expansions; so one gets an expression like (51), where β, in this case, has the meaning of an angular momentum.

If $\Phi(\vec{r},\beta)$ is analytical in its arguments, we can evaluate (51) by means of the saddle-point technique. The main contribution to the integral arises from those values (real or complex) of β for which

$$\frac{\partial\Phi}{\partial\beta} = 0 . \tag{52}$$

A complex saddle point may occur when $A(\vec{r},\beta)$ has a pole in the complex β-plane, not too far from the real axis : in this case, as pointed out by Dingle(18), the pole contribution has to be included in the rapidly varying factor. Now let us suppose that a given value $\beta_J = \beta_J(\vec{r})$ satisfies (52). If for some value of $\vec{r}$ one has $\left(\frac{\partial^2\Phi}{\partial\beta^2}\right)_{\beta_1} \neq 0$, we can evaluate (51) by the usual saddle point method, getting

$$\psi(\vec{r}) \simeq A_J(\vec{r})\, e^{ik\Phi_J(\vec{r})} \quad , \tag{53}$$

with

$$\Phi_J(\vec{r}) = \Phi\left[\vec{r}\ \beta_J(\vec{r})\right] \tag{53'}$$

and

$$A_J(\vec{r}) = \sqrt{\frac{2\pi}{k}}\, e^{i\gamma} A\left[\vec{r},\ \beta_J(\vec{r})\right] \left| \Phi_{\beta\beta}\left[\vec{r},\ \beta_J(\vec{r})\right] \right|^{-1/2} , \tag{53''}$$

where γ is a phase which depends on the steepest-discent path. It can be shown that Φ_J and A_J satisfy the eikonal and transport equations. However the usual saddle-point method does not apply if we have

$$\Phi_{\beta\beta}\left[\vec{r},\ \beta_J(\vec{r})\right] = 0 \quad , \tag{54}$$

i. e. if for some $\vec{r}$ two saddle points coincide. In those points for which eq. (54) is satisfied, the amplitude (53") becomes infinite, as is easy to see; but, as is well-known, the points in the space where the ray amplitude (53") becomes infinite belong to the cau stic : therefore we can conclude that the usual saddle point method cannot be applied uniformly in a neighborhood of any point of the caustic. To this end we recur to the method proposed by Chester et al.[19]. We define a new variable ξ such that

$$\Phi(\vec{r}, \beta) = \omega(\vec{r}) + \varrho(\vec{r})\xi - \frac{1}{3}\xi^3 , \tag{55}$$

where ω and ϱ are functions to be determined. Now let us suppose that on the caustic $\Phi_{\beta\beta}(\vec{r}, \beta) \neq 0$: then it has been proved[19] that, by suitably choosing ω and ϱ, one branch of the transformation $\beta \to \xi$, as defined in (55), can be uniformly regular near $\xi = 0$, $\varrho = 0$. For each value of $\vec{r}$ there are two saddle points in ξ, which coincide when $\varrho(\vec{r}) = 0$; therefore the points of the caustic will satisfy the equation

$$\varrho(\vec{r}) = 0 . \tag{56}$$

Substituting (55) into (51), we get

$$\psi(\vec{r}) \simeq e^{ik\omega} \int g(\vec{r}, \xi) e^{ik(\varrho\xi - \frac{1}{3}\xi^3)} d\xi , \tag{57}$$

where

$$g(\vec{r}, \xi) = A(\vec{r}, \beta) \frac{d\beta}{d\xi} . \tag{57'}$$

Developing (57) in series of descending powers of k and retaining only the first two terms in the development, we get, a constant factor apart,

$$\psi(\vec{r}) \simeq e^{ik\omega} \left[g_o(\vec{r}) \int e^{ik(\varrho\xi - \frac{1}{3}\xi^3)} d\xi + g_1(\vec{r}) \int \xi e^{ik(\varrho\xi - \frac{1}{3}\xi^3)} d\xi \right] \tag{58}$$

where ω, ϱ, g_o and g_1 are unknown functions of $\vec{r}$, to be determined in such a way that $\psi(\vec{r})$ satisfies, asymptotically for $k \to \infty$, the reduced wave equation.

4. 1. - Trial solution

The two integrals in (58) are related to the Airy function and to its derivative. However which Airy function is to be used depends on the conditions which the solution must fulfill. We have three different Airy functions :

$$V(z) = A_i(-z) = \frac{1}{2\pi} \int_{\infty e^{-i\frac{\pi}{3}}}^{\infty e^{i\frac{\pi}{3}}} e^{(\frac{1}{3}\xi^3 + z\xi)} d\xi , \tag{59}$$

$$V_+(z) = A_i(-e^{2i\frac{\pi}{3}} z) , \tag{59'}$$

$$V_-(z) = A_i(-e^{-2i\frac{\pi}{3}} z) . \tag{59"}$$

Now, as is straightforward to see from the asymptotic expansions of (59-59"), for z-values in a neighborhood of the real positive axis, V(z) represents a superposition of an incoming and an outgoing wave, while $V_+(z)$ represents an outgoing and V_- an incoming wave. Since our solution must be outgoing at infinity, we set, in (58),

$$\psi(\vec{r}) \simeq 2\sqrt{\pi}\, e^{-i\frac{\pi}{12}} k^{\frac{1}{6}} e^{ik\omega} \left[g_o V_+(k^{2/3}\varrho) + \frac{g_1}{ik^{1/3}} V_+'(k^{2/3}\varrho) \right] \tag{60}$$

where the ' in the last addend denotes differentiation with respect to the argument of V_+. For $|k^{2/3}\varrho| \gg 1$ - i. e. far from the caustic which is given by eq. (56) - $\psi(\vec{r})$ may be approximated by

$$\psi(\vec{r}) \simeq A^+ e^{ik\Phi^+} , \tag{61}$$

where

$$\Phi^+ = \omega + \frac{2}{3} \varrho^{3/2} \tag{61'}$$

and

$$A^+ = (g_o + \sqrt{\varrho}\, g_1)\, \varrho^{-1/4} . \tag{61"}$$

As we shall see in a moment, Φ^+ and A^+ satisfy the eikonal and transport equation, so that far from the caustic our solution reduces to the usual eikonal approximation.

