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AVERAGE EFFECTIVE POTENTIAL FOR THE CONFORMAL FACTOR

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Abstract

We discuss a method of quantization which produces a nontrivial effective potential for the conformal factor of the metric, both in the case of matter fields propagating in a fixed background geometry, and in the case of pure gravity. In particular, using the method of the average effective potential we compute the scale dependence of the conformal factor. We offer some speculations on the possible significance of this result.

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1. Introduction

Gravity must be described by a spontaneously broken gauge theory. This is a consequence of the fact that the vacuum expectation value (v.e.v.) of the metric, or of the vierbein, is nonzero (actually, nondegenerate). In quantum field theory the vacuum expectation value of the fields is usually determined by the effective potential (the nonderivative part of the effective action). In classical theories of gravity the possible form of a potential for the metric is severely constrained by general covariance: the only allowed local term in the Lagrangian depending on the metric but not on its derivatives is the cosmological term. Now, if the metric has the form

$$g_{\mu\nu} = \rho^2 \bar{g}_{\mu\nu} \ , \tag{1.1}$$

then the cosmological term becomes $\int d^4x \sqrt{\bar{g}}\, \rho^4$, which can be interpreted as a potential for the conformal factor ρ, but is not of the type that leads to spontaneous symmetry breaking.

In order to have a nontrivial potential (by this we mean one that leads to a nonzero v.e.v.) without breaking diffeomorphism invariance it is necessary to have two independent metrics. In [1] we have suggested a bimetric classical dynamics, in which one of the metrics is interpreted as the v.e.v. of the other, in the spirit of a mean field approximation. In this paper we will show that one need not introduce a second metric at the level of the classical action. Instead, it can appear in the process of quantization. There are two different but strictly related ways in which this can happen.

A second metric can appear in the definition of the ultraviolet regularization. Normally one uses the dynamical metric in the definition of the regulator. However, one can decide to define the regulators using a different, fixed metric. At least in the cases we shall consider here, this procedure does not lead to any pathological features that were not already present in the theory. It seems therefore to be a viable alternative.

A second metric also naturally arises in the study of the renormalization group for gravity. A convenient method of addressing this problem is the average effective action Γ_k, a continuum version of the block-spin action of lattice theories (k is a parameter with dimension of mass) [2,3]. In the case of gravity, one would compute the functional integral

$$e^{-W_k(j,\bar{g})} = \int (dg) e^{-S(g)-\Delta S_k(g,\bar{g})-(j,g)} \ , \tag{1.2}$$

where $e^{-\Delta S_k}$ is a term that forces the average of the metric g in a box of linear dimension k^{-1} to be equal to $\bar{g}$ and j is a (tensor density) source coupled linearly to g. The precise form of the constraint will be spelled out in Section 4. Gauge fixing and ghost terms are included in the action $S(g)$. If we call $g_{(\mathrm{cl})}$ the variable conjugated to j we get after Legendre transformation an effective action $\Gamma_k(g_{(\mathrm{cl})}, \bar{g})$ which can be thought of as the effective action for $g_{(\mathrm{cl})}$ in the background geometry defined by $\bar{g}$. The addition of the constraint term to the action amounts roughly speaking to putting an infrared cutoff at momentum k in the momentum integrals. This cutoff is naturally defined using the metric $\bar{g}$ (see Section 4). So we see that this second way of introducing a second metric is very closely related to the first one: in both cases the second metric appears in the definition of a cutoff.

We will restrict our attention to metrics of the form (1.1), but all that we are going to say can be generalized to arbitrary metrics. Most of the time we will consider the case when $\bar{g}$ is flat. We concentrate on the average effective potential $V_k(g_{(\mathrm{cl})}, \bar{g})$; for metrics of the form (1.1), it can be thought of as a function $V_k(\rho, \bar{g})$ and in the case when $\bar{g}$ is flat just as a function $V_k(\rho)$. We find that when the UV regulator is defined using the metric $\bar{g}$, the effective potential has the Coleman–Weinberg form [4], with the minimum occurring for nonzero ρ. We then compute the scale dependence of the minimum of the average effective potential, by varying the scale k. We find that, irrespective of the way in which the theory is regulated in the UV, the v.e.v. of ρ^2, and therefore of the metric, scales according to its canonical dimension above Planck's energy (up to logarithmic corrections).

Since the case of pure gravity presents technical complications which are inessential to the main points we want to make here, we begin in Section 2 by discussing the case of a real scalar field propagating in a bagkground gravitational field. We will see how the quantum dynamics of the scalar field can produce a nontrivial effective potential for the conformal factor ρ. In Section 3 we will repeat the same arguments in the case of a theory of gravity. The average effective action for the gravity theory is computed in Section 4. In Section 5 we offer some speculations on the physical meaning of the scaling of the metric.

2. Matter fields

Let us start from the action

$$S(\varphi, g_{\mu\nu}) = -\frac{1}{2}\int d^4x\,\sqrt{g}\left[g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + m^2\varphi^2\right], \tag{2.1}$$

describing the propagation of a scalar field φ in a background metric $g_{\mu\nu}$. Defining

$$\varphi = \rho^{-1}\phi \quad , \qquad g_{\mu\nu} = \rho^2\bar{g}_{\mu\nu} \tag{2.2}$$

the action (2.1) can be written as

$$S(\phi, \rho, \bar{g}_{\mu\nu}) = -\frac{1}{2}\int d^4x\,\sqrt{\bar{g}}\left[\bar{g}^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\rho^2\phi^2 + \ldots\right], \tag{2.3}$$

where the ellipses indicate terms containing derivatives of ρ (these terms will not be relevant for us since we shall consider only constant ρ in this paper).

The effective action $\Gamma(g_{\mu\nu})$ is minus one half of the logarithm of the determinant of the operator $\Delta_g + m^2$, where Δ_g is the covariant laplacian $\Delta_g = -\frac{1}{\sqrt{g}}\partial_\mu\sqrt{g}g^{\mu\nu}\partial_\nu$. One would expect from standard quantum gravity arguments that it has an expansion $\Gamma(g_{\mu\nu}) = \int d^4x\,\sqrt{g}\left[\Lambda_{\mathrm{eff}} + \kappa_{\mathrm{eff}}R + O(R^2)\right]$. For metrics of the form (1.1) the first term becomes $\Lambda_{\mathrm{eff}}\rho^4$ and can be interpreted as an effective potential for the conformal factor.

