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## A MULTIPLICATIVE BACKGROUND FIELD METHOD

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## Abstract

We propose that instead of the usual additive decomposition of the metric into a classical background and a quantum fluctuation, one should use a multiplicative parametrization. This, together with a discussion of the dimensions of the quantities involved, leads naturally to a non-standard approach to quantum gravity, which appears to be effectively a bimetric theory. We discuss some results of this approach, applied to the most general lagrangian with terms up to second order in curvature and torsion.

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## 1. Introduction

In General Relativity the connection is assumed to be metric and torsionfree, *i.e.*, in coordinate bases,  $\nabla_\mu g_{\rho\sigma} = 0$  and  $\Gamma_\lambda^\mu{}_\nu = \Gamma_\nu^\mu{}_\lambda$ . This is perfectly satisfactory from the observational point of view and simplifies the theory considerably. In principle, however, the motivation underlying these assumptions is not very strong. This is why generalizations of General Relativity involving torsion and/or nonmetricity have been studied for a long time [1]. In the last couple of decades this idea has become even more attractive since it makes gravitation more similar to the other interactions, which at a fundamental level all appear to be mediated by gauge fields [2,3].

If one restricts one's attention to metric connections, the most general action which is at most quadratic in curvature and torsion is

$$S(g, \Gamma) = \int d^4x \sqrt{-g} \left[ \lambda_0 + g_0 R + a_1 \Theta_{\mu\nu\rho} \Theta^{\mu\nu\rho} + a_2 \Theta_{\mu\nu\rho} \Theta^{\mu\rho\nu} + a_3 \Theta^\nu{}_{\nu\rho} \Theta_\mu{}^{\mu\rho} \right. \\ \left. + g_1 R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} + g_2 R_{\mu\nu\rho\sigma} R^{\mu\rho\nu\sigma} + g_3 R_{\mu\nu\rho\sigma} R^{\rho\sigma\mu\nu} + g_4 R_{\mu\nu} R^{\mu\nu} + g_5 R_{\mu\nu} R^{\nu\mu} + g_6 R^2 \right] \quad (1.1)$$

where  $R_{\mu\nu\rho\sigma} = \partial_\mu \Gamma_\nu^\rho{}_\sigma - \partial_\nu \Gamma_\mu^\rho{}_\sigma + \Gamma_\mu^\rho{}_\tau \Gamma_\nu^\tau{}_\sigma - \Gamma_\nu^\rho{}_\tau \Gamma_\mu^\tau{}_\sigma$ ,  $R_{\mu\nu} = R_{\rho\mu}{}^\rho{}_\nu$ ,  $R = g^{\mu\nu} R_{\mu\nu}$  and  $\Theta_\mu{}^\rho{}_\nu = \Gamma_\mu^\rho{}_\nu - \Gamma_\nu^\rho{}_\mu$ . There is a vast literature illustrating the analogy between this theory and gauge theories. It is less appreciated that gravity is, in the terminology of elementary particle physicists, a "spontaneously broken" gauge theory.

To see this, let us set the cosmological constant  $\lambda_0$  to zero for a moment; then flat space is a solution of the field equations that come from (1.1). In a certain gauge, flat space corresponds to  $\Gamma_\lambda^\mu{}_\nu = 0$  and  $g_{\mu\nu} = v^2 \eta_{\mu\nu}$ , where  $v$  is a constant. (Usually one sets  $v$  to one by rescaling the coordinates. For reasons that will become clear in the following it is better not to do so here). Flat space is the exact analog of the ground state of the Higgs model in the broken phase, where the gauge field vanishes and the Higgs field is constant:  $\phi = \phi_0 \equiv (0, \dots, 0, v)$ . When the Higgs model action is expanded to second order around its ground state, the kinetic term of the scalar fields gives a mass term for (certain components of) the gauge field. Similarly, when the action (1.1) is expanded to second order around flat space, the Einstein and torsion squared terms give a mass term for  $\Gamma$ . For generic values of the parameters, all components of the connection become massive. This is a gravitational analogue of the Higgs phenomenon [3,4]. This point is even clearer if instead of working with coordinate bases or orthonormal bases one allows for completely general linear bases. The theory then has a manifest local  $GL(4)$  invariance, which is broken by the choice either of a soldering form, or a fiber metric. In the  $GL(4)$  formalism the torsion is the exterior covariant derivative of the soldering form, and therefore the three terms with coefficients  $a_1$ ,  $a_2$  and  $a_3$  can be regarded as the kinetic terms of the soldering form (see eq.(3.1) below). Upon choosing a gauge in which the soldering form is equal to  $\delta_\mu^a$ , (this being the analog of the unitary gauge in the Higgs model), the torsion becomes just the antisymmetric part of the connection. In this way the kinetic term of the soldering form becomes a mass term for the connection.

The analogy can be generalized to include nonmetricity, and arbitrary curved backgrounds [3,4]. In the case of a curved background, it is the deviation of  $\Gamma$  from the Christoffel symbols of the background metric that becomes massive, with a mass of the

order  $v \approx m_P$  (Planck's mass). This suggests that below Planck's energy one can treat the connection as a composite field and describe gravity by the metric alone, as in standard textbook treatments. Above Planck's energy however "new physics" would appear, in the guise of new degrees of freedom related to the connection. If one allows the "internal" indices  $a, b \dots$  to run over a larger set  $0, 1, 2, 3 \dots, N - 1$ , one can describe in this way a truly unified theory of gravity and an  $O(N - 4)$  gauge theory. In this way one gets a picture of the role of Planck's energy for gravity which exactly parallels the role of the electroweak scale for the standard model.

This attractive picture has its own problems. For generic values of the parameters, the action (1.1) describes the propagation of several particles of spin 0, 1 and 2, some of which will have propagators with negative residue (ghosts) or imaginary mass (tachyons). Therefore, at the perturbative level, unitarity seems to be violated. It is true that these pathologies would only occur near Planck's scale, but nevertheless this problem has discouraged most researchers from pursuing this line of thought any further. Another major puzzle is the problem of the cosmological constant: naive dimensional arguments would suggest that the natural value of this parameter be of the order of one in Planck's units, whereas observation bounds it very close to zero. In view of these considerations it seems to be important to explore alternative approaches to the quantization of the theory (1.1). In this paper we will describe one such attempt.

In the traditional approach to quantum gravity the metric  $g_{\mu\nu}$  is usually assumed to be dimensionless. It is then decomposed as

$$g_{\mu\nu} = g_{(\text{cl})\mu\nu} + \ell h_{\mu\nu} , \quad (1.2)$$

where  $g_{(\text{cl})}$  is a classical background, to be identified with the v.e.v. of  $g$ , and  $h$  is a quantum field with canonical dimension of mass. The constant  $\ell$  is usually identified with Planck's length  $\sqrt{G}$ . The classical background is used to raise and lower indices, and therefore defines the geometry:

$$ds^2 = g_{(\text{cl})\mu\nu} dx^\mu dx^\nu . \quad (1.3)$$

There are two ways in which the decomposition (1.2) is unnatural: it does not respect the group structure of the theory, and it requires the introduction of the dimensionful constant  $\ell$ . In the next two sections we will review these points and then suggest an alternative procedure.