4. 2. - Asymptotic equations

Now we substitute $\psi(\vec{r})$ in the first member of the reduced wave equation (5), imposing that it should vanish in the limit for $k \to \infty$. As shown by Ludwig[14], this condition is fulfilled if the unknown functions ω, ϱ, g_o, g_1 satisfy the following equations:

$$(\vec{\nabla}\omega)^2 + \varrho(\vec{\nabla}\varrho)^2 - 1 = 0 , \tag{62}$$

$$2\vec{\nabla}\omega\, \vec{\nabla}\varrho = 0 , \tag{63}$$

$$2\vec{\nabla}\omega\, \vec{\nabla}g_o + \Delta\omega g_o + 2\varrho\vec{\nabla}\varrho\, \vec{\nabla}g_1 + \varrho\Delta\varrho g_1 + (\vec{\nabla}\varrho)^2 g_1 = 0 , \tag{64}$$

$$2\vec{\nabla}\varrho\, \vec{\nabla}g_o + \Delta\varrho g_o + 2\vec{\nabla}\omega\, \vec{\nabla}g_1 + \Delta\omega g_1 = 0 . \tag{65}$$

From (62) and (63) we get

$$(\vec{\nabla}\Phi^+)^2 = 1 , \tag{66}$$

where Φ^+ is given by (61'). Similarly, (64) and (65) can be combined to give[14], for $\varrho \neq 0$ (i. e. outside the caustic),

$$2\vec{\nabla}\Phi^+ \vec{\nabla}A^+ + \Delta\Phi^+ A^+ = 0 , \tag{67}$$

where A^+ is defined by (61"). Equations (66) and (67) are the eikonal and transport equatinn respectively, which now have been shown to be satisfied by Φ^+ and A^+, as defined by (61') and (61").

Equations (62-65), and therefore also (66) and (67), are valid for real values of the co-ordinates. Now we extend these equations to complex values of the co-ordinates and solve them in such a space; later on we shall that the solutions so found satisfy the original equations. We apply the uniform approximation to the system of complex rays which we have described in the preceding section, i. e. we regard the quantities A^+ and Φ^+ as referred to the "outgoing" ray, as specified in subsection 3. 2. We are able to solve the eikonal and transport equation for a single ray which arrives at a point (X, Y, Z), but this is not enough for determining the unknown functions of the system (62-65). To this end we consider also the quantities

$$\Phi^- = \omega - \frac{2}{3} \varrho^{3/2} , \tag{68}$$

and

$$A^- = \varrho^{-1/4}(g_o - \sqrt{\varrho}\, g_1) , \tag{69}$$

and we refer Φ^- and A^- to the "fictitious" ingoing ray introduced in the preceding section. By considering another suitable combination of eqs. (62-63) and (64-65), one can show that also Φ^- and A^- fulfill the eikonal equation and the transport equation. As we have shown in section 3, we can determine $\Phi^{\pm}$ and $A^{\pm}$ for each point (X, Y, Z). Now, inverting the relations (61'-61") and (68-69), we get

$$\omega = \frac{1}{2} (\Phi^+ + \Phi^-) , \tag{70}$$

$$\varrho = \left[\frac{3}{4} (\Phi^+ - \Phi^-)\right]^{2/3} , \tag{70'}$$

$$g_o = \frac{1}{2} \varrho^{1/4}(A^+ + A^-) , \tag{70''}$$

$$g_1 = \frac{1}{2} \varrho^{-1/4}(A^+ - A^-) . \tag{70'''}$$

These four quantities satisfy the system (62-65), as one can see by inverting the combinations which led from the eqs. (62-65) to the eikonal and transport equation for $\Phi^{\pm}$

and $A^{\pm}$. Furthermore, assuming that the complex caustic of eq. (18) and the function $D(U', V')$ (introduced in formula (47)) are regular, one can show (see ref. (14)) that ω, ϱ, g_o and g_1 are regular functions of U, V, W.

Now we recall eqs. (43), (44) and (46): inserting these equations in (70'), we get, for points near S',

$$\varrho \approx (2k)^{1/3} W = (\frac{k^2}{2})^{2/3} T^2 . \tag{71}$$

Now from (62), taking into account (71), it follows that

$$(\vec{\nabla}\omega)^2 = 1 , \tag{72}$$

which implies that, on the surface S', the function ω may be identified with U. On the other hand, inserting (61") and (69) in (47), and taking (49) into account, we get

$$(g_o \pm \sqrt{\varrho}\, g_1)^2 \varrho^{-1/2} T(\alpha + \beta T) = D^{\pm}(U', V') . \tag{73}$$

It follows, in the limit for $\varrho \to 0$, using (71),

$$D(U', V') = \tilde{g}_o^2 (\frac{2}{k^2})^{1/3} \alpha , \tag{74}$$

where $\tilde{g}_o$ denotes the restriction of the function g_o to the points of S'. It is worth noting that, as shown by eq. (74), $\tilde{g}_o$ is a finite quantity on the caustic; moreover $D(U', V')$ depends on the single surface ray and not on the global form of S', as we shall see in section 6, when we calculate the diffraction coefficients.