The actions (2.1) and (2.3) are classically equivalent. We are now going to show that quantizing one or the other can lead to physically inequivalent results. Let us compute directly the one-loop effective action for constant ϕ and ρ by integrating over fluctuations of the field ϕ in the action (2.3), with $\bar{g}_{\mu\nu}$ flat. It is given by

$$\Gamma^{(1)}(\rho) = -\frac{1}{2}\int d^4x \int \frac{d^4q}{(2\pi)^4}\ln(q^2 + m^2\rho^2), \tag{2.4}$$

where $q^2 = \delta^{\mu\nu} q_\mu q_\nu$. One can now proceed as one would with any theory in flat space. The integral can be regularized by imposing the cutoff $q^2 < \Lambda^2$. Adding suitable counterterms of the form $\Lambda^2\rho^2$ and $\rho^4 \ln\Lambda$ one arrives at the renormalized one-loop effective potential

$$V^{(1)}(\rho) = \frac{1}{64\pi^2} m^4 \rho^4 \left(\ln \frac{m^2\rho^2}{\mu^2} - \frac{3}{2} \right), \tag{2.5}$$

where μ is a renormalization mass. This is not the expected result: besides the quartic term in ρ it contains a further factor which is logarithmic in ρ. It is of the form found by Coleman and Weinberg [4]. Its minimum occurs for nonvanishing ρ.

Let us elaborate a little further on why this result is unexpected. The action (2.3) is invariant under the transformations

$$\bar{g}'_{\mu\nu} = \omega^2 \bar{g}_{\mu\nu}, \quad \rho' = \omega^{-1}\rho, \quad \phi' = \omega^{-1}\phi. \tag{2.6}$$

The classically equivalent action (2.1) is invariant under these transformations in a trivial way, since the combinations $g_{\mu\nu}$ and φ are not affected at all. In fact any functional of $\bar{g}$, ϕ and ρ which is invariant under (2.6) can be written as a functional of g and φ, and vice-versa. Thus the symmetry (2.6) is a "compensator" or Stückelberg type gauge invariance and there cannot be any anomaly for it (in particular, see [5], but also [6,7]). To see this it is enough to note that if we had quantized directly the field φ in the action (2.1) we would have obtained an effective action depending only on g, and hence automatically invariant under (2.6). We will call the transformations (2.6) "Stückelberg–Weyl" transformations. They should not be confused with Weyl transformations of the metric g, which are anomalous.

The effective potential (2.5) does not depend on $\bar{g}$ and ρ only through the combination g: invariance under the transformations (2.6) has been broken. It is the regularization procedure that we have chosen that breaks this invariance. In fact we have integrated over the range of momenta $q^2 = \bar{g}^{\mu\nu} q_\mu q_\nu < \Lambda^2$; this introduces a dependence of the quantum theory on $\bar{g}_{\mu\nu}$ alone, not accompanied by a factor of ρ, and is responsible for the appearance of the logarithm of ρ in (7).

There is an alternative way of regulating the theory: integrate over the range of momenta $g^{\mu\nu} q_\mu q_\nu = \rho^{-2} \bar{g}^{\mu\nu} q_\mu q_\nu < \lambda^2$. Redefining the integration variables as $q'_\mu = \rho^{-1} q_\mu$, the momentum integral in (2.4) becomes $\delta(0)\rho^4 \ln\rho^2 + \rho^4 \int \frac{d^4q'}{(2\pi)^4} \ln(q'^2 + m^2)$, the integration being now over the range $\bar{g}^{\mu\nu} q'_\mu q'_\nu < \lambda^2$. Neglecting the term $\delta(0)$ for a moment, the important point is that the integral does not depend on ρ anymore. Thus after renormalization, the effective action will be

$$V^{(1)}(\rho) = \frac{1}{64\pi^2} m^4 \left(\ln \frac{m^2}{\mu^2} - \frac{3}{2} \right) \rho^4. \tag{2.7}$$

Thus, it is this second regularization method which yields results in agreement with traditional quantum gravity. The term proportional to $\delta(0)$ could be simply discarded in dimensional regularization, but not in the cutoff regularization. We observe that it is exactly the jacobian of a change of variables of integration from ϕ to φ. Thus the result of integrating with the cutoff $g^{\mu\nu} q_\mu q_\nu < \lambda^2$ is exactly the same that one would have obtained by integrating out the field φ in the action (2.1), and we have already observed that this could not possibly lead to a nontrivial potential.

Is there any criterion to tell us which one of the two quantization procedures is the correct one? One could argue that the correct quantization is the one that preserves the Stückelberg–Weyl invariance (2.6). However in the present case it seems that breaking this invariance would not violate any physical principle. All that would happen is that the effective action, instead of depending on g and φ only, would have an additional dependence on the field ρ. If the metric was also dynamical (as will be the case in the next section), the field ρ could become an independent propagating field. This is entirely analogous to the situation occurring in those cases when a theory with anomalous local symmetry can be quantized [8,9].

The choice between the two regularization procedures can be related to which one of the metrics g and $\bar{g}$ is interpreted as giving the geometry of spacetime. In fact, the geometry enters in the definition of the modulus squared of the momentum. So if the geometry is given by g, one is led to the effective potential (2.7), while if the geometry is given by $\bar{g}$ one arrives at the effective potential (2.5).

One may still worry that the logarithmic terms in (2.5) are an artifact of the momentum cutoff regularization, which is special to flat space, and that they could not arise if an invariant regularization was used. We will therefore now rederive the effective potential (2.5) using the heat kernel regularization in a curved background. This calculation is of independent interest since it gives also the form of the curvature term in the effective action.

If we start from (2.3), the effective action can be defined by the formula

$$\Gamma^{(1)}(\rho,\bar{g}) = -\frac{1}{2}\ln\det(\Delta_{\bar{g}} + m^2\rho^2) = \frac{1}{2}\int_{\frac{1}{\Lambda^2}}^{\infty} ds\, s^{-1}\mathrm{Tr}\, e^{-s(\Delta_{\bar{g}}+m^2\rho^2)} \quad , \qquad (2.8)$$

where Λ is an ultraviolet cutoff. In order to extract the exact dependence of Γ on constant ρ, we write $e^{-s(\Delta_{\bar{g}}+m^2\rho^2)} = e^{-s\Delta_{\bar{g}}}e^{-sm^2\rho^2}$ and use the asymptotic expansion of the heat kernel of $\Delta_{\bar{g}}$. Then the r.h.s. of (2.8) becomes

$$\frac{1}{2}\int_{\frac{1}{\Lambda^2}}^{\infty} ds \int d^4x\sqrt{\bar{g}}\left(b_0(\Delta_{\bar{g}})e^{-sm^2\rho^2}s^{-3} + b_2(\Delta_{\bar{g}})e^{-sm^2\rho^2}s^{-2} + b_4(\Delta_{\bar{g}})e^{-sm^2\rho^2}s^{-1} + \ldots\right) . \qquad (2.9)$$

The integration over s can be performed explicitly (see Section 5 in [1]). Using a suitable renormalization scheme one arrives at the effective action

$$\Gamma^{(1)}(\rho,\bar{g}) = \frac{1}{64\pi^2}\int d^4x\,\sqrt{\bar{g}}\left[-m^4\rho^4\left(\ln\frac{m^2\rho^2}{\mu^2} - \frac{3}{2}\right) + \frac{1}{3}\bar{R}m^2\rho^2\left(\ln\frac{m^2\rho^2}{\mu^2} - 1\right) + \ldots\right], \qquad (2.10)$$

where the ellipses stand for terms of higher order in curvature. We see that when $\bar{g}$ is flat the potential (2.5) is reproduced.

