ISTITUTO NAZIONALE DI FISICA NUCLEARE

Sezione di Bari

INFN/AE-84/14
18 Dicembre 1984

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Servizio Documentazione
dei Laboratori Nazionali di Frascati

Istituto Nazionale di Fisica Nucleare
Sezione di Bari

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NEW SOLUTIONS FOR MATHIEU'S DIFFERENTIAL EQUATION

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ABSTRACT

New series solutions of the Mathieu's differential equation are presented, as generalization of Dougall's developments.

1.- INTRODUCTION

In a recent paper (Picca 1982) the author has shown that physical solutions of the angular equation of the Schrödinger equation for the dipolar potential can be obtained by means of the Mathieu's differential equation solutions.

The particular confluence condition to be satisfied by the physical solutions can be readily achieved by using the approach to Mathieu's equation long ago suggested by Dougall (Dougall 1923, 1926) which unfortunately has passed largely unnoticed.

The Dougall method relies upon particular functional relations between Bessel functions.

In attempting to solve the previously quoted angular equation, in the most general case, the author has found that Mathieu's equation solution can be represented through formal series of hypergeometric functions.

Since these results may be of interest for the scientific community, the relevant calculations are reported in the present paper.

2.- AN INTEGRABLE DIFFERENTIAL EQUATION

It is well known that equation:

$$\left[t^2 \frac{d^2}{dt^2} + (Pt^2 + Qt)\frac{d}{dt} + At^2 + Bt + C\right] y(t) = 0 \tag{1}$$

by factorization:

$$y(t) = t^{\varrho} e^{\lambda t} f(t) \tag{2}$$

can be transformed in a similar equation:

$$\left[t^2 \frac{d^2}{dt^2} + (P't^2 + Q't)\frac{d}{dt} + A't^2 + B't + C'\right] f(t) = 0 \tag{3}$$

where

$$\begin{aligned} P' &= P + 2\lambda \\ Q' &= Q + 2\varrho \\ A' &= A + P\lambda + \lambda^2 \\ B' &= B + P\varrho + Q\lambda + 2\lambda\varrho \\ C' &= C + Q\varrho + \varrho(\varrho-1)\,. \end{aligned} \tag{4}$$

Thanks to this special "covariance", equation (1) is integrable. In fact, if values of λ and ϱ are taken so that:

$$\begin{aligned} A' &= A + P\lambda + \lambda^2 = 0 \\ C' &= C + Q\varrho + \varrho(\varrho-1) = 0 \end{aligned} \tag{5}$$

that is:

$$\begin{aligned} \lambda &= \frac{1}{2}\left(-P \pm \sqrt{P^2 - 4A}\right) \\ \varrho &= \frac{1}{2}\left(1 - Q \pm \sqrt{(1-Q)^2 - 4C}\right) \end{aligned} \tag{6}$$

equation (1) becomes:

$$\left[t \frac{d^2}{dt^2} + (P't + Q')\frac{d}{dt} + B'\right] f(t) = 0 \tag{7}$$

hence, by changing the independent variable:

$$\xi = -P't = -(P + 2\lambda)t \tag{8}$$

the confluent hypergeometric equation:

$$\left[\xi \frac{d^2}{d\xi^2} + (Q' - \xi)\frac{d}{d\xi} - \frac{B'}{P'}\right] f(\xi) = 0 \tag{9}$$

can be obtained.

When $Q' = Q + 2\varrho$ is not an integer, independent integrals of equation (9) are:

$$f_1(\xi) = {}_1F_1\left(\frac{B+Q\lambda}{P+2\lambda}+\varrho,\ Q+2\varrho;\xi\right)$$

$$f_2(\xi) = \xi^{1-Q-2\varrho}\ {}_1F_1\left(\frac{B+Q\lambda}{P+2\lambda}+1-Q-\varrho,\ 2-Q-2\varrho;\ \xi\right) \tag{10}$$

otherwise the following solution, which may have the logarithmic term, must be considered:

$$U(\xi) = \frac{\Gamma(1-Q-2\varrho)}{\Gamma\left(\frac{B+Q\lambda}{P+2\lambda}+1-Q-\varrho\right)}\ {}_1F_1\left(\frac{B+Q\lambda}{P+2\lambda}+\varrho,\ Q+2\varrho;\xi\right) +$$

$$+\frac{\Gamma(Q+2\varrho-1)}{\Gamma\left(\frac{B+Q\lambda}{P+2\lambda}+\varrho\right)}\ \xi^{1-Q-2\varrho}\ {}_1F_1\left(\frac{B+Q\lambda}{P+2\lambda}+1-Q-\varrho,\ 2-Q-2\varrho;\ \xi\right) \tag{11}$$