4. 3. - The boundary conditions

At this point we are able to calculate our unknown function $\psi(\vec{r})$ at every point in the "shadow region" of the complex space. However $\psi(\vec{r})$ still depends on the function $W_o(\Sigma_1, \Sigma_2)$ which we have introduced in section 2 (see eq. (18)). In order to determine this unknown function, we impose the proper boundary condition on the body-surface S, i. e.

$$\left[g_o V_+(k^{2/3}\varrho) + \frac{g_1}{ik^{1/3}} V'_+(k^{2/3}\varrho) \right]_{\vec{r} \in S} = 0 . \tag{75}$$

Now g_o and g_1 depend not only on $W_o(\Sigma_1, \Sigma_2)$, but also on $D(U', V')$, which shall be determined in the following. So, initially, we begin by considering a value of k so high that the second addend in (75) becomes negligibly small; then (75) reduces to

$$V_+(k^{2/3}\varrho) = 0 , \tag{76}$$

or

$$k^{2/3}\,\varrho_n = -e^{i\frac{\pi}{3}}x_n \quad , \tag{77}$$

where $-x_n$ is the n-th zero of the Airy function, which has been already introduced in subsection 2. 2. Conditions (77) give rise to infinitely many complex caustics of the type (18), which correspond to the infinite "modes" of excitation of the surface waves.

Now, as shown in Appendix C, a set of complex functions $W_o^{(n)}(\sigma_1, \sigma_2)$ can be deduced from (77), for sufficiently high values of k, by using an iteration procedure and taking $W_o^{(n)} = 0$ as a zeroth-order approximation; the $W_o^{(n)}(\sigma_1, \sigma_2)$ so determined are analytic in their arguments. We observe that the zeroth-order approximation for the caustic does not allow us to impose tha proper condition on the real surface, since, in this case, one has $\varrho = 0$ on S, in contrast to (77). As regards the successive iteration step, we can approximate ϱ by (71) for the caustics near to S, which are the most important ones; furthermore we can confuse the curvature of S' with that of S, which we call κ_o:

$$\kappa_o = \lim_{k\to\infty} \kappa \quad ; \tag{78}$$

as a result, we get

$$kW_o^{(n)} \simeq e^{i\frac{\pi}{3}}x_n\left(\frac{k}{2\kappa_o}\right)^{1/3} \quad . \tag{79}$$

This approximation coincides with that of Levy and Keller[2], since on the surface ray we have

$$dU \simeq (1+W_o^{(n)}\kappa_o)du = \left[1+e^{i\frac{\pi}{3}}x_n\left(\frac{\kappa_o^2}{2k^2}\right)^{1/3}\right]du \ , \tag{80}$$

where du is the element of the curvilinear abscissa along the real surface ray. Further approximations can be obtained using (79) as an input for (75); this point will be further developed in section 6.

We conclude this section with some considerations about the regularity of the functions ω, ϱ, g_o and g_1. We have already seen in subsection 4. 2 that these functions are regular if $W_o(\Sigma_1, \Sigma_2)$ and D(U', V') are; but the regularity of W_o is established in Appendix C; on the other hand D(U', V') will be defined (section 6) as a regular function of $W_o(\Sigma_1, \Sigma_2)$, so that the regularity of the four above-mentioned functions is guaranteed. Moreover these functions, which are solutions of the system of equations (62-65) for complex values of the co-ordinates, satisfy the same system also for the restriction to real values of the co-ordinates, due to the analyticity of all the functions involved.

5. - THE FIELD NEAR A CAUSTIC

In this section we are faced with the problem of calculating the field in a neighborhood of the points of the surface S", which we called "second sheet" of the caustic (section 3); moreover we determine the change of phase that a ray undergoes when crossing a point of S".

As a first step, let us re-write the Helmholtz equation using U', V', T as a coordinate system. The

$$\frac{1}{J}\sum_{l=1}^{3}\frac{\partial}{\partial U_l}\left(J\sum_{i=1}^{3}G^{il}\frac{\partial\psi}{\partial U_i}\right)+k^2\psi = 0\ , \tag{81}$$

where J is the Jacobian which has been defined in (49), while G^{il} is the inverse of the tensor

$$G_{il} = \frac{\partial\vec{R}}{\partial U_i}\,\frac{\partial\vec{R}}{\partial U_l} \tag{82}$$

and

$$\vec{R} \equiv (X,Y,Z)\ ,\quad U_1 = U'\ ,\quad U_2 = V'\ ,\quad U_3 = T\ . \tag{83}$$

In all the points outside S" ψ may be approximated by (60) according to the method explained in the preceding section. In particular, for the points out of S' one can use the eikonal approximation, which has already been deduced in section 3, i. e.

$$\psi \simeq \sqrt{\frac{C(U',V')}{J}}\ e^{ik(U'+T)}\ . \tag{84}$$

But, as we have already said, the problem arises how does the field ψ behave when the ray crosses a point belonging to S"; or, in other words, how must the expression (84) be continued analytically across a focal point.

This problem can be compared with the onedimensional turning-point problem, as posed, e. g., by Knoll and Schaeffer[8]. In this case one has to solve an ordinary linear differential equation of the type

$$\frac{d^2\psi}{dX^2}+k^2\left[\omega_o(x)\right]^2\psi = 0\ , \tag{85}$$

where $\omega_o(x)$ is a regular smooth function with a simple zero in $x = x_o$, i. e.

$$\omega_o(x) \simeq \alpha_o(x-x_o) \qquad \text{for}\quad x \to x_o\ , \tag{86}$$

with α_o such that

$$0 < \arg\alpha_o < \frac{\pi}{2}\ . \tag{86'}$$

Now the ordinary WKB approximation, i. e.

$$\psi \simeq A^{+} e^{ik\Phi} + A^{-} e^{-ik\Phi} , \qquad (87)$$

with

$$\Phi = \int^{x} \omega(x')\,dx' , \qquad A^{\pm} = \frac{G^{\pm}}{\sqrt{\omega(x)}} , \qquad (87')$$

fails in a neighborhood of x_0. Moreover different values of the constant $G^{\pm}$ must be used, according to the location in the complex x-plane of the point where we wish to determine the field; in particular, we distinguish among regions which are separated from each other by lines (the so-called Stokes lines) that start at x_0. The situation in the case of a simple turning point is illustrated in Fig. 1 of ref. (8). Following Stokes[21], who gave connection rules for determining the constant $G^{\pm}$ in the various regions, we can distinguish, in the case of a simple turning point, among three different regions of analiticity: region I, where there is a superposition of an ingoing and of an outgoing wave; region II, where only the ingoing wave is present; and region III, where we have only a outgoing wave. Moreover in region I the two addends in (87) are such that the first one (outgoing) is transformed into the second one (ingoing) after a complete clockwise turn around the turning point (see Fig. 2 of ref. (8)). Conversely, an ingoing ray is transformed into an outgoing one by a counterclockwise turn around x_0. As a consequence, if an incoming ray travels in the direction of increasing Re x, the analytical continuation has to be made taking a cut in the complex x-plane from x_0 to $+i\infty$; from our assumptions (86), (86'), and from the expression of A^{+} given in (87'), it follows that the phase of A is changed by $-\frac{\pi}{2}$ in crossing a turning point.