Alternatively, one could start from the action (2.1) and integrate out the field φ. In this case the relevant operator is $\Delta_g + m^2$, and one arrives at an effective action $\Gamma^{(1)}(g_{\mu\nu})$ which is identical to (2.10) except for the replacement of $m^2\rho^2$, $\bar{g}$ and $\bar{R}$ by m^2, g and

R. In this effective action, ρ only appears within the metric g and the Stückelberg–Weyl invariance is preserved. When $\bar{g}$ is flat and ρ is constant, the effective potential (2.7) is reproduced.

As in the case of the flat space calculations with cutoff, the choice between the two quantization procedures depends on whether g or $\bar{g}$ is interpreted as the geometric metric. In fact in the heat kernel regularization one isolates the divergences by looking at the coincidence limit of a Green function with the distance between the points measured with respect to a certain geometry. In the first calculation, this geometry was given by $\bar{g}$, and this led to the appearance of a nontrivial potential for ρ, in the second calculation it was given by g and the effective potential is given just by a cosmological term. The difference between the two quantization procedures outlined above can also be interpreted as the addition of a Wess-Zumino term [6,10].

Effective potentials of the form (2.5) had been obtained earlier [6,10,11] and were attributed to a dilatation anomaly. We have given another derivation of these potentials, from the point of view of the coupling of matter to gravity. In our approach the presence of the potential is not the consequence of a conformal anomaly. Strictly speaking, one should say that a theory is anomalous only when there is no way to quantize it which preserves all classical symmetries. We have seen that the Stückelberg–Weyl symmetry is not anomalous. Instead, we have used a quantization procedure that explicitly breaks the Stückelberg–Weyl invariance.

The dynamics for the conformal factor induced by the anomaly of matter fields has been investigated recently in [12]. There, the standard regularization was used and therefore no nontrivial effective potential was obtained. In a similar context, an effective potential for the conformal factor was obtained and discussed in [13].

3. Gauge theory of gravity

Having discussed how a nontrivial potential for the conformal factor can appear in quantum field theory in a fixed background metric, we are now ready to see the same phenomenon happening in pure gravity. It should be clear from the previous discussion that a crucial ingredient is the appearance of an operator of the form of a laplacian plus a constant times ρ^2. Thus in the case of pure gravity we will also need an action which upon linearization yields an operator of this form. This is not the case for the ordinary Einstein action. We will choose instead an action quadratic in curvature and torsion. This type of action is also necessary in a unified theory of gravity [14].

In the model we shall consider, the independent dynamical variables are the vierbein $\theta^a{}_\mu$ and an $O(4)$ gauge field $A_\mu{}^a{}_b$ (we shall concentrate on the Euclidean theory, where $a, b = 1, 2, 3, 4$ are internal indices and μ, $\nu = 1, 2, 3, 4$ are spacetime indices). With θ and A we can construct metric, curvature and torsion fields:

$$g_{\mu\nu} = \theta^a{}_\mu \theta^b{}_\nu \delta_{ab} , \tag{3.1a}$$

$$F_{\mu\nu}{}^a{}_b = \partial_\mu A_\nu{}^a{}_b - \partial_\nu A_\mu{}^a{}_b + eA_\mu{}^a{}_c A_\nu{}^c{}_b - eA_\nu{}^a{}_c A_\mu{}^c{}_b , \tag{3.1b}$$

$$\Theta_\mu{}^a{}_\nu = \partial_\mu \theta^a{}_\nu - \partial_\nu \theta^a{}_\mu + eA_\mu{}^a{}_b \theta^b{}_\nu - eA_\nu{}^a{}_b \theta^b{}_\mu , \tag{3.1c}$$

where e is the gauge coupling constant. As an action we take

$$S(\theta, A) = \frac{1}{4}\int d^4x \, \sqrt{|\det g|}\, g^{\mu\rho} g^{\nu\sigma} \left[\delta_{ac}\delta^{bd} F_{\mu\nu}{}^a{}_b F_{\rho\sigma}{}^c{}_d + \delta_{ab}\Theta_\mu{}^a{}_\nu \Theta_\rho{}^b{}_\sigma\right]. \tag{3.2}$$

It is manifestly invariant under local $O(4)$ and general coordinate transformations.

In the previous section, it was implicitly assumed that $g_{\mu\nu}$, ρ and $\bar{g}_{\mu\nu}$ were all dimensionless. On the other hand now $\theta^a{}_\mu$ and $A_{\mu ab}$ have canonical dimension of mass, and $g_{\mu\nu}$ of mass squared. Therefore, from now on, we will assume that in (1.1) ρ has dimension of mass. One can adjust the dimensions of the fields φ and ϕ to be consistent with this choice [15]. Note that on the other hand the geometric metric $\tilde{g}$ which enters in the definition of the line element $ds^2 = \tilde{g}_{\mu\nu} dx^\mu dx^\nu$ has to be dimensionless (we are taking here the coordinates to have dimension of length, as is customary in quantum field theory). Therefore, $\tilde{g}_{\mu\nu}$ must be related to the v.e.v. $\langle g_{\mu\nu}\rangle$ by a constant factor with dimension of lenght squared, which can be naturally related to the Planck length [1].

We will now assume $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$ in (1.1) and evaluate the one-loop effective potential for the conformal factor ρ using the background field method. We first expand S up to second order around the classical solution of the field equations $A_{(\mathrm{cl})\mu}{}^a{}_b = 0$, $\theta_{(\mathrm{cl})}{}^a{}_\mu = \rho\,\delta^a_\mu$, with ρ constant. The linearized action has the form

$$\begin{aligned} S^{(2)} = \frac{1}{2}\int d^4x \Big[&\delta A_{\mu ab}\delta^{ac}\big(\delta^{bd}(-\delta^{\mu\rho}\partial^2 + \partial^\mu\partial^\rho) + e^2\rho^2(\delta^{bd}\delta^{\mu\rho} - \delta^{b\rho}\delta^{d\mu})\big)\delta A_{\rho cd} \\ &- 2e\rho\,\delta\theta^a{}_\mu(\delta^{d\mu}\partial^\rho - \delta^{\mu\rho}\partial^d)\,\delta^{ac}\delta A_{\rho cd} + \delta\theta^a{}_\mu\delta_{ac}(-\delta^{\mu\rho}\partial^2 + \partial^\mu\partial^\rho)\,\delta\theta^c{}_\rho\Big]. \end{aligned} \tag{3.3}$$

In this expression, indices are raised and lowered with $\delta_{\mu\nu}$. This linearized action is invariant under the linearized gauge transformations and linearized coordinate transformations. We add to the linearized action the gauge-fixing terms

$$\int d^4x \left[\frac{1}{2\alpha}(\partial_\mu \delta A^\mu_{ab})^2 + \frac{1}{2\beta}(\partial_\mu \delta\theta^{a\mu})^2\right]. \tag{3.4}$$

The effective action is one half the logarithm of the determinant of the differential operator appearing in (3.3), plus gauge fixing and ghost terms. The operator can be diagonalized using the method of the spin projectors, see the Appendix. We find

$$\Gamma^{(1)}(\rho) = \frac{1}{2}\int d^4x \int \frac{d^4q}{(2\pi)^4}\left[(5+3)\ln(q^2 + \frac{1}{2}e^2\rho^2) + 3\ln(q^2 + e^2\rho^2) + \ln(q^2 + 2e^2\rho^2)\right], \tag{3.5}$$

plus terms independent of ρ (we used the notation $q^2 = \delta^{\mu\nu}q_\mu q_\nu$). The first term comes from the modes with spin 2^- and 1^-, the second from those with spin 1^+, the last from those with spin 0^-. The ghost contribution turns out to be independent of ρ.