## 2. A Multiplicative Parametrization of the Metric

Let us consider again a "spontaneously broken" gauge theory; for definiteness take an  $SO(n)$  gauge theory with scalars in the fundamental representation. Assume that the gauge invariant potential has its minima on the orbit of the field  $\phi_0$ . In order to deduce the masses of the physical particles described by the theory, it is necessary to linearize the Lagrangian around the vacuum state. Therefore, one expands linearly the field around its v.e.v. as  $\phi = \phi_0 + \varphi$ . For many purposes, however, it is better to consider a different parametrization of the scalar fields, one that respects more the group action on the scalar fields. Any field configuration can be parametrized as

$$\phi(x) = g^{-1}(x)\eta(x) , \quad (2.1)$$

where  $\eta(x) = (0, \dots, 0, v + \sigma(x))$ . This decomposition is not unique: the group-valued field  $g(x)$  is defined up to left multiplication by a field  $h(x)$  with values in  $SO(n-1)$ . The independent degrees of freedom are the radial coordinate  $\sigma$  (the physical Higgs field) and a  $SO(n)/SO(n-1)$ -valued field. A convenient parametrization of the latter can be given in the following way. In the fundamental representation the generators  $L_{ij} = -L_{ji}$  of the Lie algebra of  $SO(n)$  are  $(L_{ij})_{kl} = \delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}$ , all indices running from 1 to  $n$ . We denote  $K_i = L_{in}$  the generators which are not in the subalgebra of  $SO(n-1)$ . Then, we can choose a gauge such that  $g(x) = \exp(-\xi_i(x)K_i/v)$ . The Higgs fields  $\sigma$  and the "Goldstone fields"  $\xi_i$  give a unique parametrization of any field configuration  $\phi$ . This decomposition shows immediately that the fields  $\xi_i$  are gauge degrees of freedom, while  $\sigma$  is a physical variable.

There exists an analogous decomposition also for the metric. The analogs of the groups  $SO(n)$  and  $SO(n-1)$  in the case of gravity are the general linear group  $GL(4)$  and the Lorentz group  $SO(1,3)$  [5]. The space of nondegenerate symmetric covariant two-tensors of signature  $-, +, +, +$  is an orbit of the group  $GL(4)$ . This is simply a restatement of Sylvester's law. Thus one can decompose any metric as

$$g_{\mu\nu}(x) = \theta^a{}_{\mu}(x)\theta^b{}_{\nu}(x)\gamma_{ab}(x), \quad (2.2)$$

where  $\gamma_{ab}$  is any fixed metric and  $\theta^a{}_{\mu}(x)$  belongs to the group  $GL(4)$ . For reasons that will become clear later, we are using here different types of indices on the metrics  $g$  and  $\gamma$ , but they should all be interpreted as coordinate indices in the tangent bundle. As in the case of (2.1), also (2.2) is an overparametrization: the ten components of  $g$  are parametrized by the sixteen components of  $\theta$ . For example if  $\gamma_{ab} = \eta_{ab}$ ,  $\theta$  is defined modulo left multiplication by some element of the Lorentz group. One can choose the Lorentz gauge such that  $\theta = \exp(\xi)$ , where  $\xi$  is a symmetric matrix. This matrix is the analog of the Goldstone fields  $\xi_i$ . Note that there is no analog of the physical Higgs field here. If  $\gamma_{\mu\nu} \neq \eta_{\mu\nu}$  the residual invariance is a subgroup of  $GL(4)$  conjugate to the Lorentz group. Namely, if  $\tau^a{}_b$  is a matrix such that  $\gamma_{bc} = \tau^d{}_b\tau^f{}_c\eta_{df}$  (a vierbein for  $\gamma$ ), then  $\theta$  is defined up to left multiplication by a matrix of the form  $\tau^{-1}\Lambda\tau$ , where  $\Lambda$  is a Lorentz transformation. Further motivation for the use of the multiplicative decomposition (2.2) comes from the canonical approach [6].

Consider now any gravitational action  $S(g, \Gamma)$ . If we use in this action the parametrization (2.2) and the analogous parametrization of the connection:

$$\Gamma_{\lambda}{}^{\mu}{}_{\nu} = \theta_a{}^{\mu}A_{\lambda}{}^a{}_b\theta^b{}_{\nu} + \theta_a{}^{\mu}\partial_{\lambda}\theta^a{}_{\nu}, \quad (2.3)$$

(where  $\theta_a{}^{\mu}$  is the inverse matrix of  $\theta^a{}_{\mu}$ ) then the action can be written  $S(g, \Gamma) = S'(\theta, \gamma, A)$ . The action  $S'$  is invariant under diffeomorphisms, as well as under local  $GL(4)$  transformations. If  $\epsilon^a{}_b$  is an arbitrary matrix (an element of the Lie algebra of  $GL(4)$ ), the infinitesimal transformation of the fields is given by:

$$\delta_{\epsilon}\theta^a{}_{\mu} = -\epsilon^a{}_b\theta^b{}_{\mu}, \quad (2.4a)$$

$$\delta_{\epsilon}\gamma_{ab} = \epsilon^c{}_a\gamma_{cb} + \epsilon^c{}_b\gamma_{ac}, \quad (2.4b)$$

$$\delta_{\epsilon}A_{\lambda}{}^a{}_b = \partial_{\lambda}\epsilon^a{}_b + A_{\lambda}{}^a{}_c\epsilon^c{}_b - \epsilon^a{}_cA_{\lambda}{}^c{}_b. \quad (2.4c)$$

This invariance is due to the fact that  $S'$  depends on  $\theta$ ,  $\gamma$  and  $A$  only through the combinations  $g$  and  $\Gamma$ , and one can easily check from the parametrizations (2.2) and (2.3) that these quantities are invariant under the transformations (2.4). Note that the  $GL(4)$  transformations only affect the fields  $\theta$ ,  $A$  and  $\gamma$ , and not other spacetime fields. So for example if  $v^\mu$  and  $w^\mu$  are two vectors,  $\gamma_{\mu\nu}v^\mu w^\nu$  is not invariant under  $GL(4)$ , but is invariant under the (Lorentz) subgroup of  $GL(4)$  that leaves  $\gamma$  invariant.