Hence integrals of equation (1) are

$$Y_1(t) = e^{\lambda t}\, t^{\varrho}\ {}_1F_1\left(\frac{B+Q\lambda}{P+2\lambda}+\varrho,\ Q+2\varrho;\ -(P+2\lambda)t\right) \tag{12}$$

$$Y_2(t) = e^{\lambda t}\, t^{1-Q-\varrho}\ {}_1F_1\left(\frac{B+Q\lambda}{P+2\lambda}+1-Q-\varrho,\ 2-Q-2\varrho;\ -(P+2\lambda)t\right) \tag{13}$$

$$Y_o(t) = \frac{\Gamma(1-Q-2\varrho)}{\Gamma\left(\frac{B+Q\lambda}{P+2\lambda}+1-Q-\varrho\right)}\, y_1(t) + \frac{\Gamma(Q+2\varrho-1)}{\Gamma\left(\frac{B+Q\lambda}{P+2\lambda}+\varrho\right)}\, y_2(t) \tag{14}$$

respectively, where λ and ϱ are given by (6).

Moreover, by

$$\eta = \frac{1}{t} \tag{15}$$

eq.(1) can be transformed into

$$\left[\eta^2\frac{d^2}{d\eta^2} + (-P-(Q-2)\eta)\frac{d}{d\eta} + \frac{A}{\eta^2} + \frac{B}{\eta} + C\right] y(\eta) = 0 \tag{16}$$

whose solutions are still given by eqs.(12),(13),(14) after making the required variable change.

3.- SOME BASIC FUNCTIONAL RELATIONS

Now let us suppose

$$2\,\frac{B+Q\lambda}{P+2\lambda} = Q \tag{17}$$

that is

$$2B = PQ \tag{18}$$

thus eq.(1) can be written:

$$\left[t^2\frac{d^2}{dt^2} + (Pt^2+Qt)\frac{d}{dt} - \lambda(P+\lambda)t^2 + \frac{PQ}{2}t - \varrho(Q+\varrho-1)\right] y(t) = 0 \tag{19}$$

whose solutions are linear combination of eqs. (12), (13) and/or (14) that is of the following functions:

$$\mathscr{F}^{(1)}(P,Q,\lambda,\varrho;t) = \frac{e^{\lambda t}(-(P+2\lambda)t)^{\varrho}}{4^{\varrho}\Gamma(\frac{Q+1}{2}+\varrho)}\ {}_1F_1(\tfrac{Q}{2}+\varrho;\ Q+2\varrho;\ -(P+2\lambda)t) \tag{20}$$

$$\mathscr{F}^{(2)}(P,Q,\lambda,\varrho;t) = \frac{e^{\lambda t}(-(P+2\lambda)t)^{1-Q-\varrho}}{4^{\varrho}\Gamma(\frac{-Q+3}{2}-\varrho)}\ {}_1F_1(1-\tfrac{Q}{2}-\varrho,\ 2-Q-2\varrho;\ -(P+2\lambda)t) \tag{21}$$

$$\mathscr{F}^{(0)}(P,Q,\lambda,\varrho;t) = 4^{\varrho}\frac{\Gamma(\frac{Q+1}{2}+\varrho)\Gamma(1-Q-2\varrho)}{\Gamma(1-\frac{Q}{2}-\varrho)}\mathscr{F}^{(1)}(P,Q,\lambda,\varrho;t) + 4^{\varrho}\frac{\Gamma(\frac{3-Q}{2}-\varrho)\Gamma(Q+2\varrho-1)}{\Gamma(\frac{Q}{2}+\varrho)}\mathscr{F}^{(2)}(P,Q,\lambda,\varrho;t) \tag{22}$$

Well- known functional relations satisfied by contigous confluent hypergeometric functions (Luke 1975), by a little algebra, give:

$$t\left[\mathscr{F}^{(i)}(P,Q,\lambda,\varrho-1;t) - \mathscr{F}^{(i)}(P,Q,\lambda,\varrho+1;t)\right] = h^{(i)}\mathscr{F}^{(i)}(P,Q,\lambda,\varrho;t) \qquad i=0,1,2 \tag{23}$$

where

$$h^{(1,2)} = \frac{2(Q+2\varrho-1)}{P+2\lambda}, \qquad h^{(0)} = \frac{Q+2\varrho}{P+2\lambda} \tag{24}$$

Of course, any linear combination of eqs.(20),(21) and/or (22) satisfies the relation (23) with the corresponding linear combination of $h^{(i)}$.