As in the case of the scalar field, one has a choice in the definition of the regularization. The standard result of quantum gravity, that $\Gamma(g) \approx \Lambda_{\mathrm{eff}} \int d^4x \sqrt{\bar{g}}\rho^4$, is obtained if we define the cutoff with the metric g. In what follows we describe the result of regularizing the integral (3.5) with the cutoff $q^2 < \Lambda^2$. Adding suitable counterterms of the form $\Lambda^2\rho^2$ and $\rho^4 \ln \Lambda$ one arrives at the renormalized effective potential

$$\Gamma^{(1)}(\rho,\bar{g}) = \int d^4x \sqrt{\bar{g}}\, V_0(\rho), \tag{3.6a}$$

$$V_0(\rho) = \frac{9}{64\pi^2} e^4 \rho^4 \left(\ln \frac{e^2\rho^2}{\mu^2} - \frac{1}{2} \right), \tag{3.6b}$$

where μ is a renormalization constant with dimensions of mass; we have written the result for an arbitrary constant $\bar{g}_{\mu\nu}$. This potential has the same form of the one we computed previously in the mean field approach [1]. It has a minimum for $\rho = \rho_0 = \mu/e$. Not that since we had to add counterterms involving the metric $\bar{g}$, the quantum theory is effectively a bimetric theory of the type discussed in [1].

Finally, let us comment on diffeomorphism invariance. The calculation leading to the effective potential (3.6) was performed in flat spacetime and therefore diffeomorphism invariance is not manifest. As discussed in the introduction, one has to think of the effective action as a functional of two metrics, and if both are transformed at the same time, this functional is diffeomorphism invariant. For metrics conformally related as in (1.1) this is made clear by the heat kernel calculation in Section 2.

4. Average effective potential

We turn now to the average effective action [2,3]. It is an effective action depending on a momentum scale k, which can be used to compute the renormalization group flow of various quantities of interest. Our main interest will be in the scale dependence of the effective potential, and hence of the v.e.v. of the operator $g_{\mu\nu}$.

To define the average effective action, one begins by adding to the classical action (3.2) quadratic terms which constrain the averages of the fields θ and A in volumes of size k^{-4} centered around the point x to take certain values $\bar{\theta}(x)$ and $\bar{A}(x)$ (up to small fluctuations):

$$\begin{aligned} S_{\rm constr} = \int d^4x \sqrt{\bar{g}} \Big[& \frac{1}{4} (\bar{F}_{\mu\nu ab} - f_k F_{\mu\nu ab}) \frac{\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}}{1 - f_k^2} (\bar{F}_{\rho\sigma}{}^{ab} - f_k F_{\rho\sigma}{}^{ab}) \\ & + \frac{1}{2\alpha} \bar{g}^{\mu\nu} \bar{\nabla}_\mu (A - \bar{A})_{\nu ab} \frac{1}{1 - f_k^2} \bar{g}^{\rho\sigma} \bar{\nabla}_\rho (A - \bar{A})_\sigma{}^{ab} \\ & + \frac{1}{2} \bar{\nabla}_\mu (\bar{\theta} - f_k\theta)^a{}_\nu \frac{\bar{g}^{\mu\rho}\bar{g}^{\nu\sigma}}{1 - f_k^2} \bar{\nabla}_\rho (\bar{\theta} - f_k\theta)^a{}_\sigma \Big] . \end{aligned} \tag{4.1}$$

In this formula $f_k = f_k(-\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu)$, where $\bar{\nabla}_\mu\theta^a{}_\nu = \partial_\mu\theta^a{}_\nu + e\bar{A}_\mu{}^a{}_b\theta^b{}_\nu - \Gamma^\lambda_{\mu\nu}\theta^a{}_\lambda$ and Γ are the Christoffel symbols for the metric $\bar{g}_{\mu\nu}$. The differential operator $f_k(-\bar{g}^{\mu\nu}\bar{\nabla}_\mu\bar{\nabla}_\nu)$ will perform the desired averaging operation if we take $f_k(x) = exp(-a(x/k^2)^b)$, with a, b constant parameters. Note that the explicit introduction of the fields $\bar{\theta}$ and $\bar{A}$ breaks both coordinate and gauge invariance, so no further gauge fixing is needed (for similar remarks see [16]). In (4.1) we have contracted all spacetime indices with the metric $\bar{g}_{\mu\nu}$. This is necessary in order that the constraint term be only quadratic in the fields θ and A. It is also in line with the assumption that it is the metric $\bar{g}$ that dictates the geometry.

In order to compute the average effective potential we choose the average fields $\bar{A}=0$ and $\bar{\theta}^a{}_\mu = \rho\,\delta^a_\mu$ with ρ constant. If the parameter b in f_k is chosen larger than 2, the Ansatz $A_{\mathrm{cl}\mu ab} = \bar{A}_{\mu ab}$ and $\theta_{\mathrm{cl}}{}^a{}_\mu = \bar{\theta}^a{}_\mu$ gives a solution of the classical equations of motion of the total action. Proceeding as before, we arrive at the average action

$$\begin{aligned}\Gamma_k(\rho) = \frac{1}{2}\int d^4x \int \frac{d^4q}{(2\pi)^4}\Big[& 8\ln\left(P_k + \frac{1}{2}e^2\rho^2\right) + 6\ln\left(P_k + \frac{1}{2}e^2\rho^2 f_k^2\right) \\ & + 3\ln\left(P_k^2 + e^2\rho^2 P_k\left(1+\left(\alpha+\frac{1}{2}\right)f_k^2\right) + \alpha e^4\rho^4 f_k^2\right) \\ & + 3\ln\left(P_k + \frac{1}{2}\alpha e^2\rho^2 f_k^2\right) + \ln\left(P_k + 2e^2\rho^2\right)\Big]\,, \end{aligned} \tag{4.2}$$