Similarly if we write an infinitesimal diffeomorphism  $x'^\mu = x^\mu - v^\mu$  for some vectorfield  $v$ , then the infinitesimal transformations of the fields are

$$\delta_v \theta^a{}_\mu = v^\lambda \partial_\lambda \theta^a{}_\mu - \theta^c{}_\mu \partial_c v^a + \theta^a{}_\lambda \partial_\mu v^\lambda, \quad (2.5a)$$

$$\delta_v \gamma_{ab} = v^\lambda \partial_\lambda \gamma_{ab} + \gamma_{\mu b} \partial_a v^\mu + \gamma_{a\mu} \partial_b v^\mu, \quad (2.5b)$$

$$\delta_v A_\lambda{}^a{}_b = v^\tau \partial_\tau A_\lambda{}^a{}_b + A_\tau{}^a{}_b \partial_\lambda v^\tau - A_\lambda{}^c{}_b \partial_c v^a + A_\lambda{}^a{}_c \partial_b v^c + \partial_\lambda \partial_b v^a. \quad (2.5c)$$

Except for the last term in (2.5c), these variations are just the Lie derivatives of the fields, regarded as world tensors on all indices.

The parametrizations (2.2) and (2.3) have led us to a  $GL(4)$ -invariant action  $S'$ . Mathematically, the equations (2.2-5) are identical to equations that were given in the  $GL(4)$ -invariant formulation of gravity [3,4]. The physical interpretation, however, is now different: we do not treat  $\theta$ ,  $\gamma$  and  $A$  as independent variables. Instead,  $\gamma$  is to be treated as a fixed background. For this reason we shall henceforth write the action as  $S'(\theta, A; \gamma)$  to emphasize that  $\theta$  and  $A$  are the dynamical variables, whereas  $\gamma$  is a fixed background.

Since  $\gamma$  is now an "absolute element", the gauge group has to be restricted to those transformations that leave it invariant, *i.e.*  $(\delta_v + \delta_\epsilon)\gamma_{ab} = 0$ . Combining (2.4b) and (2.5b) we get a condition on the infinitesimal parameters  $v$  and  $\epsilon$ . For any  $v$ , the most general solution is given by

$$\epsilon^a{}_b = -\partial_b v^a - \tau^{-1a}{}_c v^\lambda \partial_\lambda \tau^c{}_b + \tau^{-1a}{}_c \alpha^c{}_d \tau^d{}_b, \quad (2.6)$$

where  $\tau^c{}_b$  satisfies  $\gamma_{bc} = \tau^d{}_b \tau^f{}_c \eta_{df}$  and  $\alpha$  satisfies  $\eta_{cf} \alpha^f{}_d + \eta_{df} \alpha^f{}_c = 0$  (it represents therefore an infinitesimal Lorentz transformation). At the infinitesimal level, the gauge group is parametrized by vectorfields  $v$  and Lorentz-algebra-valued fields  $\alpha$ . The transformations of the fields under these local Lorentz transformations and modified diffeomorphisms are

$$\delta_\alpha \theta^a{}_\mu = -\bar{\alpha}^a{}_b \theta^b{}_\mu, \quad (2.7a)$$

$$\delta_\alpha A_\lambda{}^a{}_b = \partial_\lambda \bar{\alpha}^a{}_b + A_\lambda{}^a{}_c \bar{\alpha}^c{}_b - \bar{\alpha}^a{}_c A_\lambda{}^c{}_b, \quad (2.7b)$$

where  $\bar{\alpha}^a{}_b = \tau^{-1a}{}_d \alpha^d{}_e \tau^e{}_b$ , and

$$\delta_v \theta^a{}_\mu = v^\lambda \partial_\lambda \theta^a{}_\mu + \theta^a{}_\lambda \partial_\mu v^\lambda - \bar{\epsilon}^a{}_b \theta^b{}_\mu, \quad (2.8a)$$

$$\delta_v A_\lambda{}^a{}_b = v^\tau \partial_\tau A_\lambda{}^a{}_b + A_\tau{}^a{}_b \partial_\lambda v^\tau + \partial_\lambda \bar{\epsilon}^a{}_b + A_\lambda{}^a{}_c \bar{\epsilon}^c{}_b - \bar{\epsilon}^a{}_c A_\lambda{}^c{}_b, \quad (2.8b)$$

where  $\bar{\epsilon}^a{}_b = -\tau^{-1a}{}_c v^\lambda \partial_\lambda \tau^c{}_b$ .

The formalism simplifies somewhat if the background metric is flat:  $\gamma_{ab} = \eta_{ab}$ . In this case, and only in this case, one sees from (2.2) that  $\theta$  can be interpreted as the vierbein of

$g$ ; we can take  $\tau^a_b = \delta^a_b$ , so (2.6) reduces to  $\epsilon^a_b = -\partial_b v^a + \alpha^a_b$ . In (2.7) one can replace  $\bar{\alpha}$  by  $\alpha$  and in (2.8)  $\bar{\epsilon} = 0$ . The resulting gauge transformation properties of the fields are exactly the familiar ones of the vierbein formalism. Note that the latin indices are inert under the modified diffeomorphisms. It is important to stress again that all indices can be interpreted as referring to coordinate bases in the tangent bundle. The distinction between indices  $a, b, \dots$  and  $\mu, \nu, \dots$  is kept here only because it allows one to remember better how the gauge group acts: invariant quantities can be formed by contracting tensor indices  $a, b, \dots$  with  $\gamma_{ab}$  and  $\mu, \nu, \dots$  with  $g_{\mu\nu}$ .

### 3. Dimensions

In quantum gravity there is always a clash between the geometrical requirement that the metric be dimensionless, as required by the formula for the line element (1.3), and the dynamical requirement that the metric, or vierbein, have the canonical dimension of bosonic quantum field. To see this in a concrete example, consider the action (1.1) and apply to it the multiplicative decomposition (2.2-3), choosing  $\gamma_{\mu\nu} = \delta_{\mu\nu}$  (from here on we work in the Euclidean regime). We get an action  $S'(\theta, A; \gamma)$  which is identical to the action (1.1) written in vierbein formalism. We have an independent vierbein  $\theta^a_\mu$  and  $SO(4)$  gauge field  $A_\mu^a_b$ . The curvature and torsion fields are:

$$F_{\mu\nu}^a_b = \theta^a_\rho \theta_b^\sigma R_{\mu\nu}^{\rho\sigma} = \partial_\mu A_\nu^a_b - \partial_\nu A_\mu^a_b + A_\mu^a_c A_\nu^c_b - A_\nu^a_c A_\mu^c_b, \quad (3.1a)$$

$$\Theta_\mu^a_\nu = \theta^a_\lambda \Theta_\mu^{\lambda\nu} = \partial_\mu \theta^a_\nu - \partial_\nu \theta^a_\mu + A_\mu^a_b \theta^b_\nu - A_\nu^a_b \theta^b_\mu. \quad (3.1b)$$