Now we assume that the same effect occurs in the three-dimensional case: since a ray propagates in the direction of increasing Re T, we take a cut in the complex T-plane from the focal point (where $J = 0$) to $+i\infty$; moreover we choose the orientation of the parameter V along the curve C in such a way that the expression (47) of the amplitude A undergo a change of phase of $-\frac{\pi}{2}$ in crossing a focal point.

The same change of phase occurs every time a surface ray crosses a point of S", i. e. a point where α vanishes, as can be seen from eq. (74); in this case, one has to cut the complex U-plane from the focal point to $+i\infty$, the sign of α being uniquely determined by the orientation assigned to the parameter V along the curve C. Here we do not tackle the problem of determining the field uniformly near a caustic in the most general case. We limit ourselves to considering the case in which S" is - at least locally - sufficiently well-behaved, so that we can apply the uniform approximation described in the preceding section. In other words, we use the tentative solution (58), choosing an integral path in the ξ-plane, such that it represents a sum of an ingoing plus an outgoing wave, i. e.

$$\psi(\vec{r}) \simeq 2\sqrt{\pi}k^{1/6}e^{ik\omega_a}\left[g_{oa}V(k^{2/3}\varrho_a) + g_{1a}\frac{1}{ik^{1/3}}V'(k^{2/3}\varrho_a)\right] , \tag{88}$$

where V(z) is defined by (59), whereas ω_a, ϱ_a, g_{oa}, g_{1a} are determined in such a way to fulfill the system (62-65) and in such a way that (88) go over smoothly into the eikonal approximation for points sufficiently far from the caustic.

In the next section we shall consider the case of an axial caustic, employing a completely different method.

6. - THE GREEN FUNCTION

At this point we have still to determine the Green function, as defined in section 2. We write it, tentatively, as a linear combination of functions (60): our problem amounts, therefore, to determining the coefficients of such a combination, which, as we shall see, are related to the diffraction coefficients. At first we consider diffraction of waves emitted by a linear source on a smooth convex right cylinder, with its generants parallel to the source. Then we study diffraction of waves emitted by a point -like source on a smooth convex object.

6. 1. - Diffraction by cylinders (linear source)

Recalling the reciprocity theorem relative to the Green function for the Helmholtz operator, it seems to be suitable to set

$$G(\vec{r};\vec{r}_o) \simeq k^{-1}\sum_{s,n}\psi_{s,n}(\vec{r})\,\psi_{s,n}(\vec{r}_o) , \tag{89}$$

where $\psi_{s,n}(\vec{r})$ are given by (60), i. e.

$$\psi_{s,n}(\vec{r}) = 2\sqrt{\pi}k^{1/6}e^{-i\frac{\pi}{12}}e^{ik\omega_{s,n}}\chi_{s,n}(\vec{r}) , \tag{90}$$

and

$$\chi_{s,n}(\vec{r}) = g_{os,n}V_+(k^{2/3}\varrho_{s,n}) + \frac{1}{ik^{1/3}}V'_+(k^{2/3}\varrho_{s,n}) ; \tag{91}$$

s denotes a generical branch of the solution of the eikonal equation, while n refers to the modes of excitation of surface waves, as determined in section 4.

Using formula (70) relative to ω and taking into account all branches of the eiko nal function, we get, in the case of a right smooth cylinder

$$G(\vec{r}, \vec{r}_o) \simeq 4\pi e^{-i\frac{\pi}{6}} k^{1/3} k^{-1} \sum_n \frac{\cos\left[k(\frac{1}{2}\tilde{U}_n - U_n)\right]}{\sin(\frac{1}{2}k\tilde{U}_n)} \chi_n(\vec{r})\,\chi_n(\vec{r}_o) \quad , \qquad (92)$$

where U_n is the curvilinear abscissa of the point where the ray leaves the complex caustic, while $\tilde{U}_n$ refers to a complete turn around the cylinder.

The determination of the Green function amounts to determining the functions $D_{s,n}(U_n, V_n)$, contained in the expressions of $g_{os,n}$ and of $g_{1s,n}$. To this end we ob serve that these functions do not depend neither on the location of the source, nor on the point of observation; so we find it convenient to take both P_o and P as near as possible to the surface of the cylinder (see also ref. (3)). P_o and P are located also near to the complex caustics that are closest to the real surface, which, as we have already seen, give the largest contribution to the field. Since, near to the caustics, the second added of (91) is negligible as compared to the first one, near to the real surface one has

$$\chi_n(\vec{r}) \simeq (2\kappa)^{-1/6} \left[D_n(U', V')\right]^{1/2} V_+(k^{2/3}\varrho_n) \, , \qquad (93)$$

where (46), (47), (49), (70") and (71) have been taken into account. Now (93) does not contain any dependence on the global form of S, but only on local properties of the real surface of the body; so we compare it with the expression obtained in the case of a circular cylinder, which has been deduced from the exact one (see section 2). More precisely, we specialize (92) to the case of a circular cylinder and compare it with (5); as a result we obtain

$$D_n = - \frac{e^{-i\frac{\pi}{6}}}{4} \left(\frac{ka}{2}\right)^{1/3} \left[A_i'(-x_n)\right]^{-2} \quad . \qquad (94)$$

This expression is valid for a generical convex cylinder, provided we substitute the radius a with

$$a_n(U_n', V_n') = \left[\kappa_n(U_n', V_n')\right]^{-1} . \qquad (95)$$

It is worth observing that (93) coincides asymptotically with the expression used by Ludwig[6] for describing, near the surface, the diffraction by cylinders: this can be checked by substituting the Hankel functions used in ref. (6) with their uniform appro- mations[22].