Thus eq.(23) is the basic relation between contiguous solutions of eq.(19). From eq.(23) a functional relation between the derivative of the solutions and the contiguous solutions of eq.(19) can be easily deduced. In fact, from:

$$\left[t^2\frac{d^2}{dt^2} + (Pt^2+Qt)\frac{d}{dt} - \lambda(P+\lambda)t^2 + \frac{PQ}{2}t - \varrho(Q+\varrho-1)+Q+2(\varrho-1)\right]\mathscr{F}_{\varrho-1}(t) = 0$$

$$\left[t^2\frac{d^2}{dt^2} + (Pt^2+Qt)\frac{d}{dt} - \lambda(P+\lambda)t^2 + \frac{PQ}{2}t - \varrho(Q+\varrho-1)-Q-2\varrho\right]\mathscr{F}_{\varrho+1}(t) = 0$$

where $\mathscr{F}_{\varrho}(t)$ stands for a generic linear combination of $\mathscr{F}^{(i)}(P,Q,\lambda,\varrho;t)$, we get

$$\left[t^2\frac{d^2}{dt^2} + (Pt^2+Qt)\frac{d}{dt} - \lambda(P+\lambda)t^2 - \frac{PQ}{2}t - \varrho(Q+\varrho-1)\right](\mathscr{F}_{\varrho-1}(t) - \mathscr{F}_{\varrho+1}(t)) +$$

$$+ (Q+2(\varrho-1))\mathscr{F}_{\varrho-1}(t) + (Q+2\varrho)\mathscr{F}_{\varrho+1}(t) = 0$$

hence by relation (23)

$$(Q-1+\varrho)\mathscr{F}_{\varrho+1}(t) + \varrho\mathscr{F}_{\varrho-1}(t) = -h\left[\frac{d}{dt}\mathscr{F}_{\varrho}(t) + \frac{P}{2}\mathscr{F}_{\varrho}(t)\right] \tag{25}$$

where h is the linear combination of $h^{(i)}$ relative to $\mathscr{F}_{\varrho}(t)$.

4.- A FUNDAMENTAL DIFFERENTIAL IDENTITY

Let us consider the pair of differential equations:

$$\left[t^2\frac{d^2}{dt^2}+(P_1t^2+Q_1t)\frac{d}{dt}-\lambda_1(P_1+\lambda_1)t^2+\frac{P_1Q_1}{2}t-\mu(Q_1+\mu-1)\right]{}_1\mathcal{F}_\mu(t)=0$$

$$\left[t^2\frac{d^2}{dt^2}+(-P_2-(Q_2-2)t)\frac{d}{dt}-\lambda_2(P_2+\lambda_2)\frac{1}{t^2}+\frac{P_2Q_2}{2t}-\nu(Q_2+\nu-1)\right]{}_2\mathcal{F}_\nu\left(\frac{1}{t}\right)=0 \qquad (26)$$

where ${}_1\mathcal{F}_\varrho$ and ${}_2\mathcal{F}_\varrho$ stand simply for $\mathcal{F}(P_i,Q_i,\lambda_i,\varrho_i;t)$ that is to indicate a different parametric dependence.

By cross-multiplying the first equation of (25) by ${}_2\mathcal{F}_\nu(\frac{1}{t})$ and the second one by ${}_1\mathcal{F}_\mu(t)$ and then summing up the results, after some simple calculation we get:

$$\left[t^2\frac{d^2}{dt^2}+(P_1t^2+(Q_1-Q_2+1)t-P_2)\frac{d}{dt}-\lambda_1(P_1+\lambda_1)t^2-\frac{\lambda_2(P_2+\lambda_2)}{t^2}+\frac{P_1Q_1}{2}t+\right.$$