We will freely raise and lower indices  $a, b, \dots$  with  $\delta_{ab}$ . The action (1.1) becomes

$$S'(\theta, A) = \int d^4x \det \theta [\lambda_0 + g_0 \theta_a^\mu \theta_b^\nu F_{\mu\nu}^{ab} + H^{\mu\nu\rho\beta\sigma} \Theta_{\mu\nu} \Theta_{\rho\beta\sigma} + G^{\mu\nu\alpha\beta\rho\sigma cd} F_{\mu\nu\alpha\beta} F_{\rho\sigma cd}], \quad (3.2)$$

where

$$G^{\mu\nu\alpha\beta\rho\sigma cd} = \text{SYMM}\{g_1 g^{\mu\rho} g^{\nu\sigma} \eta^{ac} \eta^{bd} + g_2 g^{\mu\rho} \theta^{c\nu} \theta^{a\sigma} \eta^{bd} + g_3 \theta^{c\mu} \theta^{d\nu} \theta^{a\rho} \theta^{b\sigma} + \theta^{a\mu} \theta^{c\rho} (g_4 g^{\nu\sigma} \eta^{bd} + g_5 \theta^{d\nu} \theta^{b\sigma} + g_6 \theta^{b\nu} \theta^{d\sigma})\}, \quad (3.3a)$$

$$H^{\mu\nu\rho\beta\sigma} = \text{SYMM}\{g^{\mu\rho} (a_1 \eta^{ab} g^{\nu\sigma} + a_2 \theta^{a\sigma} \theta^{b\nu} + a_3 \theta^{a\nu} \theta^{b\sigma})\}, \quad (3.3b)$$

where the prefix "SYMM" indicates that one has to take the proper combinations so that  $G$  is antisymmetric in  $(\mu, \nu)$ ,  $(\rho, \sigma)$ ,  $(a, b)$ ,  $(c, d)$  and symmetric under the simultaneous interchange of  $\mu\nu\alpha\beta$  and  $\rho\sigma cd$  and similarly  $H$  is antisymmetric in  $(\mu, \nu)$ ,  $(\rho, \sigma)$  and symmetric under the simultaneous interchange of  $\mu\nu$  and  $\rho\sigma$ .

Let us now discuss the dimensions of all quantities considered so far. We take the coordinates to have the dimension of length, as is customary in field theory. While the dimensions of all objects depend on this choice, the general conclusions that we shall reach would be the same if we had chosen the coordinates to be dimensionless, as would be more natural in quantum gravity (see the appendix in [4]). The kinetic terms of  $\theta$  and  $A$  are the terms quadratic in torsion and curvature respectively. In these terms indices are contracted with four inverse powers of  $\theta$ , which cancel with four powers of  $\theta$  in  $\det \theta$ .

Thus, the dimension of  $\theta$  and  $A$  is that of mass. One can now easily check that all coupling constants appearing in the action are dimensionless, including the parameters  $\lambda_0$  and  $g_0$ . This is desirable from the point of view of quantum field theory, where one would like to see Newton's constant arise from a phenomenon of dimensional transmutation in an otherwise scale invariant theory, rather than appearing as a bare coupling constant in the Lagrangian [7].

Since  $\theta$  has dimension of mass, the "metric"  $g_{\mu\nu}$ , as defined by (2.2), would have dimension of mass squared, which is not appropriate to define a line element. One could rescale  $g$  by a dimensionful constant, but this seems to be artificial, since there is no dimensionful parameter in the theory. The interpretation that we shall follow is that the geometry is not given by  $g$  but rather by the dimensionless background  $\gamma$ . Thus, the line element would not be given by (1.3) but rather by

$$ds^2 = \gamma_{\mu\nu} dx^\mu dx^\nu . \quad (3.4)$$

In the next section we will apply this point of view to the quantum theory and see that it leads to some nonstandard results.

#### 4. Quantization

We will now consider the formal path integral quantization of the theory. We have to functionally integrate  $\exp(-S'(\theta, A; \gamma)/\hbar)$  over  $\theta$  and  $A$ . This requires that we fix the gauge for the modified gauge transformations (2.7-8). Since we are assuming that the geometry is given by  $\gamma$ , the functional measure  $(d\theta dA)$  will also be defined with  $\gamma$ . Here we depart from the standard procedure, which uses the metric  $g$  in the definition of the measure. Our functional measure will be invariant under the original diffeomorphisms (2.5), but not under the  $GL(4)$  transformations (2.4). This is because it depends on  $\gamma$  alone, not on the  $GL(4)$ -invariant combination  $g$ . As a consequence, the measure will also not be invariant under the modified diffeomorphisms (2.8), although it will be invariant under the Lorentz transformations (2.7), as follows from the remark made after (2.4).

This leads to a slight complication in the gauge fixing procedure for the modified diffeomorphisms. According to the standard Faddeev-Popov procedure, one inserts in the functional integral a factor  $1 = \Delta_{\text{FP}}(\theta, A) \int (df) \delta(G_D(\theta^f, A^f))$ , where  $G_D$  is the gauge fixing,  $f$  denotes a diffeomorphism,  $\theta^f$  and  $A^f$  are transformed fields, and  $\Delta_{\text{FP}} = \det \mathcal{O}_{\text{FP}}$  is the invariant Faddeev-Popov determinant. Then one changes the variables of integration from  $\theta$  and  $A$  to  $\theta^f$  and  $A^f$ . Due to the noninvariance of the measure, this generates a Jacobian  $\det \left[ \frac{\delta(\theta^f, A^f)}{\delta(\theta, A)} \right]$ . However, it appears from (2.8) that this Jacobian depends only on  $f$ , and therefore one can still factor a global group integral, independent of the fields.

We introduce tensor density sources  $J_\rho{}^\mu$  and  $K^{\mu ab}$  coupled linearly to the quantum fields  $\theta$  and  $A$  respectively. We will use the notations  $(J, \theta) = \int d^4x J_\rho{}^\mu \theta^\rho{}_\mu$  and  $(K, A) = \int d^4x K^{\mu ab} A_\mu{}^a{}_b$ . With all this in mind, the generating functional for connected Green functions  $W(J, K; \gamma)$  can be written

$$e^{-\frac{1}{\hbar} W(J, K; \gamma)} = \int (d\theta dA) \Delta_{\text{FP}} e^{-\frac{1}{\hbar} (S(\theta, A; \gamma) + (J, \theta) + (K, A))} , \quad (4.1)$$



where  $S = S' + S_{\text{GF}}$ , with  $S_{\text{GF}}$  a gauge fixing term for the modified diffeomorphisms and local Lorentz transformations. For definiteness we will assume that the gauge fixing term has the form

$$S_{\text{GF}}(\theta, A) = \int d^4x \sqrt{\tilde{g}} \left[ \frac{1}{2\alpha_L} \gamma_{ac} \gamma^{bd} G_L^a{}_b G_L^c{}_d + \frac{1}{2\alpha_D} \tilde{g}^{\rho\sigma} G_{D\rho} G_{D\sigma} \right], \quad (4.2a)$$

$$G_L^a{}_b = \tilde{g}^{\mu\nu} \tilde{\nabla}_\mu (A - \tilde{A})_\nu^a{}_b + \beta_G \tilde{\theta}_b{}^\mu (\theta - \tilde{\theta})^a{}_\mu, \quad (4.2b)$$

$$G_{D\rho} = \tilde{\theta}_a{}^\mu \tilde{\nabla}_\mu (\theta - \tilde{\theta})^a{}_\rho + \beta_D \tilde{\theta}_a{}^\mu \tilde{\nabla}_\rho (\theta - \tilde{\theta})^a{}_\mu. \quad (4.2c)$$

In this formula quantities with tildas are fixed background fields and  $\tilde{\nabla}$  denotes the covariant derivatives with respect to the background fields  $\tilde{g}$  and  $\tilde{A}$ . The functionals  $W$  and  $\Gamma$  below will depend on the choice of these backgrounds.

