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A BI-METRIC THEORY OF KLEIN-KALUZA TYPE

P. K. Smrz(*)
Istituto Dipartimentale di Fisica dell'Università di Catania

ABSTRACT

A unified theory of Klein-Kaluza type with a de Sitter structure is considered. A condition binding the external and internal spaces reduces the Lagrangian of the theory to the Einstein Lagrangian plus a small quadratic term.

It has been suggested that a theory of strong interactions could be modelled along the lines of general relativity[1,2], i. e. that the basic field describing strong interactions could be a tensor field. In the region where both the strong and the gravitational interaction is present the theory should possess two tensor fields of the same kind: the usual metric tensor describing gravitation and the second tensor attributed to strong interactions. The basic equations or the Lagrangian of such theories are usually postulated without giving them a specific geometric meaning. There exists, however, a very natural geometric pattern that unifies various fields with gravitation, namely the theories of Klein-Kaluza type[3]. It is, therefore, interesting to see whether a bi-metric theory can be constructed in such a way. The present note represents the first step in such a direction.

Unified theories of Klein-Kaluza type are based on a principal fibre bundle with a pseudo-Riemannian space-time for the base manifold and a structure group depending on the type of the field which is to be unified with gravitation. A connection in such a

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(*) Permanent address : Faculty of Mathematics, University of Newcastle, N. S. W. 2308, Australia.

fibre bundle defines in a geometrically natural way a pseudo-Riemannian metric in the bundle manifold. Every horizontal vector is considered as orthogonal to every vertical vector, the inner product of vertical vectors is taken as the invariant Killing product in the structure group, and the product of horizontal vectors is defined by the metric of the base manifold. The scalar curvature of the bundle manifold can then be simply calculated using for example the non-coordinate basis technique of ref. (4), and yields[3]

$$R = \tilde{R} - \frac{\ell^2}{4} F^a_{\mu\nu} F_a^{\mu\nu} + \frac{1}{4\ell^2} G_{ab} G^{ab} , \tag{1}$$

where $\tilde{R}$ is the scalar curvature of the base manifold, $F^a_{\mu\nu}$ is the curvature of the connection, and $G_{ab} = C^f_{ad} C^d_{bf}$ is the invariant metric of the structure group that serves also for the raising and lowering of the group index. A fundamental length ℓ must be introduced in order to make the three terms of expression (1) of the same physical dimension. Its necessity follows from the fact that we combine in a single manifold space-time coordinates (all of dimension "length" with $c = 1$) and dimensionless group coordinates.

In the usual Klein-Kaluza theory[5] unifying gravitation and electromagnetism the structure group is $U(1)$, the third term of expression (1) is absent, and ℓ^2 has the meaning of the universal gravitation constant.

Let us now choose a structure group that can in principle introduce another metric tensor, besides the one connected with the base manifold. The basic fields defining the connection in the bundle are the gauge potentials A^a_μ, equipped with one space-time and one group index. They could approach the form of a tensor (or rather that of a tetrad) if the group index has the meaning of a local Lorentz index. That could be achieved by using the Poincaré group, since the gauge potentials corresponding to translations could play such a role. However, in order to have a non-singular metric, the structure group must be semi-simple. Thus the obvious choice falls on a de Sitter group.

We consider a space-time manifold with a de Sitter group (serving as an "internal space") attached to every point, forming a 14-dimensional bundle manifold. Using the usual basis of the de Sitter Lie algebra with a skew-symmetric pair of indices corresponding to the Lorentz subalgebra and a single index to the de Sitter translation, we can write the group metric G_{ab} as

$$G_{(ij)(k\ell)} = 6(g_{jk}g_{i\ell} - g_{ik}g_{j\ell}),$$
$$G_{ij} = \pm 6\, g_{ij}, \qquad G_{i(k\ell)} = 0. \tag{2}$$

The ($\pm$) sign corresponds to the choice of (3, 2) or (4, 1) de Sitter group. Eq. (1) can then be rewritten as

$$R = \tilde{R} - \frac{3}{4}\ell^2 F^{ij}_{\mu\nu} F^{\mu\nu}_{ij} \pm \frac{3}{2}\ell^2 F^{i}_{\mu\nu} F^{\mu\nu}_{i} + \frac{5}{2\ell^2}, \tag{3}$$

where:

$$F^{ij}_{\mu\nu} \equiv \partial_\mu A^{ij}_\nu - \partial_\nu A^{ij}_\mu + g_{k\ell}(A^{ik}_\mu A^{\ell j}_\nu - A^{ik}_\nu A^{\ell j}_\mu) \pm (A^i_\mu A^j_\nu - A^i_\nu A^j_\mu),$$
$$F^{i}_{\mu\nu} \equiv \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + g_{k\ell}(A^{ik}_\mu A^\ell_\mu - A^{ik}_\nu A^\ell_\mu)$$

and the raising and lowering of Latin indices is now performed with help of the ordinary four-dimensional Minkowski metric. Non-abelian unified theories of Klein-Kaluza type differ in general from the abelian Einstein-Maxwell theory by the presence of the constant cosmological term in the scalar curvature. It means that they do not describe pure gravitation when the gauge field equals zero, unless, of course, ℓ is very large. Since we hope to describe also short-distance phenomena besides gravitation, it is natural to consider ℓ as a small fundamental length. Then the theory without the gauge fields could in the spirit of refs. (1, 2) describe the interior of hadrons. The case of pure gravity outside the particle must be described by a different condition than $F^a_{\mu\nu} = 0$. Such a condition is suggested by the content of ref. (6), where fibre bundles with de Sitter structure were considered as generalizations of the Lorentz structured bundles describing gravitation. Indeed, if we assume in the spirit of ref. (6) that the gauge potentials A^i_μ actually measure the macroscopic translations experienced in the space-time manifold, then the purely gravitational case should correspond to

$$A^i_\mu = k \cdot h^i_\mu, \tag{4}$$

where h^i_μ is the tetrad field of the base manifold.

In fact, with $k^2 = \sqrt{5}/6\ell^2$, $F^i_{\mu\nu} = 0$, and A^{ij}_μ given the meaning of the Lorentz components of the metric connection, eq. (3) reduces to

$$R = (1 \pm \frac{\sqrt{5}}{2})\, \tilde{R} - \frac{3}{4}\ell^2 \tilde{R}^{ij}_{\mu\nu} \tilde{R}^{\mu\nu}_{ij}, \tag{5}$$

where $\tilde{R}^{ij}_{\mu\nu}$ is the space-time curvature. The fact that the cosmological term can be made to disappear by a specific choice of k is not quite obvious, since there is not much freedom in the formula for R. In particular, the signatures of the two terms that cancel are fixed. Expression (5) differs from usual Einstein's Lagrangian by the presence of the quadratic term, but this term is multiplied by a small constant ℓ^2.

While the interior region of a hadron is characterized by the zero gauge field, and the region of pure gravitation far from the hadron by a gauge field correlated with the space-time metric, there should be also an intermediate region, where the gauge field and the metric are independent. The full system of equations should be then investigated, obtained by applying the variational principle to R of eq. (3) while considering $g_{\mu\nu}$, A_{μ}^{ij}, and A_{μ}^{i} as independent. The system is, of course, quite complicated and further work is required to analyze its properties. Also, condition (4) is not fully gauge invariant and thus it should not form part of the theory. It could, however, come in as a consequence of a particular solution of the equations of motion. For example, it should be found out whether an assumption of asymptotic flatness of space-time could not bring automatically a gradual employment of condition (4).

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