$$\left.+\frac{P_2Q_2}{2t}-\mu(Q_1+\mu-1)-\nu(Q_2+\nu-1)\right]{}_1\mathcal{F}_\mu(t)\,{}_2\mathcal{F}_\nu\left(\frac{1}{t}\right)-2t^2\frac{d\,{}_1\mathcal{F}_\mu(t)}{dt}\cdot \qquad (27)$$

$$\cdot\frac{d\,{}_2\mathcal{F}_\nu\left(\frac{1}{t}\right)}{dt}-(P_1t^2+(Q_1-1)t)\cdot{}_1\mathcal{F}_\mu(t)\frac{d\,{}_2\mathcal{F}_\nu\left(\frac{1}{t}\right)}{dt}+((Q_2-1)t+P_2)\cdot{}_2\mathcal{F}_\nu\left(\frac{1}{t}\right)\frac{d\,{}_1\mathcal{F}_\mu(t)}{dt}=0$$

Now from eq.(25):

$${}_1\dot{\mathcal{F}}_\mu(t)\,{}_2\dot{\mathcal{F}}_\nu\left(\frac{1}{t}\right)+\frac{P_1P_2}{4}\,{}_1\mathcal{F}_\mu(t)\,{}_2\mathcal{F}_\nu\left(\frac{1}{t}\right)+\frac{P_1}{2}\,{}_1\mathcal{F}_\mu(t)\,{}_2\dot{\mathcal{F}}_\nu\left(\frac{1}{t}\right)+\frac{P_2}{2}\,{}_1\dot{\mathcal{F}}_\mu(t)\,{}_2\mathcal{F}_\nu\left(\frac{1}{t}\right)=$$

$$=h_1h_2\left[\mu\nu\,{}_1\mathcal{F}_{\mu-1}(t)\,{}_2\mathcal{F}_{\nu-1}\left(\frac{1}{t}\right)+(Q_1+\mu-1)(Q_2+\nu-1)\,{}_1\mathcal{F}_{\mu+1}(t)\,{}_2\mathcal{F}_{\nu+1}\left(\frac{1}{t}\right)+\right. \qquad (28)$$

$$\left.+\mu(Q_2+\nu-1)\,{}_1\mathcal{F}_{\mu-1}(t)\,{}_2\mathcal{F}_{\nu+1}\left(\frac{1}{t}\right)+\nu(Q_1+\mu-1)\,{}_1\mathcal{F}_{\mu+1}(t)\,{}_2\mathcal{F}_{\nu-1}\left(\frac{1}{t}\right)\right]$$

where $\dot{\mathcal{F}}$ stands for the derivate of $\mathcal{F}$ with respect to the actual argument shown in brackets.

From eq.(23) we get

$${}_1\mathcal{F}_\mu(t)\,{}_2\mathcal{F}_\nu\left(\frac{1}{t}\right)=h_1h_2\left[{}_1\mathcal{F}_{\mu-1}(t)\,{}_2\mathcal{F}_{\nu-1}\left(\frac{1}{t}\right)+{}_1\mathcal{F}_{\mu+1}(t)\,{}_2\mathcal{F}_{\nu+1}\left(\frac{1}{t}\right)-\right. \qquad (29)$$

$$\left.-{}_1\mathcal{F}_{\mu-1}(t)\,{}_2\mathcal{F}_{\nu+1}\left(\frac{1}{t}\right)-{}_1\mathcal{F}_{\mu+1}(t)\,{}_2\mathcal{F}_{\nu-1}\left(\frac{1}{t}\right)\right]$$

In the same fashion combining equation of kind (23) with equation of kind (25) by multiplication we obtain:

$$t\left[{}_1\dot{\mathcal{F}}_\mu(t)\,{}_2\mathcal{F}_\nu\left(\frac{1}{t}\right)+\frac{P_1}{2}\,{}_1\mathcal{F}_\mu(t)\,{}_2\mathcal{F}_\nu\left(\frac{1}{t}\right)\right]=h_1h_2\left[\mu\,{}_1\mathcal{F}_{\mu-1}(t)\,{}_2\mathcal{F}_{\nu-1}\left(\frac{1}{t}\right)-\right. \qquad (30)$$

$$\left.-(Q_1+\mu-1)\,{}_1\mathcal{F}_{\mu+1}(t)\,{}_2\mathcal{F}_{\nu+1}\left(\frac{1}{t}\right)-\mu\,{}_1\mathcal{F}_{\mu-1}(t)\,{}_2\mathcal{F}_{\nu+1}\left(\frac{1}{t}\right)+(Q_1+\mu-1)\,{}_1\mathcal{F}_{\mu+1}(t)\,{}_2\mathcal{F}_{\nu-1}\left(\frac{1}{t}\right)\right]$$