The expectation values of  $\theta$  and  $A$  in the presence of the sources are given by

$$\theta_{(\text{cl})}^a{}_\mu = \frac{\delta W}{\delta J_a{}^\mu}, \quad A_{(\text{cl})\mu}{}^a{}_b = \frac{\delta W}{\delta K^{\mu a}{}_b}. \quad (4.3)$$

These equations in principle can be inverted to yield  $J$  and  $K$  as functionals of  $\theta_{(\text{cl})}$  and  $A_{(\text{cl})}$ . The effective action  $\Gamma(\theta_{(\text{cl})}, A_{(\text{cl})}; \gamma)$  is the Legendre transform of  $W$ :

$$\Gamma(\theta_{(\text{cl})}, A_{(\text{cl})}; \gamma) = W(J, K; \gamma) - (J, \theta_{(\text{cl})}) - (K, A_{(\text{cl})}). \quad (4.4)$$

It is the generator of one-particle irreducible diagrams. From (4.3) and (4.4) one has

$$\frac{\delta \Gamma}{\delta \theta_{(\text{cl})}^{\rho\mu}} = -J_\rho{}^\mu, \quad \frac{\delta \Gamma}{\delta A_{(\text{cl})\mu}{}^a{}_b} = -K^{\mu a}{}_b, \quad (4.5)$$

showing that the stationary points of  $\Gamma$  are the expectation values of the fields in the absence of sources.

Since the functional measure is defined with  $\gamma$  instead of  $g$ , the  $GL(4)$  invariance of the classical action  $S'(\theta, A; \gamma)$  will be broken at the quantum level. We recall that this invariance amounts simply to the statement that  $S'$  depends on its arguments only through the invariant combinations  $g$  and  $\Gamma$ . The quantum effective action will not be expressible in terms of these combinations alone, because the functional measure depends on  $\gamma$  alone. Therefore the quantum action will have separate functional dependences on the background metric  $\gamma$  and the (generalized) vierbein  $\theta$ : it will be a bimetric action.

However, there are some remaining symmetries. The local Lorentz transformations (2.7) are broken only by the gauge fixing term. This is because all other terms in (3.1) are constructed with  $g$  and  $\gamma$ , both of which are invariant under these transformations. As in ordinary gauge theories, this invariance will be recovered in the effective action  $\Gamma$ . Since local  $GL(4)$  invariance is broken, the modified diffeomorphisms (2.8) will be broken too. However, invariance under the original diffeomorphisms (2.5) will be preserved, with our choice of gauge, provided the background fields  $\tilde{\theta}$  and  $\tilde{A}$  are transformed too. This is certainly the case if in the end we set  $\tilde{\theta} = \theta_{(\text{cl})}$  and  $\tilde{A} = A_{(\text{cl})}$ , as will be implicitly assumed from now on. In conclusion, the effective action  $\Gamma(\theta_{(\text{cl})}, A_{(\text{cl})}; \gamma)$  will be invariant under

Lorentz transformations; it will not be invariant under diffeomorphisms if  $\gamma$  is kept fixed, but it will be invariant if  $\gamma$  is transformed too. These rather formal arguments have been verified in a special case by explicit calculation [8].

It is important to realize that these conclusion do not depend on any approximation. In practice, however, if we want to compute the effective action (4.4), we will have to resort to some approximation. For example, if we want to compute  $\Gamma$  at one loop, we can use the WKB method. In practice, this amounts to performing again an additive type of decomposition of the quantum fields. Let  $\bar{\theta}$  and  $\bar{A}$  be solutions of the classical field equations in the presence of sources  $J$  and  $K$ :

$$\left. \frac{\delta S}{\delta \theta^\rho{}_\mu} \right|_{\bar{\theta}, \bar{A}} = -J_\rho{}^\mu; \quad \left. \frac{\delta S}{\delta A_\mu{}^{ab}} \right|_{\bar{\theta}, \bar{A}} = -K^\mu{}_{ab}. \quad (4.6)$$

We expand

$$\theta^\rho{}_\mu = \bar{\theta}^\rho{}_\mu + \psi^\rho{}_\mu, \quad A_\mu{}^{ab} = \bar{A}_\mu{}^{ab} + \omega_\mu{}^{ab}. \quad (4.7)$$

Note that unlike in the decomposition (1.2), it is not necessary to introduce dimensionful constants at this stage. Next, the action is expanded in Taylor series around the classical fields:

$$S(\theta, A; \gamma) + (J, \theta) + (K, A) = S(\bar{\theta}, \bar{A}; \gamma) + (J, \bar{\theta}) + (K, \bar{A}) + S^{(2)}(\psi, \omega, \bar{\theta}, \bar{A}; \gamma) + \dots \quad (4.8)$$

The linear part is absent because of (4.6), and the quadratic part of the action can be written

$$\begin{aligned} S^{(2)}(\psi, \omega; \bar{\theta}, \bar{A}; \gamma) &= \frac{1}{2} \int d^4x \sqrt{\gamma} \operatorname{tr} \left[ (\omega \ \psi) \begin{pmatrix} \mathcal{O}_{[\omega\omega]} & \mathcal{O}_{[\omega\psi]} \\ \mathcal{O}_{[\psi\omega]} & \mathcal{O}_{[\psi\psi]} \end{pmatrix} \begin{pmatrix} \omega \\ \psi \end{pmatrix} \right] \\ &= \frac{1}{2} \int d^4x \sqrt{\gamma} \left[ \omega_\mu{}^{ab} \mathcal{O}_{[\omega\omega]}{}^{\mu\nu}{}_{cd} \omega_\nu{}^c{}_d \right. \\ &\quad \left. + 2\psi^a{}_\mu \mathcal{O}_{[\psi\omega]}{}^{\mu\nu}{}_{cd} \omega_\nu{}^c{}_d + \psi^a{}_\mu \mathcal{O}_{[\psi\psi]}{}^{\mu\nu}{}_{ab} \psi^b{}_\nu \right]. \end{aligned} \quad (4.9)$$

The part of this linearized action coming from  $S'$  is invariant under the linearized gauge transformations: the fields  $\psi^\rho{}_\mu = \delta\theta^\rho{}_\mu$  and  $\omega_\mu{}^{ab} = \delta A_\mu{}^{ab}$  given by Eqs. (2.7-8), with  $A = \bar{A}$  and  $\theta = \bar{\theta}$ , are null vectors for the operator  $\mathcal{O}$  which is the block matrix operator appearing in (4.9). This degeneracy is removed however by taking into account the contribution of the gauge fixing term  $S_{\text{GF}}$ .