6.2. - Diffraction by a smooth, convex object (point-like source)

In this case we set, in a way similar to the case of a cylinder,

$$G(\vec{r}, \vec{r}_o) \simeq k^{-1} \sum_{s,n} e^{-ip_s \frac{\pi}{2}} \psi_{s,n}(\vec{r}) \, \psi_{s,n}(\vec{r}_o) \, , \qquad (96)$$

where $\psi_{s,n}(r)$ is given by (90), while p_s denotes the number of focal points crossed by each ray (more precisely, by the s-th branch of the eikonal). Also in this case we have to determine the functions $D_{s,n}(U_n, V_n)$, which are contained in $g_{os,n}$ and in $g_{1s,n}$; now we shall compare (96) with the asymptotic solution which is obtained for a sphere (see ref. (2)). Therefore, at first, we specialize our formula to the case of a smooth convex body of revolution, with the point-like source set on the symmetry axis. We are faced with a difficulty, since the symmetry axis constitutes the second sheet S" of the caustic (more precisely, it constitutes its restriction to real values of the co-ordinates); therefore, as we have seen, formulae (90-91) are not adequate for describing the field at $\vec{r}$. Now we give a prescription for describing the field near to an axial caustic. To this end, let us write the Green function for a body of revolution, taking into account the sum over all the branches s of the eikonal and the phase shifts $-p_s \frac{\pi}{2}$, which each ray indergoes when it crosses points of the symmetry axis. We get

$$G(\vec{r}, \vec{r}_o) \simeq 4\pi e^{-i\frac{\pi}{6}} e^{-i\frac{\pi}{4}} k^{-2/3} \sum_n \frac{\cos\left[k(\frac{1}{2}\tilde{U}_n - U_n) - \frac{\pi}{4}\right]}{\cos(\frac{1}{2} k\tilde{U}_n)} \bar{\chi}_n(\vec{r}) \, \bar{\chi}_n(\vec{r}_o) \, , \qquad (97)$$

where $\bar{\chi}_n(r)$ coincides with $\chi_n(r)$ for points far from the symmetry axis. On the contrary, near to the axis it is convenient to set

$$\bar{\chi}_n(\vec{r}) = \frac{\sqrt{2\pi k R_{1n} \varepsilon} \; J_o\left[k(\frac{1}{2}\tilde{U}_n - U_n)\right]}{\cos\left[k(\frac{1}{2}\tilde{U}_n - U_n) - \frac{\pi}{4}\right]} \, \chi_n(\vec{r}) \, , \qquad (98)$$

where R_{1n} is the radius of curvature of the n-th caustic surface at the point where it meets the axis, while ε is the angle between the axis and the straight line OP, where O is the centre of curvature of the real geodesics passing through the point L where the real surface crosses the axis (cfr. Fig. 2). For

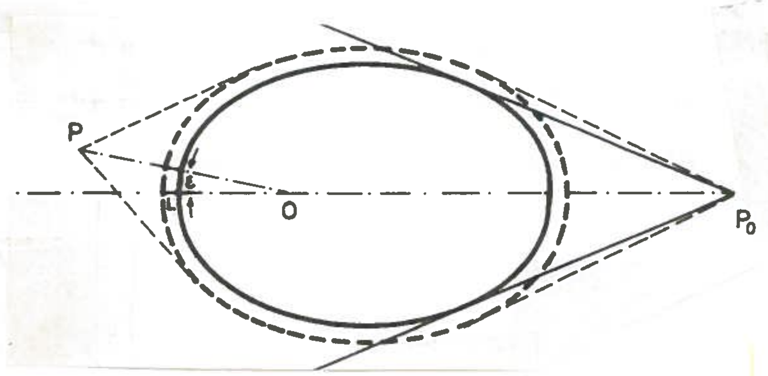

FIG. 2 - Diffraction by a body of revolution. The dashed lines represent the compex rays.

mula (98) holds true as long as

$$\varepsilon \lesssim (ka_1)^{-1} ,$$

where a_1 is the radius of curvature of the geodesics at the point L. Now, specializing (97) to the case of a sphere and comparing the expression so obtained with the one given by Levy and Keller[2], we get

$$D_n = \frac{1}{a^2} \frac{\sqrt[6]{2}\, e^{i\frac{\pi}{12}}}{16\pi^2} (ka)^{1/3} \left[A_i'(-x_n)\right]^{-2} , \tag{99}$$

where a is the radius of the sphere.

Now, as in the case of the cylinder, the Green function does not depend on the global form of the surface S of the object when the points P_o and P are close to S; therefore the result (99) can be extended to any convex smooth object by making the substitution (95). Obviously the expression (96) of the Green function has to be suitably modified for points (if any) which belong to the sheet S" of the caustic.

As a conclusion of this section, let us observe that the location of the complex caustics and the functions $D_{s,n}(U_n, V_n)$ are related, respectively, to the decay exponents and to the diffraction coefficients of the modes of excitation of surface waves. Moreover (94) and (99), together with (95), ensure us that $D_{s,n}(U_n, V_n)$ are regular functions of the curvature of the complex caustics, thus legitimating the iteration procedure - as illustrated in section 4 and in Appendix C - for solving eq. (75), i. e. for determining the complex caustics.

7. - CONCLUSION

Let us summarize here the main results that we have obtained in this paper.

1) Firstly, the use of the uniform approximation at complex caustics allows us to write the expression of the field both near the surface of the object and far from it, avoiding complications which arise in connecting the diffracted field to the surface field[2]; moreover we need write only few simple equations for the field, without considering, for example, the "surface" transport equation[2,3].