and

$$\frac{1}{t}\left[{}_1\mathcal{F}_\mu(t)\ {}_2\dot{\mathcal{F}}_\nu(\tfrac{1}{t}) + \frac{P_2}{2}\ {}_1\mathcal{F}_\mu(t)\ {}_2\mathcal{F}_\nu(\tfrac{1}{t})\right] = h_1 h_2 \left[\nu\, {}_1\mathcal{F}_{\mu-1}(t)\ {}_2\mathcal{F}_{\nu-1}(\tfrac{1}{t}) - \right. \tag{31}$$

$$\left. - (Q_2+\nu-1)\ {}_1\mathcal{F}_{\mu-1}(t)\ {}_2\mathcal{F}_{\nu-1}(\tfrac{1}{t}) + (Q_2+\nu-1)\ {}_1\mathcal{F}_{\mu-1}(t)\ {}_2\mathcal{F}_{\nu+1}(\tfrac{1}{t}) - \nu\, {}_1\mathcal{F}_{\mu+1}(t)\ {}_2\mathcal{F}_{\nu-1}(\tfrac{1}{t})\right]$$

Finally by summing eq.(28) multiplied by 2, eq.(29) by

$$2\mu\nu + \mu(Q_2-1) + \nu(Q_1-1) + (Q_1-1)(Q_2-1)\,,$$

eq.(30) by Q_2-1 and eq.(31) by Q_1-1 respectively, we obtain:

$$2\ {}_1\dot{\mathcal{F}}_\mu(t)\ {}_2\dot{\mathcal{F}}_\nu(\tfrac{1}{t}) + (P_1 + \frac{Q_1-1}{t})\ {}_1\mathcal{F}_\mu(t)\ {}_2\dot{\mathcal{F}}_\nu(\tfrac{1}{t}) + (P_2+(Q_2-1)t)\ {}_1\dot{\mathcal{F}}_\mu(t)\ {}_2\mathcal{F}_\nu(\tfrac{1}{t}) +$$

$$+ (\frac{P_1(Q_2-1)}{2}\, t + \frac{P_2(Q_1-1)}{2}\, \frac{1}{t} + \frac{P_1P_2}{2} + \mu\nu + (\mu+Q_1-1)(\nu+Q_2-1)\ {}_1\mathcal{F}_\mu(t)\ {}_2\mathcal{F}_\nu(\tfrac{1}{t}) =$$

$$= h_1 h_2 \left[{}_1\mathcal{F}_{\mu-1}(t)\ {}_2\mathcal{F}_{\nu-1}(\tfrac{1}{t}) + {}_1\mathcal{F}_{\mu+1}(t)\ {}_2\mathcal{F}_{\nu+1}(\tfrac{1}{t})\right] \tag{32}$$

By identity

$${}_2\dot{\mathcal{F}}_\nu(\tfrac{1}{t}) = -t^2\, \frac{d}{dt}\, {}_2\mathcal{F}_\nu(\tfrac{1}{t}) \tag{33}$$

eq.(27) can be written as:

$$\left[t^2 \frac{d^2}{dt^2} + (P_1t^2+(Q_1-Q_2+1)t - P_2)\frac{d}{dt} - \lambda_1(P_1+\lambda_1)t^2 - \lambda_2(P_2+\lambda_2)\frac{1}{t^2} + \right.$$

$$\left. + \frac{P_1(Q_1-Q_2+1)}{2}\, t + \frac{P_2(Q_2-Q_1+1)}{2}\, \frac{1}{t} - \frac{P_1P_2}{2} - (\mu+\nu+Q_1-1)(\mu+\nu+Q_2-1)\right] \cdot \tag{34}$$

$$\cdot\ {}_1\mathcal{F}_\mu(t)\ {}_2\mathcal{F}_\nu(\tfrac{1}{t}) = h_1 h_2 \left[{}_1\mathcal{F}_{\mu-1}(t)\ {}_2\mathcal{F}_{\nu-1}(\tfrac{1}{t}) + {}_1\mathcal{F}_{\mu+1}(t)\ {}_2\mathcal{F}_{\nu+1}(\tfrac{1}{t})\right]$$

which is the very starting point for the new functional series.