The generating functional  $W$  is given in the WKB approximation by

$$e^{-\frac{1}{\hbar} W(J, K; \gamma)} = e^{-\frac{1}{\hbar} (S(\bar{\theta}, \bar{A}; \gamma) + (J, \bar{\theta}) + (K, \bar{A}))} \int (d\psi d\omega) \Delta_{\text{FP}} e^{-\frac{1}{\hbar} S^{(2)}(\psi, \omega, \bar{\theta}, \bar{A}; \gamma)}. \quad (4.10)$$

The functional integral is now Gaussian and we find

$$W(J, K; \gamma) = S(\bar{\theta}, \bar{A}; \gamma) + (J, \bar{\theta}) + (K, \bar{A}) + \frac{\hbar}{2} \ln \det \mathcal{O} - \hbar \ln \det \mathcal{O}_{\text{FP}}. \quad (4.11)$$

From (4.3) and (4.11), the expectation values of  $\theta$  and  $A$  in the presence of sources  $J$  and  $K$  are

$$\theta_{(\text{cl})}{}^a{}_\mu = \bar{\theta}^a{}_\mu + O(\hbar) , \quad A_{(\text{cl})\mu}{}^a{}_b = \bar{A}_\mu{}^a{}_b + O(\hbar) . \quad (4.12)$$

Expanding  $S$  to first order around  $\theta_{(\text{cl})}$  and  $A_{(\text{cl})}$  and using (4.6) one finds that  $S(\bar{\theta}, \bar{A}; \gamma) = S(\theta_{(\text{cl})}, A_{(\text{cl})}; \gamma) + (J, \theta_{(\text{cl})} - \bar{\theta}) + (K, A_{(\text{cl})} - \bar{A})$ . From the definition (4.4), discarding terms of order  $\hbar^2$ , and dropping the subscripts “(cl)” for notational simplicity, we find the usual result

$$\Gamma(\theta, A; \gamma) = S(\theta, A; \gamma) + \frac{\hbar}{2} \ln \det \mathcal{O} - \hbar \ln \det \mathcal{O}_{\text{FP}} . \quad (4.13)$$

The nonstandard aspect of our approach is now in the definition of the determinants. We shall see this explicitly in the next section.

## 5. One-loop calculation

We illustrate the general ideas of the previous section with a specific calculation. We will compute the one-loop effective potential for  $\theta$ , which is defined by

$$\Gamma(\theta) = \int d^4x \sqrt{\gamma} V(\theta) , \quad (5.1)$$

with  $\theta$  constant and  $A = 0$ . We restrict our attention to the case  $\gamma_{ab} = \delta_{ab}$  and  $\bar{\theta}^a{}_\mu = \rho \delta^a{}_\mu$ , so the effective potential will only be a function of the conformal factor  $\rho$ . This choice of  $\theta$  breaks the classical local  $GL(4)$  invariance to Weyl scalings. We shall see explicitly how this invariance is violated by our quantization procedure.

The part of the operators appearing in (4.9) that comes from varying the action (3.2) is

$$\mathcal{O}_{[\omega\mu]_{\mu ab}{}^{\rho cd}} = -8\tilde{G}_\mu{}^\nu{}_{ab}{}^{\rho\sigma cd} \partial_\nu \partial_\sigma + 8\tilde{H}_{\mu ab}{}^{\rho cd} \rho^2 + 2g_0 \rho^2 \delta_b^c (\delta_{a\mu} \delta^{d\rho} - \delta_a^\rho \delta_\mu^d) , \quad (5.2a)$$

$$\mathcal{O}_{[\psi\mu]_{a\mu}{}^{\rho cd}} = -8\tilde{H}^{\gamma}{}_{a\mu}{}^{\rho cd} \partial_\gamma + 2g_0 \rho (\delta_{a\mu} \delta^{d\rho} \delta^{c\gamma} + \delta_\mu^d \delta^{c\rho} \delta_a^\gamma - \delta_\mu^d \delta_a^\rho \delta^{c\gamma}) \partial_\gamma , \quad (5.2b)$$

$$\mathcal{O}_{[\psi\psi]_{a\mu}{}^{b\rho}} = -8\tilde{H}_{\mu a}{}^{\nu\rho b\sigma} \partial_\nu \partial_\sigma , \quad (5.2c)$$

where

$$\begin{aligned} \tilde{G}^{\mu\nu ab\rho\sigma cd} = & \text{SYMM} \{ g_1 \delta^{\mu\rho} \delta^{\nu\sigma} \delta^{ac} \delta^{bd} + g_2 \delta^{\mu\rho} \delta^{c\nu} \delta^{a\sigma} \delta^{bd} + g_3 \delta^{c\mu} \delta^{d\nu} \delta^{a\rho} \delta^{b\sigma} \\ & + \delta^{a\mu} \delta^{c\rho} (g_4 \delta^{\nu\sigma} \delta^{bd} + g_5 \delta^{d\nu} \delta^{b\sigma} + g_6 \delta^{b\nu} \delta^{d\sigma}) \} , \end{aligned} \quad (5.3a)$$

$$\tilde{H}^{\mu a \nu \rho b \sigma} = \text{SYMM} \{ \delta^{\mu\rho} (a_1 \delta^{ab} \delta^{\nu\sigma} + a_2 \delta^{a\sigma} \delta^{b\nu} + a_3 \delta^{a\nu} \delta^{b\sigma}) \} . \quad (5.3b)$$

Here and in the following indices are raised and lowered with  $\gamma_{\mu\nu} = \delta_{\mu\nu}$ . In (5.2) we have set  $\lambda_0$  to zero for simplicity. To this, one has to add the contribution of the gauge fixing terms, which now read

$$S_{\text{GF}} = \int d^4x \left[ \frac{1}{2\alpha_G} (\partial_\mu \omega^\mu{}_{ab} + \beta_G \rho \psi_{ab})^2 + \frac{1}{2\alpha_D} (\partial_\mu \psi_a{}^\mu - \beta_D \partial_a \psi^\mu{}_\mu)^2 \right] . \quad (5.4)$$

A partial diagonalization of the operator  $\mathcal{O}$  in (5.2) is achieved by decomposing the fields in irreducible parts with definite spin and parity. For convenience, we make the following definitions:  $\varphi_{\mu\nu} = \psi_{(\mu\nu)} = (\psi_{\mu\nu} + \psi_{\nu\mu})/2$  and  $\chi_{\mu\nu} = \psi_{[\mu\nu]} = (\psi_{\mu\nu} - \psi_{\nu\mu})/2$ . There are two modes with spin-parity  $2^+$ , coming from  $\omega$  and  $\varphi$ , one  $2^-$  mode from  $\omega$ , three  $1^+$  modes, two of which come from  $\omega$  and one from  $\chi$ , four  $1^-$  mode, two from  $\omega$  and one each from  $\varphi$  and  $\chi$ , three  $0^+$  of which one comes from  $\omega$  and two from  $\varphi$  and finally one  $0^-$  mode from  $\omega$ .