where $f_k^2 = \left(f_k(q^2)\right)^2$ and $P_k(q^2) = q^2/(1-f_k^2)$. Note that in the limit $k \to 0$, the function f_k becomes zero and P_k becomes equal to q^2. One can then easily check that up to field-independent terms, $\Gamma_0 = \Gamma_{k=0}$ reduces to the old effective action (3.5). One can split $\Gamma_k(\rho) = \Gamma_0(\rho) + \Delta\Gamma_k(\rho)$, where $\Gamma_0(\rho)$ contains the divergences but is independent of k and

$$\begin{aligned}\Delta\Gamma_k(\rho) = \frac{1}{2}\int d^4x \int \frac{d^4q}{(2\pi)^4}\Bigg[& 8\ln\left(\frac{P_k+\frac{1}{2}e^2\rho^2}{q^2+\frac{1}{2}e^2\rho^2}\right) + 6\ln\left(\frac{P_k+\frac{1}{2}e^2\rho^2 f_k^2}{q^2}\right) \\ & + 3\ln\left(\frac{P_k^2 + e^2\rho^2 P_k\left(1+\left(\alpha+\frac{1}{2}\right)f_k^2\right) + \alpha e^4\rho^4 f_k^2}{q^2(q^2+e^2\rho^2)}\right) \\ & + 3\ln\left(\frac{P_k+\frac{1}{2}\alpha e^2\rho^2 f_k^2}{q^2}\right) + \ln\left(\frac{P_k+2e^2\rho^2}{q^2+2e^2\rho^2}\right)\Bigg]\,, \end{aligned} \tag{4.3}$$

which contains all the k dependence, is both IR and UV convergent. The part $\Gamma_0(\rho)$ can be renormalized in either one of the two ways mentioned in Section 3. If the UV cutoff is defined using the metric $\bar{g}$, we are led again to the effective potential $V_0(\rho)$ given in (3.6). We shall briefly discuss the other option later. Define the average effective potential $V_k(\rho) = V_0(\rho) + \Delta V_k(\rho)$ by $\Gamma_k(\rho) = \int d^4x\,\sqrt{\bar{g}}\,V_k(\rho)$. To find its minimum we have to solve the equation

$$0 = \frac{\partial V_k(\rho)}{\partial(\rho^2)} = \frac{e^2}{64\pi^2}\left[18e^2\rho^2\ln\frac{e^2\rho^2}{\mu^2} + k^2 F\left(\frac{e^2\rho^2}{k^2}\right)\right]\,, \tag{4.4}$$

where, using the dimensionless variables $x = q^2/k^2$, $t = e^2\rho^2/k^2$ and $\tilde{P}(x) = P_k(q^2)/k^2 = x/(1+f^2)$, $f^2 = \exp(-2ax^b)$, the function F is given by

$$F(t) = \frac{64\pi^2}{e^2k^2}\frac{\partial \Delta V_k(\rho)}{\partial(\rho^2)} = 2\int_0^\infty dx\, x\, f^2 \Bigg[-\frac{4\tilde{P}}{(\tilde{P}+\frac{t}{2})(x+\frac{t}{2})} + \frac{3}{\tilde{P}+\frac{1}{2}tf^2}$$

$$+3\frac{\tilde{P}(\frac{1}{2}x-\tilde{P})+\alpha(\tilde{P}x+t(2x+t))}{(x+t)(\tilde{P}^2+\tilde{P}t(1+(\alpha+\frac{1}{2})f^2)+\alpha t^2f^2)} \tag{4.5}$$

$$+\frac{3}{2}\frac{\alpha}{\tilde{P}+\frac{\alpha}{2}tf^2} - \frac{2\tilde{P}}{(\tilde{P}+2t)(x+2t)}\Bigg]$$

This function can be studied numerically. Choosing $a = 1$, $b = 3.19$ in f^2, (see [2]) and setting $\alpha = 0$, $F(t)$ grows from $F(0) = -c_1 \sim -12$ to zero for $t \sim 5$, it reaches a maximum of order 0.2 for $t \sim 15$ and decreases slowly to zero for large t like K/t for K very slowly varying. The minimum of the effective potential can be plotted numerically. One can only study analytically the behavior for t very large and very small. Let us denote ρ_k the minimum of V_k. For $k = 0$, $\rho_0 = \mu/e$. For $t \gg 1$ (which corresponds to $k \ll \mu$) we can expand $\rho_k = \rho_0 + \epsilon$, and use the asymptotic behavior $F(t) \sim K/t$. Inserting in (4.4) one finds

$$\rho_k^2 = \rho_0^2\left(1 - \frac{K}{18}\frac{k^4}{\mu^4}\right) , \qquad k^2 \ll \rho^2 \approx \mu^2 . \tag{4.6}$$

On the other hand for $t \ll 1$ we can expand the function F in Taylor series around $t = 0$: $F(t) = -c_1 + c_2 t + \ldots$ Equation (4.4) shows that ρ_k^2 grows slower than $c_1 k^2$ and faster than $c_1 k^2/(18\ln(c_1k^2/\mu^2)+c_2)$. For $k \gg \mu$ the denominator becomes large and this justifies a posteriori the approximation $t \ll 1$. In fact, this can also be checked numerically.

For $\alpha \neq 0$ but not too large, the behavior of the potential is essentially the same. We note that the behavior for large and small k agrees with the one found using a mean field approximation and treating k simply as a sharp infrared cutoff [17].

By a reasoning which was discussed in [1], the scale μ has to be identified with Planck's mass. We have thus found that the v.e.v. of $g_{\mu\nu}$ is given by

$$\langle g_{\mu\nu}\rangle_k = \rho_k^2 \delta_{\mu\nu} , \tag{4.7}$$

where ρ_k is essentially constant for k smaller than Planck's mass and grows linearly for k larger than Planck's mass. Note that the scaling behavior for large k is the one that is expected on the basis of dimensional considerations. In a more sophisticated approach one would have to take into account also the corrections due to the anomalous dimension of the metric.

Let us discuss briefly what would happen if the UV regulator was defined with the metric g. Then $V_0(\rho) = \Lambda_{\text{eff}}\rho^4$, where Λ_{eff} is an undetermined parameter, possibly zero. The minimum of the effective potential at $k = 0$ is for $\rho = 0$. For k sufficiently large, however, V_k would develop a nontrivial minimum. This is analogous to the result of [18]. For $k \gg \mu$ the v.e.v. of ρ scales again linearly with k, up to small corrections. Thus irrespective of how the theory is regulated in the UV, the metric scales like k^2 above Planck's energy. Modulo anomalous dimensions, this result can be trusted. The behavior for small k should be studied more carefully using an "improved" form of the renormalization group equation, along the lines of [3]. We plan to discuss this elsewhere.

5. Implications and conclusions

The most striking aspect of the foregoing analysis is the scaling of the v.e.v. of the metric g. This can be understood in terms of a fractal structure of spacetime. Let us recall the well known example of the length of a coastline [19]. This concept is not well defined by itself: it depends on the choice of a parameter L representing the length of the yardstick that we use to perform the measurement. It is essentially the smallest size of the features of the coast that we can resolve. The result of an actual measurement will be larger if we use a shorter yardstick, since we are then able to follow more faithfully all the features of the coast. Thus the distance between two points on the coast is a decreasing function of L.