5.- THE PRINCIPLES OF THE METHOD

Let us consider the differential equation:

$$\left[t^2 \frac{d^2}{dt^2} + (p_1t^2+qt+p_2)\frac{d}{dt} + a_1t^2 + \frac{a_2}{t^2} + b_1t + \frac{b_2}{t} + c\right] y = 0 \tag{35}$$

where

$$2b_1 = p_1 q$$

$$2b_2 = p_2(q-2) \tag{36}$$

If we take:

$$P_1 = p_1\,, \quad Q_1 = \hat{q}\,, \quad -\lambda_1(P_1+\lambda_1) = a_1\,, \quad \mu = n + \varrho_1 \tag{37}$$

$$P_2 = -p_2\,, \quad Q_2 = \hat{q} - q + 1\,, \quad -\lambda_2(P_2+\lambda_2) = a_2\,, \quad \nu = n + \varrho_2 \tag{38}$$

where $\hat{q}$, ϱ_1, ϱ_2 are arbitrary (real or complex) numbers and n a positive or negative integer, it is easy to verify that eq. (35) is formally satisfied by:

$$y(t) = \sum_{-\infty}^{+\infty} D_n^{(\varrho_1,\varrho_2)} \mathscr{F}_{1\,n+\varrho_1}(t)\; \mathscr{F}_{2\,n+\varrho_2}\left(\frac{1}{t}\right) \tag{39}$$

when the recurrent relation

$$\left[c - \frac{p_1 p_2}{2} + (2n+\varrho_1+\varrho_2+\hat{q}-1)(2n+\varrho_1+\varrho_2+\hat{q}-q)\right] D_n^{(\varrho_1,\varrho_2)} = h_1 h_2 \left[D_{n-1}^{(\varrho_1,\varrho_2)} + D_{n+1}^{(\varrho_1,\varrho_2)}\right] \tag{40}$$

holds.

Formula (40) is peculiar to the trigonometrical developments of Mathieu's equation solutions. In this sense our functions (39) can be considered generalized Dougall's developments.

Of course, so that eq.(39) can make sense, the convergence of the series must be proven. A well-known result (Luke 1975) asserts that

$${}_1F_1(a,c;\xi) \longrightarrow e^{\frac{1}{2}\xi}$$

holds uniformuly in ξ when c is large and $\frac{c}{2}-a$ is bounded.

Hence in our case we have

$$\frac{\mathscr{F}_{1\,n+\varrho_1+1}(t)\; \mathscr{F}_{2\,n+\varrho_2+1}\left(\frac{1}{t}\right)}{\mathscr{F}_{1\,n+\varrho_1}(t)\; \mathscr{F}_{2\,n+\varrho_2}\left(\frac{1}{t}\right)} \longrightarrow 1$$

thus, for $n \longrightarrow +\infty$

$$\left| \frac{D_{n+1}^{(\varrho_1,\varrho_2)}\, \mathscr{F}_{1\,n+\varrho_1+1}(t)\; \mathscr{F}_{2\,n+\varrho_2+1}\left(\frac{1}{t}\right)}{D_n^{(\varrho_1,\varrho_2)}\, \mathscr{F}_{1\,n+\varrho_1}(t)\; \mathscr{F}_{2\,n+\varrho_2}\left(\frac{1}{t}\right)} \right| \longrightarrow \left| \frac{D_{n+1}^{(\varrho_1,\varrho_2)}}{D_n^{(\varrho_1,\varrho_2)}} \right|$$

Now from eq.(40) we get :

$$\frac{D_{n+1}^{(\varrho_1,\varrho_2)}}{D_n^{(\varrho_1,\varrho_2)}} = \cfrac{1}{\cfrac{c-\frac{p_1p_2}{2}+(2n+\varrho_1+\varrho_2+\hat{q}+1)(2n+\varrho_1+\varrho_2+\hat{q}-q+2)}{h_1h_2} - \cfrac{D_{n+2}^{(\varrho_1,\varrho_2)}}{D_{n+1}^{(\varrho_1,\varrho_2)}}} =$$

$$= -\sum_{k=0}^{\infty} \frac{1|}{\left| \frac{c-\frac{p_1p_2}{2}+(2n+\varrho_1+\varrho_2+\hat{q}+1)(2n+\varrho_1+\varrho_2+\hat{q}-q+2)}{h_1h_2}\right.} \tag{41}$$

It follows that series (41) will converge (for positive integers) by virtue of the Pringsheim's theorem (Pringsheim 1898, Perron 1929) when:

$$\forall k \in \mathbb{N}: \left| \frac{c-\frac{p_1p_2}{2}+\left[2(n+k)+\varrho_1+\varrho_2+\hat{q}+1\right]\left[2(n+k)+\varrho_1+\varrho_2+\hat{q}-q+2\right]}{h_1h_2} \right| > 2 \tag{42}$$

but this condition in always satisfied (for n sufficiently high).