The linearized quadratic action (4.9) can be rewritten as

$$S^{(2)} = \frac{1}{2} \int d^4q \Phi_A(-q) \cdot a_{ij}^{AB}(J^P) P_{ij}^{AB}(J^P) \cdot \Phi_B(q), \quad (5.5)$$

where  $\Phi = (\omega, \psi, \chi)$ , the indices  $A, B$  run over the letters  $\omega, \psi$  and  $\chi$ , and the dot signifies contraction over the greek indices;  $P_{ij}^{AB}(J^P)$  are spin projection operators, which can be found in the literature [8,9], and  $a_{ij}^{AB}(J^P)$  are coefficient matrices. For the operator given in (5.2), one explicitly finds:

$$a(2^+) = \begin{bmatrix} G_1 q^2 + B_1 \rho^2 & -i\sqrt{2}B_1 |q|\rho \\ i\sqrt{2}B_1 |q|\rho & B_2 q^2 \end{bmatrix}, \quad (5.6a)$$

$$a(2^-) = G_2 q^2 + B_1 \rho^2, \quad (5.6b)$$

$$a(1^+) = \begin{bmatrix} G_3 q^2 + B_3 \rho^2 & -\sqrt{2}B_4 \rho^2 & -i\sqrt{2}B_4 |q|\rho \\ -\sqrt{2}B_4 \rho^2 & B_5 \rho^2 & iB_5 |q|\rho \\ i\sqrt{2}B_4 |q|\rho & -iB_5 |q|\rho & B_5 q^2 \end{bmatrix}, \quad (5.6c)$$

$$a(1^-) = \begin{bmatrix} G_4 q^2 + B_6 \rho^2 & \sqrt{2}B_7 \rho^2 & i\sqrt{2}B_7 |q|\rho & i\sqrt{2}B_7 |q|\rho \\ \sqrt{2}B_7 \rho^2 & B_8 \rho^2 & iB_8 |q|\rho & iB_8 |q|\rho \\ -i\sqrt{2}B_7 |q|\rho & -iB_8 |q|\rho & B_8 q^2 & B_8 q^2 \\ -i\sqrt{2}B_7 |q|\rho & -iB_8 |q|\rho & B_8 q^2 & B_8 q^2 \end{bmatrix}. \quad (5.6d)$$

$$a(0^+) = \begin{bmatrix} G_5 q^2 + B_9 \rho^2 & -i\sqrt{2}B_9 |q|\rho & 0 \\ i\sqrt{2}B_9 |q|\rho & B_{10} q^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad (5.6e)$$

$$a(0^-) = G_6 q^2 + B_{11} \rho^2, \quad (5.6f)$$

where we have used the following abbreviations:

$$\begin{aligned}
G_1 &= 4g_1 + 2g_2 + 4g_3 + g_4 + g_5, & B_4 &= 2a_1 - 3a_2 - g_0, \\
G_2 &= 4g_1 + g_2, & B_5 &= 4a_1 - 2a_2, \\
G_3 &= 4g_1 - 4g_3 + g_4 - g_5, & B_6 &= 2a_1 + a_2 + 2a_3 - g_0, \\
G_4 &= 4g_1 + g_2 + 2g_4, & B_7 &= a_3 - g_0, \\
G_5 &= 4g_1 + 2g_2 + 4g_3 + 4g_4 + 4g_5 + 12g_6, & B_8 &= 2a_1 + a_2 + a_3, \\
G_6 &= 4g_1 - 2g_2, & B_9 &= 2a_1 + a_2 + 3a_3 - 2g_0, \\
B_1 &= 2a_1 + a_2 + g_0, & B_{10} &= 4a_1 + 2a_2 + 6a_3, \\
B_2 &= 4a_1 + 2a_2, & B_{11} &= 8a_1 - 8a_2 - 2g_0. \\
B_3 &= 6a_1 - 5a_2 - g_0, & &
\end{aligned} \tag{5.7}$$

The matrices  $a(1^+)$ ,  $a(1^-)$  and  $a(0^+)$  are degenerate. This is a consequence of the gauge invariance of the classical action: diffeomorphism invariance requires that the third and fourth rows and columns of  $a(1^-)$  be proportional and the third row and column of  $a(0^+)$  be zero; local Lorentz invariance requires that the second and third rows and columns of  $a(1^+)$ , and the second and fourth rows and columns of  $a(1^-)$  be proportional. These degeneracies are removed by the contributions of the gauge fixing terms, which are:

$$a(2^+)_{\text{GF}} = \begin{bmatrix} 0 & 0 \\ 0 & \frac{\beta_G^2}{\alpha_G} \rho^2 \end{bmatrix}, \tag{5.8a}$$

$$a(1^+)_{\text{GF}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{q^2}{\alpha_G} & -i \frac{\beta_G}{\alpha_G} |q| \rho \\ 0 & i \frac{\beta_G}{\alpha_G} |q| \rho & \frac{\beta_G^2}{\alpha_G} \rho^2 \end{bmatrix}, \tag{5.8b}$$

$$a(1^-)_{\text{GF}} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{q^2}{\alpha_G} & 0 & i \frac{\beta_G}{\alpha_G} |q| \rho \\ 0 & 0 & \frac{q^2}{2\alpha_D} + \frac{\beta_G^2}{\alpha_G} \rho^2 & -\frac{q^2}{2\alpha_D} \\ 0 & -i \frac{\beta_G}{\alpha_G} |q| \rho & -\frac{q^2}{2\alpha_D} & \frac{q^2}{2\alpha_D} + \frac{\beta_G^2}{\alpha_G} \rho^2 \end{bmatrix}. \tag{5.8c}$$

$$a(0^+)_{\text{GF}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 \frac{\beta_D^2}{\alpha_D} q^2 + \frac{\beta_G^2}{\alpha_G} \rho^2 & \sqrt{3} \beta_D \frac{\beta_D - 1}{\alpha_D} q^2 \\ 0 & \sqrt{3} \beta_D \frac{\beta_D - 1}{\alpha_D} q^2 & \frac{(\beta_D - 1)^2}{\alpha_D} q^2 + \frac{\beta_G^2}{\alpha_G} \rho^2 \end{bmatrix}. \tag{5.8d}$$