2) Secondly, our method reduces considerably the number of assumptions, made in preceding diffraction theories, about the diffracted field.

3) Thirdly, the decay exponents and diffraction coefficients can be calculated - through an iterative process - with higher precision than in the preceding treatment.

4) Lastly, the second addend of (60), which vanishes in the case of a sphere or of

a circular cylinder, is sensitive - for not too large values of k - to the degree of asymmetry of the diffracting object.

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APPENDIX A

A first-order real partial differential equation

$$F(x, y, \Phi, \Phi_x, \Phi_y) = 0 , \tag{A.1}$$

with given initial conditions on a curve C on the (xy)-plane, can be shown (ref. (23), pag. 75) to be equivalent to the following system of ordinary differential equations:

$$\frac{dx}{d\sigma} = F_p , \quad \frac{dy}{d\sigma} = F_q , \quad \frac{d\Phi}{d\sigma} = pF_p + qF_q ,$$
$$\frac{dp}{d\sigma} = -(pF_\Phi + F_x) , \quad \frac{dq}{d\sigma} = -(qF_\Phi + F_y) , \tag{A.2}$$

with a suitable initial strip on C.

Now let us suppose the first member of (A.1) to be given by complex function

$$F = F_1 + iF_2 . \tag{A.3}$$

Then also Φ, p and q will be complex functions:

$$\Phi = \Phi_1 + i\Phi_2 , \quad p = p_1 + ip_2 , \quad q = q_1 + iq_2 . \tag{A.4}$$

Equation (A.1) goes over into the following system:

$$F_1(x, y, \Phi_1, \Phi_2, p_1, p_2, q_1, q_2) = 0 , \tag{A.5}$$

$$F_2(x, y, \Phi_1, \Phi_2, p_1, p_2, q_1, q_2) = 0 , \tag{A.5'}$$

whose characteristic equation is given by

$$\det \left[\frac{\partial F_i}{\partial p_j} - \tau \frac{\partial F_i}{\partial q_j} \right] = 0 \qquad (i, j = 1, 2) \tag{A.6}$$

where

$$\tau = \frac{dx}{d\sigma} / \frac{dy}{d\sigma} . \tag{A.7}$$

Now let us suppose F to be analytic in p, q, Φ : then, due to the Cauchy-Riemann equations, (A.6) becomes

$$\left(\frac{\partial F_1}{\partial p_1} - \tau \frac{\partial F_1}{\partial q_1}\right)^2 + \left(\frac{\partial F_2}{\partial p_1} - \tau \frac{\partial F_2}{\partial q_1}\right)^2 = 0 , \tag{A.8}$$

which is solved by

$$\tau_1 = F_p/F_q , \qquad \tau_2 = (F_p/F_q)^* . \tag{A.9}$$

Now, if we assume F to be analytic also in x, y and look for a solution Φ analytic in its arguments, the only acceptable solution of (A.8) is τ_1. Then, taking into account (A.7) and choosing suitably the parameter σ, one gets

$$\frac{dx}{d\sigma} = F_p , \qquad \frac{dy}{d\sigma} = F_q . \tag{A.10}$$

Moreover it is easy to show that the system of differential equations (A.2) holds true in a way formally identical to the case in which F is real. This system is equivalent to the original (complex) partial differential equation, provided that suitable initial conditions are imposed.

APPENDIX B

Here we prove the following

THEOREM: "If $F(\xi, \eta, \zeta)$ is an analytic function of its arguments, the partial differential equation

$$\psi_t + t^2 F\left(\frac{\psi_s}{t}, t, s\right) = 0 \tag{B.1}$$

has one and only one solution $\psi(s,t)$ analytic in its arguments, which vanishes for $t=0$; furthermore one has $\psi_t(0,s) = \psi_{tt}(0,s) = 0$."

Proof. Let us write the characteristic system relative to (B.1):

$$\frac{dt}{d\sigma} = 1 , \qquad \frac{ds}{d\sigma} = \frac{\partial F}{\partial\left(\frac{q}{t^2}\right)} , \qquad \frac{d\psi}{d\sigma} = p\frac{dt}{d\sigma} + q\frac{ds}{d\sigma} ,$$

$$\frac{dp}{d\sigma} = \frac{\partial}{\partial t}\left[t^2 F\left(\frac{q}{t^2}, s, t\right)\right] , \qquad \frac{dq}{d\sigma} = t^2 \frac{\partial F}{\partial \sigma} , \tag{B.2}$$

with the following initial conditions:

$$t(0) = 0, \quad s(0) = s_o, \quad \psi(0,s) = 0, \quad p(0,s) = p_o, \quad q(0,s) = 0 . \tag{B.3}$$

Then, by Picard's iteration method, one can show that the system admits one and only one solution $\psi(t,s)$, analytic in its arguments.

As a second step, we have to show that equation (B.1) is equivalent to the system (B.2). This second step of the proof can be carried out by means of considerations quite analogous to those made in Appendix A and by a reasoning nearly identical to the non singular case (see ref. (23), pag. 79-82). The proof of this theorem follows closely the one given in Appendix A of ref. (14).

APPENDIX C

Here we give a sufficient condition for the existence of complex caustic of type (18), such that they should satisfy (41) and such that the boundary condition on the real surface S should be fulfilled.