Our result on the scaling of the metric can be interpreted in the same way. If we perform experiments at an energy scale k, we are unable to resolve features of the fluctuating spacetime geometry with lengths smaller that $L = k^{-1}$. Fluctuations of the metric with wavelengths shorter than k will have to be integrated over, so the effective metric is $\langle g_{\mu\nu}\rangle_k$. According to the results of the previous section, the distance between two points measured with this metric increases with k. This is exactly the behavior that one expects on the basis of the example of the coastline. For other comments on the fractal structure of spacetime, see *e.g.* [20].

We now turn to the discussion of the physical implications of our results. Consider a scalar test particle coupled to the dynamical metric $g_{\mu\nu}$. The action for the scalar field is given by (2.1), where φ and m are dimensionless. The first term in (2.1) represents an interaction between the scalar field and the metric. The standard perturbative kinetic term is obtained by replacing $g_{\mu\nu}$ with its v.e.v. and rescaling φ as in (2.2) so that it acquires canonical dimension of mass. In this way one recovers the action (2.3), which for $\bar{g}_{\mu\nu} = \delta_{\mu\nu}$ and ρ constant can be written in momentum space

$$S(\phi) = -\frac{1}{2}\int d^4p\, \phi(-p)\left[p^2 + m^2\rho^2\phi^2\right]\phi(p)\,. \tag{5.1}$$

As we argued above, a particle of momentum p is insensitive to fluctuations of the metric with wavelengths smaller than p^{-1}, and therefore it propagates according to the metric $\langle g_{\mu\nu}\rangle_k$ with $k = p$. This amounts to evaluating the factor ρ^2 in (5.1) at the scale $k = p$. The inverse propagator of ϕ would then have the form $p^2 + m^2\rho_p^2$, with ρ_p^2 approximately constant for $p^2 < \rho_0^2$ and growing roughly like p^2 for $p^2 > \rho_0^2$. The physical pole of the propagator occurs at mass approximately equal to $m\rho_0$ for $m^2 < 1$, but is shifted to exponentially large values for $m^2 > 1$. In fact, a positive anomalous dimension for ρ could make the pole disappear altogether. As already mentioned, the mass ρ_0 has to be identified with Planck's mass [1]. Thus, particles with masses larger than Planck's mass would essentially disappear from the spectrum. One may hope that a mechanism of this type is capable of removing the ghosts of the gravitational sector. This seems to be a restatement of the criterion given in [21].

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Appendix

In this appendix we shall explain how the expression (3.5) for the effective action can be computed starting from the quadratic action (3.3) and (3.4). The effective action is minus one half the functional determinant of the operator $\mathcal{O}$ appearing in the linearized action. Since this operator is Lorentz covariant, it does not mix irreducible representations of the Lorentz group with different values of spin J and parity $\mathcal{P}$. A partial diagonalization is therefore achieved by decomposing $\mathcal{O}$ in diagonal blocks corresponding to definite spin and parity.

Since the background vierbein is δ_μ^a, there is no need to keep the distinction between latin and greek indices. For convenience, we also make the following redefinitions: $\omega_{\mu\nu\rho} = \delta A_{\mu\nu\rho}$, $\varphi_{\mu\nu} = \delta\theta_{(\mu\nu)} = (\delta\theta_{\mu\nu} + \delta\theta_{\nu\mu})/2$ and $\chi_{\mu\nu} = \delta\theta_{[\mu\nu]} = (\delta\theta_{\mu\nu} - \delta\theta_{\nu\mu})/2$.

The linearized quadratic action (3.3), together with the gauge-fixing (3.4) (the ghost contribution is field independent and thus decouples) can be written as

$$S^{(2)} = -\frac{1}{2}\int d^4q\ \Phi_A(-q)\cdot\mathcal{O}_{AB}(q)\cdot\Phi_B(q)\ , \tag{A.1}$$

where $\Phi = (\omega, \varphi, \chi)$; the indices A, B run over the letters ω, φ and χ, and the dot signifies contraction over the greek indices.

We decompose $\omega_{\alpha\beta\gamma}$, $\varphi_{\alpha\beta}$ and $\chi_{\alpha\beta}$ into irreducible representations of the Lorentz group. If we choose coordinates such that x_L is in the direction of the momentum and x_i, $i = 1,2,3$ are transverse, then the irreducible representations can be listed as follows: $\omega_{\underline{ij}k}$ (2^-), ω_{iik} (1^-), $\omega_{[ijk]}$ (0^-), $\omega_{L[ij]}$ (1^+), $\omega_{(\underline{ij})L}$ (2^+), $\omega_{[ij]L}$ (1^+), ω_{iiL} (0^+), ω_{LiL} (1^-), $\varphi_{\underline{ij}}$ (2^+), φ_{ii} (0^+), φ_{iL} (1^-), φ_{LL} (0^+), $\chi_{[ij]}$ (1^+) χ_{iL} (1^-). We have used square and round brackets around indices to denote symmetrization and antisymmetrization, repeated indices denote a trace and underlined indices denote tracelessness. Each spin-2 part has five independent components, the spin-1 parts have three independent components and the spin-0 parts one component, so that the tensor $\omega_{\mu\nu\rho}$ posesses 24 independent components, while the tensor $\varphi_{\mu\nu} + \chi_{\mu\nu}$ has 16 independent components, as it should be.

The 40×40 matrix $\mathcal{O}_{AB}$ appearing in (A.1) decomposes into two 1×1 blocks, corresponding to $J^\mathcal{P} = 2^-, 0^-$, one 2×2 block, with $J^\mathcal{P} = 2^+$, two 3×3 blocks, with $J^\mathcal{P} = 1^+, 0^+$ and one 4×4 block, corresponding to $J^\mathcal{P} = 1^-$. In order to obtain this decomposition, we shall use the formalism of the spin-projector operators.

We will use indices i, j to label isomorphic representations occurring more than once. For example for spin-parity 2^+, $i = 1,2$; for 1^+, $i = 1,2,3$ *etc.* One can construct a complete set of projection operators $P_{ii}^{AA}(J^\mathcal{P})$ which project out of a field a given irreducible representation, and intertwiners $P_{ij}^{AB}(J^\mathcal{P})$ (with $i \neq j$) which give isomorphisms between the different representations occurring more than once. (Note that the indices A, B in these projectors are redundant since i, j already label the representations. It is nevertheless convenient to keep them in order to remember by what field a certain representation is carried, *e.g.* for $J^\mathcal{P} = 1^+$ the representations $i = 1$ and $i = 2$ are carried by ω and $i = 3$ is carried by χ).