The ghost action is

$$\begin{aligned}
S_{\text{FP}}(\bar{l}, l, \bar{d}, d) &= \frac{1}{2} \int d^4 x \text{tr} \left[ (\bar{l} \ \bar{d}) \begin{pmatrix} \mathcal{O}_{[ll]} & \mathcal{O}_{[ld]} \\ \mathcal{O}_{[dl]} & \mathcal{O}_{[dd]} \end{pmatrix} \begin{pmatrix} l \\ d \end{pmatrix} \right] \\
&= \frac{1}{2} \int d^4 x \left[ \bar{l}^{ab} \mathcal{O}_{[ll]_{ab}}{}^{cd} l_{cd} + \bar{l}^{ab} \mathcal{O}_{[ld]_{ab}}{}^{\nu} d_{\nu} \right. \\
&\quad \left. + \bar{d}^{\mu} \mathcal{O}_{[dl]_{\mu}}{}^{cd} l_{cd} + \bar{d}^{\mu} \mathcal{O}_{[dd]_{\mu}}{}^{\nu} d_{\nu} \right]. \tag{5.9}
\end{aligned}$$

where

$$\mathcal{O}_{[ll]ab}{}^{cd} = \delta_a^c \delta_b^d (-\partial^2 + \beta_G \rho^2) , \quad (5.10a)$$

$$\mathcal{O}_{[ld]ab}{}^\nu = -\beta_G \rho^2 \delta_a^\nu \partial_b , \quad (5.10b)$$

$$\mathcal{O}_{[dl]_\mu}{}^{cd} = \rho \delta_\mu^c \partial^d , \quad (5.10c)$$

$$\mathcal{O}_{[dd]_\mu}{}^\nu = \rho (-\delta_\mu^\nu \partial^2 + \beta_D \partial_\mu \partial^\nu) . \quad (5.10d)$$

The determinant of  $\mathcal{O}_{AB}$  as a  $40 \times 40$  matrix is equal to the product over spin  $J$  and parity  $\mathcal{P}$  of the determinants of the matrices  $a$ . Taking into account the multiplicity of these contributions, the one-loop effective action is

$$\Gamma^{(1)}(\rho) = \frac{1}{2} \int d^4x \int \frac{d^4q}{(2\pi)^4} \sum_{J,\mathcal{P}} (2J+1) \ln(\det a(J^\mathcal{P})) - \ln \det \mathcal{O}_{\text{FP}} . \quad (5.11)$$

The determinants are homogeneous polynomials in  $q^2$  and  $\rho^2$  of degree up to eight. The evaluation of the momentum integrals would be quite difficult in general. To simplify it we choose the gauge  $\beta_G = \beta_D = 0$ . With these assumptions, one easily finds:

$$\det a(2^+) = q^2 [B_2 G_1 q^2 + B_1 (B_2 - 2B_1) \rho^2] , \quad (5.12a)$$

$$\det a(1^+) = \frac{q^4}{2\alpha_G} [B_5 G_3 q^2 + (B_3 B_5 - 2B_4) \rho^2] , \quad (5.12b)$$

$$\det a(1^-) = \frac{2q^6}{\alpha_G \alpha_D} [B_8 G_4 q^2 + (B_6 B_8 - 2B_7^2) \rho^2] , \quad (5.12c)$$

$$\det a(0^+) = \frac{q^4}{\alpha_D} [B_{10} G_5 q^2 + B_9 (B_{10} - 2B_9) \rho^2] , \quad (5.12d)$$

while the ghost determinant becomes independent of  $\rho$  after a rescaling of the ghost variables. The integrals in (5.11) can now be performed exactly. With an appropriate choice of the renormalization conditions, the result is

$$V(\rho) = \frac{C}{64\pi^2} \rho^4 \left( \ln \frac{\rho^2}{\mu^2} - \frac{1}{2} \right) , \quad (5.13)$$

where  $\mu$  is a renormalization constant with dimensions of mass and the constant  $C$  is related to the coupling constants in the Lagrangian by

$$C = 5 \frac{B_1^2 (B_2 - 2B_1)^2}{B_2^2 G_1^2} + 5 \frac{B_1^2}{G_2^2} + 3 \frac{(B_3 B_5 - 2B_4^2)^2}{B_5^2 G_3^2} \\ + 3 \frac{(B_6 B_8 - 2B_7^2)^2}{B_8^2 G_4^2} + \frac{B_9^2 (B_{10} - 2B_9)^2}{B_{10}^2 G_5^2} + \frac{B_{11}^2}{G_6^2} . \quad (5.14)$$

There are various restrictions on the coupling constants due to the requirement that the ratios of the coefficients of  $\rho^2$  and  $q^2$  in (5.12) be positive. We shall not discuss this here.

The minimum of the potential occurs for  $\rho = \mu$ . In order to relate this undetermined mass to physical quantities one has to consider the case when  $\gamma$  is not flat and compute more terms in the derivative expansion of  $\Gamma$ . The results of [8,10] show that for  $A = 0$  and  $\rho$  constant,  $\Gamma$  will have the general form

$$\Gamma(\rho; \gamma) = \int d^4x \sqrt{\gamma} \left[ c_0 \rho^4 \left( \ln \left( \frac{\rho^2}{\mu^2} \right) + d_0 \right) + c_1 \rho^2 \left( \ln \left( \frac{\rho^2}{\mu^2} \right) + d_1 \right) R(\gamma) + O(R(\gamma)^2) \right], \quad (5.15)$$

where  $c_i$  and  $d_i$  are constants and  $R(\gamma)$  is the curvature of the metric  $\gamma$ . The minimum of the potential will be affected by the presence of curvature, but for weak fields it will always be of the order of  $\mu$ . Reinserting the minimum value in (5.15) one obtains a functional of the metric of the form

$$\Gamma_{\text{eff}}(\gamma) = \int d^4x \sqrt{\gamma} [C_0 \mu^4 + C_1 \mu^2 R(\gamma) + \dots], \quad (5.16)$$

showing that  $\mu$  has to be of the order of Planck's mass. The effective action  $\Gamma_{\text{eff}}$  describes the long distance dynamics of the gravitational field. As described in section 1, the gauge field  $A$  acquires a mass of the order of  $\mu$ , and therefore can be neglected below Planck's energy. The only dynamical variable at low energy is the dimensionless metric  $\gamma$ ; the coefficients of the Einstein and cosmological terms in the action now have the familiar dimensions. It is in this regime that we expect to recover the graviton. However, this issue requires further analysis.

The cosmological term in (5.16) seems to be undetermined as in the standard approach. However, the fact that  $\Gamma$  is essentially a bimetric action could have some bearing on this issue. A hint in this direction comes from the observation [10] that in the presence of a second metric  $\gamma$ ,  $g_{\mu\nu} = \delta_{\mu\nu}$  is a solution of the field equations also in the presence of the  $\lambda_0$  term in the action. One way to address this issue is to compute the renormalization group flow of  $\Gamma$ . Some preliminary investigations in this direction [10] show that the v.e.v. of the metric  $g_{\mu\nu}$  is constant, equal to  $\mu^2 \delta_{\mu\nu}$ , for scales smaller than Planck's mass, but grows roughly like the momentum squared for larger scales. This behaviour may affect the propagation of particles with very high energy, in particular the massive ghosts of the gravitational sector.

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