To this end, firstly, we define a co-ordinate system σ_1, σ_2, σ_3, where σ_3 is the distance from S taken along the direction of the normal versor n defined by (19); moreover we choose $\sigma_1 = u$ and $\sigma_2 = v$, where u and v are defined on S in a way similar to the parameters U and V (these have been introduced in section 3 and tend to u and v in the limit in which S' tends to S). Secondly we consider the normed linear space $\mathcal{N}$ of the complex regular functions $W_o(\sigma_1, \sigma_2)$ defined on a compact set I of the (σ_1, σ_2) space, with the norm

$$\left\| W_o(\sigma_1, \sigma_2) \right\| = \max_{\sigma_1, \sigma_2 \in I} \left| W_o(\sigma_1, \sigma_2) \right| . \qquad (C.1)$$

Now, by means of (18), each $W_o(\sigma_1, \sigma_2)$ corresponds to a complex surface, on which we can introduce a set of co-ordinates U, V, W; moreover we consider the complex extension Σ_k (k = 1, 2, 3) of the σ_k-co-ordinates. As a first step we want to show that, for every compact subset $\mathcal{S}$ of the Σ_k-space, there is a real positive constant δ such that, for $\| W_o(\sigma_1, \sigma_2) \| < \delta$, all the points of $\mathcal{S}$ can be put in a one-to-one, regular correspondence with the points of a subset $\mathcal{U}$ of the (U, V, W)-space. To see that, we observe that the Jacobian

$$J' = \frac{\partial(U, V, W)}{\partial(\Sigma_1, \Sigma_2, \Sigma_3)} \qquad (C.2)$$

reduces, for $\| W_o \| = 0$, identically to 1 throughout the whole Σ_k-space. Moreover, if we set

$$W_o(\sigma_1, \sigma_2) = \delta \Xi(\sigma_1, \sigma_2) , \qquad (C.3)$$

where $\delta > 0$ and $\Xi(\sigma_1, \sigma_2)$ has a unitary norm, one has

$$J' = J'\left[\Sigma_1, \Sigma_2, \Sigma_3, \delta \Xi(\Sigma_1, \Sigma_2)\right] , \qquad (C.4)$$

which is uniformly continuous in its arguments for $0 \leq \delta \leq \delta_o$ (δ_o is a real number) and for $(\Sigma_1, \Sigma_2, \Sigma_3)$ belonging to a compact set $\mathcal{S}$. Therefore, given an $\mathcal{S}$, we can choose δ so small that J' is regular and nonvanishing on $\mathcal{S}$. Then our thesis follows by defining $\mathcal{U}$ as the image of $\mathcal{S}$ in the (U, V, W)-space.

As a consequence of what we have shown, ϱ, g_o and g_1 (which are defined by eqs. (70'-70''')) can be regarded as regular functions of $\Sigma_1, \Sigma_2, \Sigma_3$ on $\mathcal{S}$, for a δ sufficiently small, say $\delta < \delta_1$; to our ends it is convenient to choose $\mathscr{S}$ in such a way that it includes a neighborhood of S.

At this point let us turn our attention to condition (75). We have to determine $W_o(\sigma_1, \sigma_2)$ in such a way that the functions

$$\varrho\left[\Sigma_1, \Sigma_2, \Sigma_3, W_o(\Sigma_1, \Sigma_2)\right] , \tag{C.5}$$

$$g_o\left[\Sigma_1, \Sigma_2, \Sigma_3, W_o(\Sigma_1, \Sigma_2)\right] , \tag{C.6}$$

$$g_1\left[\Sigma_1, \Sigma_2, \Sigma_3, W_o(\Sigma_1, \Sigma_2)\right] , \tag{C.7}$$

should satisfy, for $\Sigma_3 = 0$ and Σ_1, Σ_2 real, the relation

$$g_o V_+(k^{2/3}\varrho) + \frac{1}{ik^{1/3}} g_1 V'_+(k^{2/3}\varrho) = 0 . \tag{C.8}$$

We want to solve (C.8) iteratively with respect to W_o, starting from $\|W_o\| = 0$. To this end, primarily, let us study the function

$$\xi\left[\sigma_1, \sigma_2, W_o(\Sigma_1, \Sigma_2)\right] = \varrho\left[\sigma_1, \sigma_2, 0, W_o(\sigma_1, \sigma_2)\right] . \tag{C.9}$$

If we call $\overline{W}(\sigma_1, \sigma_2)$ the values of the W-coordinates relative to the points of the real surface S, we get from (71) and from (C.9)

$$\frac{\partial\xi}{\partial W_o} = \frac{\partial\xi}{\partial\overline{W}}\frac{\partial\overline{W}}{\partial W_o} = (2k)^{1/3}\frac{\partial\overline{W}}{\partial W_o}\left[1 + O(\overline{W})\right] , \tag{C.10}$$

which tends to a positive quantity in the limit of $\|W_o\| \to 0$; therefore we can choose δ so small (say $\delta < \delta_2 < \delta_1$) that the derivative (C.10) is different from zero on $\mathscr{S}$ and, therefore, the relation between ξ and W_o is invertible for $(\sigma_1, \sigma_2) \in I$:

$$W_o(\sigma_1, \sigma_2) = f\left[\sigma_1, \sigma_2, \xi(\sigma_1, \sigma_2)\right] . \tag{C.11}$$

Moreover from (71) and from (C.10) it follows that ξ is proportional to W_o.

Now let us consider the operator

$$\varphi(W_o) = f\left[\sigma_1, \sigma_2, \eta\right] , \tag{C.12}$$

with

$$\eta = k^{-2/3} e^{i\frac{\pi}{3}} A_i^{-1}(\zeta) \tag{C.12'}$$

and

$$\zeta = i \frac{g_1}{k^{1/3} g_o} A_i' \left\{ e^{-i\frac{\pi}{3}} k^{2/3} \xi \left[\sigma_1, \sigma_2, W_o(\sigma_1, \sigma_2)\right] \right\} \quad . \qquad (C.12'')$$