The series (39) for negative n is likewise convergent, since by changing n into -n, (40) can be written as:

$$\left[c-\frac{p_1p_2}{2}+(2n-\varrho_1-\varrho_2-\hat{q}+1)(2n-\varrho_1-\varrho_2-\hat{q}+q)\right]D_{-n}^{(\varrho_1,\varrho_2)} =$$
$$= h_1h_2\left[D_{-n+1}^{(\varrho_1,\varrho_2)} + D_{-n-1}^{(\varrho_1,\varrho_2)}\right] \tag{43}$$

which is identical to (40) for positive n except for an inessential sign change in the parameters; thus the same conclusions are true.

6.- SOLUTIONS OF MATHIEU'S EQUATION

The close link between eq.(35) and Mathieu's equation must now be shown.

Let us observe that by factorization

$$y(t) = e^{\lambda_1 t+\frac{\lambda_2}{t}}\, t^{\sigma} f(t) \tag{44}$$

eq.(35) transforms into:

$$\left[t^2\frac{d^2}{dt^2} + (p'_1t^2+q't+p'_2)\frac{d}{dt} + a'_1\, t^2 + \frac{a'_2}{t^2} + b'_1t + \frac{b'_2}{t} + c'\right] f(t) = 0 \tag{45}$$

where

$$p'_1 = p_1+2\lambda_1\,,\quad q' = q+2\sigma\,,\quad p'_2 = p_2-2\lambda_2,\quad a'_1 = a_1+p_1\lambda_1+\lambda_1^2\,,$$

$$a'_2 = a_2-p_2\lambda_2+\lambda_2^2\,,\quad b'_1 = b_1+q\lambda_1+p_1\sigma+2\lambda_1\sigma,\quad b'_2 = b_2+(2-q)\lambda_2+p_2\sigma-2\lambda_2\sigma,$$

$$c' = c-p_1\lambda_2+p_2\lambda_1+q\sigma-2\lambda_1\lambda_2+\sigma(\sigma-1) \tag{46}$$

As special property of (44), for every λ_1, λ_2 and σ, we have

$$2b_1 = p_1 q \;\Rightarrow\; 2b'_1 = p'_1 q'$$
$$2b_2 = p_2(2-q) \;\Rightarrow\; 2b'_2 = p'_2(2-q') \qquad (47)$$

Now let us choose

$$\lambda_1 = -\frac{p_1}{2}, \qquad \lambda_2 = \frac{p_2}{2}, \qquad \sigma = \frac{1-q}{2} \qquad (48)$$

so that eq.(45) becomes:

$$\left[t^2 \frac{d}{dt^2} + t\frac{d}{dt} + (a_1 - \frac{p_1^2}{4})t^2 + (a_2 - \frac{p_2^2}{4})\frac{1}{t^2} + c - (\frac{1-q}{2})^2 - \frac{p_1 p_2}{2}\right] f(t) = 0 \qquad (49)$$

hence with

$$t = \sqrt{\frac{a_2 - \frac{p_2^2}{4}}{a_1 - \frac{p_1^2}{4}}}\;\tau \qquad (50)$$

we get Mathieu's equation in its algebraic form:

$$\left[\tau^2 \frac{d^2}{d\tau^2} + \tau\frac{d}{d\tau} + p(\tau^2 + \frac{1}{\tau^2}) + c - (\frac{1-q}{2})^2 - \frac{p_1 p_2}{2}\right] f(\tau) = 0 \qquad (51)$$

where

$$p = \sqrt{(a_1 - \frac{p_1^2}{4})(a_2 - \frac{p_2^2}{4})}$$

Therefore series (39) represent solutions of Mathieu's equation.

By choosing opportunely $\hat{q}$, ϱ_1 and ϱ_2 we can construct pairs of linearly independent integrals satisfying particular confluence conditions.

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