The projection operators have been computed and listed in the literature (see [22]). Their explicit expression is reported below. The operators $P_{ij}^{AB}(J^\mathcal{P})$ are orthonormal and complete:

$$P^{AB}_{ij}(J^{\mathcal{P}}) \cdot P^{CD}_{kl}(I^{\mathcal{Q}}) = \delta_{IJ}\,\delta_{\mathcal{P}\mathcal{Q}}\,\delta_{jk}\,\delta_{BC}\; P^{AD}_{il}(J^{\mathcal{P}}) ,$$
$$\sum_{J,\mathcal{P},A,i} P^{AA}_{ii}(J^{\mathcal{P}}) = \mathbf{1} . \tag{A.2}$$

Expanding the operator $\mathcal{O}_{AB}$ in terms of these projection operators, one can rewrite the action $(A.1)$ as

$$S^{(2)} = -\frac{1}{2}\int d^4q\;\Phi_A(-q)\cdot a^{AB}_{ij}(J^{\mathcal{P}})\,P^{AB}_{ij}(J^{\mathcal{P}})\cdot\Phi_B(q) , \tag{A.3}$$

where $a^{AB}_{ij}(J^{\mathcal{P}})$ are coefficient matrices. These can be computed by acting with $\mathcal{O}_{AB}$ on the diagonal spin projectors $P^{AA}_{ii}(J^{\mathcal{P}})$ and subsequently reexpressing the result in terms of $P^{AB}_{ij}(J^{\mathcal{P}})$. In the present case, one explicitly finds:

$$a(2^-) = q^2 + e^2\rho^2/2, \tag{A.4a}$$
$$a(0^-) = q^2 + 2e^2\rho^2, \tag{A.4b}$$

$$a(2^+) = \begin{bmatrix} q^2 + e^2\rho^2/2 & -ie\rho|q|/\sqrt{2} \\ ie\rho|q|/\sqrt{2} & q^2 \end{bmatrix}, \tag{A.4c}$$

$$a(0^+) = \begin{bmatrix} q^2 + e^2\rho^2/2 & -ie\rho|q|/\sqrt{2} & 0 \\ ie\rho|q|/\sqrt{2} & q^2 & 0 \\ 0 & 0 & q^2/\beta \end{bmatrix}, \tag{A.4d}$$

$$a(1^+) = \begin{bmatrix} q^2 + 3e^2\rho^2/2 & -e^2\rho^2/\sqrt{2} & -ie\rho|q|/\sqrt{2} \\ -e^2\rho^2/\sqrt{2} & q^2/\alpha + e^2\rho^2 & ie\rho|q| \\ ie\rho|q|/\sqrt{2} & -ie\rho|q| & q^2 \end{bmatrix}, \tag{A.4e}$$

$$a(1^-) = \begin{bmatrix} q^2 + e^2\rho^2/2 & 0 & 0 & 0 \\ 0 & q^2/\alpha + e^2\rho^2/2 & ie\rho|q|/2 & ie\rho|q|/2 \\ 0 & -ie\rho|q|/2 & (1+1/\beta)q^2/2 & (1-1/\beta)q^2/2 \\ 0 & -ie\rho|q|/2 & (1-1/\beta)q^2/2 & (1+1/\beta)q^2/2 \end{bmatrix} . \tag{A.4f}$$

One can easily check that the determinant of $\mathcal{O}_{AB}$ as a 40×40 matrix is equal to the product over spin J and parity $\mathcal{P}$ of the determinants of the matrices a. The determinants of the above matrices are readily computed; only those with $J^{\mathcal{P}} = 2^-, 1^+, 1^-, 0^-$ depend on ρ. Taking into account the multiplicities of these contributions, one finally arrives at (3.5).

Similar computations lead to the effective action (4.2). In this case one starts from the linearized action (3.3), to which the constraint contribution (4.1) is added. No gauge-fixing is now needed because of the presence in (4.1) of the background fields $\bar{A}$ and $\bar{\theta}$. The matrices a are now as follows

$$a(2^-) = P_k + e^2\rho^2/2 , \tag{A.5a}$$

$$a(0^-) = P_k + 2e^2\rho^2\ , \tag{A.5b}$$

$$a(2^+) = \begin{bmatrix} P_k + e^2\rho^2/2 & -ie\rho|q|/\sqrt{2} \\ ie\rho|q|/\sqrt{2} & P_k \end{bmatrix}\ , \tag{A.5c}$$

$$a(0^+) = \begin{bmatrix} P_k + e^2\rho^2/2 & -ie\rho|q|/\sqrt{2} & 0 \\ ie\rho|q|/\sqrt{2} & P_k & 0 \\ 0 & 0 & f_k P_k \end{bmatrix}\ , \tag{A.5d}$$

$$a(1^+) = \begin{bmatrix} P_k + 3e^2\rho^2/2 & -e^2\rho^2/\sqrt{2} & -ie\rho|q|/\sqrt{2} \\ -e^2\rho^2/\sqrt{2} & P_k/\alpha + e^2\rho^2 & ie\rho|q| \\ ie\rho|q|/\sqrt{2} & -ie\rho|q| & P_k \end{bmatrix}\ , \tag{A.5e}$$

$$a(1^-) = \begin{bmatrix} P_k + e^2\rho^2/2 & 0 & 0 & 0 \\ 0 & P_k/\alpha + e^2\rho^2/2 & ie\rho|q|/2 & ie\rho|q|/2 \\ 0 & -ie\rho|q|/2 & q^2/2 + f_k P_k & (1-1/\beta)q^2/2 \\ 0 & -ie\rho|q|/2 & (1-1/\beta)q^2/2 & q^2/2 + f_k P_k \end{bmatrix} . \tag{A.5f}$$

The calculation of the functional determinants of these matrices immediately gives the result (4.2).

Finally, let us collect the explicit expressions of the spin-projector operators $P^{AB}_{ij}(J^{\mathcal{P}})$. It is useful to introduce the following notations:

$$\hat{q}^\mu = q^\mu/\sqrt{q^2}\ , \qquad L^\nu_\mu = \hat{q}_\mu \hat{q}^\nu\ , \qquad T^\nu_\mu = \delta^\nu_\mu - L^\nu_\mu\ , \tag{A.6}$$

obeying the relations:

$$L^\mu_\nu T^\rho_\mu = 0\ , \qquad T^\nu_\mu T^\rho_\nu = L^\rho_\mu\ , \qquad L^\nu_\mu L^\rho_\nu = L^\rho_\mu\ . \tag{A.7}$$