As can be seen from (C.12'), $\varphi(W_o)$ is a multivaluted function, with infinitely many branches, each corresponding to a zero x_r' of the derivative of the Airy function. Let us consider the first n branches of $\varphi(W_o)$: we show that there are two positive constants $\bar{k}$ and $\bar{a}$ (the latter depending on $\bar{k}$) such that, for $k > \bar{k}$ and $\|W_o\| < \bar{a}$, the equation

$$W_o = \varphi(W_o) \qquad (C.13)$$

has exactly one regular solution for each of the n branches. As is easy to see, each solution of (C.13) is also a solution of (C.8). Therefore our thesis will be proved if we show that there are two positive constants $\bar{a}$ and $\bar{k}$ such that, for $\|W_o\| < \bar{a}$ and $k > \bar{k}$, φ acts as a contraction operator on $\mathcal{N}$, i. e. such that

i) φ is regular ,

ii) $\|\varphi(W_o)\| < \bar{a}$ (C.14)

iii) $\|\varphi(W_{o2}) - \varphi(W_{o1})\| \leq \nu \|W_{o2} - W_{o1}\|$, $(0 < \nu < 1)$, (C.15)

whenever $\|W_{oi}\| < \bar{a}$ $(i = 1, 2)$ and $k > \bar{k}$. Once i), ii) and iii) have been proved, the fixed-point theorem guarantees that (C.13) can be solved iteratively, starting from $W_o(\sigma_1, \sigma_2) = 0$, and that the iterative process converges to a unique, regular solution. Then let us examine the three questions exposed above.

As for the first two questions, we observe, primarily, that we can choose the set $\mathcal{S}$ of complex Σ_k-coordinates in such a way that the function ζ, as defined by (C.12''), has no singularities ; in fact the function g_o, which appears in the denominator of the r. h. s. of (C.12''), reduces, for $\|W_o\| = 0$, to a nonvanishing function (see eq. (40'') in a suitable compact set which includes the points of the surface S, and therefore it keeps different from zero in this set, provided that $\|W_o\|$ is sufficiently small.

Secondly, since ξ has been found to be proportional to W_o, the argument of the derivative of the Airy function (see (C.12'')) is limited if we choose $\|W_o\|$ suitably small, depending on $k^{2/3}$; consequently ζ can be made as small as we like for k sufficiently large, as can be seen from (C.12''). Thirdly, we require ζ to be small enough for the branches $A_i^{-1}(\zeta)$ to be regular; furthermore, if we consider the r-th branch of the inverse of the Airy function $(r \leq n)$, $A_i^{-1}(\zeta)$ lies in a neighborhood of $-x_r$ (which is the r-th zero of the Airy function); therefore, as can be seen from (C.12--C.12'), for sufficiently large values k the function η - and consequently $f(\sigma_1, \sigma_2, \eta)$, which is proportional to η - can be made to satisfy (C.14) whenever $\|W_o\| < \bar{a}$ ($\bar{a}$ being a suitable positive number).

Now let us turn our attention to question iii). From the mean-value theorem it follows that

$$\varphi(W_{o2}) - \varphi(W_{o1}) = \left(\frac{\partial\varphi}{\partial W_o}\right)_{\overline{W}_o} (W_{o2} - W_{o1}) \,, \qquad (C.16)$$

where $\overline{W}_o$ lies on the segment which joins W_{o1} to W_{o2} in the complex W-plane. Now, if $\|W_{o1}\|$ and $\|W_{o2}\|$ are less than a positive quantity ε_o, also $\|\overline{W}_o\| < \varepsilon_o$; therefore the partial derivative $\partial\varphi/\partial W_o$ is limited; moreover it can be made, in modulus, less than a positive quantity $\nu < 1$, provided that we choose k sufficiently large. Then from (C.16) it follows (C.15).

Our thesis follows by calling $\bar{k}$ and $\bar{a}$ respectively the smallest value of k and the largest value of $\|W_o\|$ for which the above conditions are satisfied.

REFERENCES

(1) - J.B. Keller, Proceedings Symp. Applied Mathematics 8, 27 (1958).
(2) - B.R. Levy and J.B. Keller, Comm. Pure Appl. Math. 12, 159 (1959).
(3) - R.M. Lewis, N. Bleistein and D. Ludwig, Comm. Pure Appl. Math. 20, 295 (1967).
(4) - S. Hong, Journ. Math. Phys. 8, 1223 (1967).
(5) - F. Ursell, Proc. Camb. Phil. Soc. 62, 227 (1966).
(6) - D. Ludwig, SIAM Rev. 17, 1 (1975).
(7) - J. Knoll and R. Schaeffer, Phys. Letters B52, 131 (1974).
(8) - J. Knoll and R. Schaeffer, Ann. Phys. (N.Y.) 97, 307 (1976).
(9) - R. Balian and C. Bloch, Ann. Phys. (N.Y.) 63, 592 (1971); 85, 514 (1974).
(10) - B.J.B. Crowley, Phys. Rep. 57, 47 (1980).
(11) - E. Heyman and L.B. Felsen, SIAM J. Appl. Math. 43, 885 (1983).
(12) - E. Heyman and L.B. Felsen, to be published in the Handbook on Acoustic Electromagnetic and Elastic Wave Scattering, Proceedings of a Workshop held at Ohio State University, October 1982.
(13) - S. Choudary and L.B. Felsen, IEEE Trans. Antennas and Propagation, AP-21, 827 (1973); L.B. Felsen, J. Opt. Soc. Am. 66, 751 (1976).
(14) - D. Ludwig, Comm. Pure Appl. Math. 19, 215 (1966).
(15) - B. Schrempp and F. Schrempp, Nuclear Phys. B163, 397 (1980).
(16) - W.D. Wang and G.A. Deschamps, Proc. IEEE 62, 1541 (1974).
(17) - H.M. Nussenzveig, Ann. Phys. (N.Y.) 34, 23 (1965).
(18) - R.B. Dingle, Asymptotic Expansions: Their Derivation and Interpretation (London and New York, 1973), p. 255.
(19) - C. Chester, B. Friedman and F. Ursell, Proc. Camb. Phil. Soc. 54, 599 (1957).
(20) - S. Helgason, Differential Geometry and Symmetric Spaces (New York and London, 1962), p. 387.
(21) - G.G. Stokes, Trans. Camb. Phil. Soc. 10, 106 (1857).
(22) - F.W.S. Olver, Phil. Trans. Roy. Soc. London, Ser. A 247, 328 (1954).
(23) - R. Courant, Methods of Mathematical Physics (New York, 1962), Vol. 2.