In terms of $\hat{q}^\mu$, L^ν_μ, and T^ν_μ, one finds:

$$[P^{\omega\omega}(2^-)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} = \frac{2}{3} T^\alpha_\tau T^{[\beta}_{[\rho} T^{\gamma]}_{\sigma]} + \frac{2}{3} T^{[\gamma}_\tau T^{\beta]}_{[\rho} T^\alpha_{\sigma]} - T_{\tau[\rho} T^{[\gamma}_{\sigma]} T^{\beta]\alpha}\ ,$$

$$[P^{\omega\omega}(0^-)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} = \frac{1}{3} T^\alpha_\tau T^{[\beta}_{[\rho} T^{\gamma]}_{\sigma]} - \frac{2}{3} T^{[\gamma}_\tau T^{\beta]}_{[\rho} T^\alpha_{\sigma]} \equiv T^{[\alpha}_{[\tau} T^{\beta}_{\rho} T^{\gamma]}_{\sigma]}\ ,$$

$$[P(2^+)] = \begin{bmatrix} [P^{\omega\omega}_{11}(2^+)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} & [P^{\omega\varphi}_{12}(2^+)]_{\tau\rho\sigma}{}^{\alpha\beta} \\ [P^{\varphi\omega}_{21}(2^+)]_{\rho\sigma}{}^{\alpha\beta\gamma} & [P^{\varphi\varphi}_{22}(2^+)]_{\rho\sigma}{}^{\alpha\beta} \end{bmatrix}\ ,$$

$$[P^{\omega\omega}_{11}(2^+)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} = T^\alpha_\tau T^{[\beta}_{[\rho} L^{\gamma]}_{\sigma]} + T^{[\gamma}_\tau L^{\beta]}_{[\rho} T^\alpha_{\sigma]} - \frac{2}{3} T_{\tau[\rho} L^{[\gamma}_{\sigma]} T^{\beta]\alpha}\ ,$$

$$[P^{\omega\varphi}_{12}(2^+)]_{\tau\rho\sigma}{}^{\alpha\beta} = \sqrt{2}\, T^{(\alpha}_\tau T^{\beta)}_{[\rho}\, \hat{q}_{\sigma]} - \frac{\sqrt{2}}{3} T^{\alpha\beta} T_{\tau[\rho}\, \hat{q}_{\sigma]}\ ,$$

$$[P^{\varphi\omega}_{21}(2^+)]_{\rho\sigma}{}^{\alpha\beta\gamma} = \sqrt{2}\, T^\alpha_{(\rho} T^{[\beta}_{\sigma)} \hat{q}^{\gamma]} - \frac{\sqrt{2}}{3} T_{\rho\sigma} T^{\alpha[\beta} \hat{q}^{\gamma]}\ ,$$

$$[P^{\varphi\varphi}_{22}(2^+)]_{\rho\sigma}{}^{\alpha\beta} = T^{(\alpha}_{(\rho} T^{\beta)}_{\sigma)} - \frac{1}{3} T_{\rho\sigma} T^{\alpha\beta}\ ,$$

$$[P(0^+)] = \begin{bmatrix} [P_{11}^{\omega\omega}(0^+)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} & [P_{12}^{\omega\varphi}(0^+)]_{\tau\rho\sigma}{}^{\alpha\beta} & [P_{13}^{\omega\varphi}(0^+)]_{\tau\rho\sigma}{}^{\alpha\beta} \\ [P_{21}^{\varphi\omega}(0^+)]_{\rho\sigma}{}^{\alpha\beta\gamma} & [P_{22}^{\varphi\varphi}(0^+)]_{\rho\sigma}{}^{\alpha\beta} & [P_{23}^{\varphi\varphi}(0^+)]_{\rho\sigma}{}^{\alpha\beta} \\ [P_{31}^{\varphi\omega}(0^+)]_{\rho\sigma}{}^{\alpha\beta\gamma} & [P_{32}^{\varphi\varphi}(0^+)]_{\rho\sigma}{}^{\alpha\beta} & [P_{33}^{\varphi\varphi}(0^+)]_{\rho\sigma}{}^{\alpha\beta} \end{bmatrix} ,$$

$$[P_{11}^{\omega\omega}(0^+)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} = \frac{2}{3}\, T_{\tau[\rho}\, L_{\sigma]}^{[\gamma}\, T^{\beta]\alpha} ,$$

$$[P_{12}^{\omega\varphi}(0^+)]_{\tau\rho\sigma}{}^{\alpha\beta} = \frac{\sqrt{2}}{3}\, T^{\alpha\beta}\, T_{\tau[\rho}\, \hat{q}_{\sigma]} ,$$

$$[P_{13}^{\omega\varphi}(0^+)]_{\tau\rho\sigma}{}^{\alpha\beta} = \sqrt{\frac{2}{3}}\, L^{\alpha\beta}\, T_{\tau[\rho}\, \hat{q}_{\sigma]} ,$$

$$[P_{21}^{\varphi\omega}(0^+)]_{\rho\sigma}{}^{\alpha\beta\gamma} = \frac{\sqrt{2}}{3}\, T_{\rho\sigma}\, T^{\alpha[\beta}\, \hat{q}^{\gamma]} ,$$

$$[P_{22}^{\varphi\varphi}(0^+)]_{\rho\sigma}{}^{\alpha\beta} = \frac{1}{3}\, T_{\rho\sigma}\, T^{\alpha\beta} ,$$

$$[P_{23}^{\varphi\varphi}(0^+)]_{\rho\sigma}{}^{\alpha\beta} = \frac{1}{\sqrt{3}}\, T_{\rho\sigma}\, L^{\alpha\beta} ,$$

$$[P_{31}^{\varphi\omega}(0^+)]_{\rho\sigma}{}^{\alpha\beta\gamma} = \sqrt{\frac{2}{3}}\, L_{\rho\sigma}\, T^{\alpha[\beta}\, \hat{q}^{\gamma]} ,$$

$$[P_{32}^{\varphi\varphi}(0^+)]_{\rho\sigma}{}^{\alpha\beta} = \frac{1}{\sqrt{3}}\, L_{\rho\sigma}\, T^{\alpha\beta} ,$$

$$[P_{33}^{\varphi\varphi}(0^+)]_{\rho\sigma}{}^{\alpha\beta} = L_{\rho\sigma}\, L^{\alpha\beta} ,$$

$$[P(1^+)] = \begin{bmatrix} [P_{11}^{\omega\omega}(1^+)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} & [P_{12}^{\omega\omega}(1^+)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} & [P_{13}^{\omega\chi}(1^+)]_{\tau\rho\sigma}{}^{\alpha\beta} \\ [P_{21}^{\omega\omega}(1^+)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} & [P_{22}^{\omega\omega}(1^+)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} & [P_{23}^{\omega\chi}(1^+)]_{\tau\rho\sigma}{}^{\alpha\beta} \\ [P_{31}^{\chi\omega}(1^+)]_{\rho\sigma}{}^{\alpha\beta\gamma} & [P_{32}^{\chi\omega}(1^+)]_{\rho\sigma}{}^{\alpha\beta\gamma} & [P_{33}^{\chi\chi}(1^+)]_{\rho\sigma}{}^{\alpha\beta} \end{bmatrix} ,$$

$$[P_{11}^{\omega\omega}(1^+)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} = T_\tau^\alpha\, T_{[\rho}^{[\beta}\, L_{\sigma]}^{\gamma]} - T_\tau^{[\gamma}\, L_{[\rho}^{\beta]}\, T_{\sigma]}^\alpha ,$$

$$[P_{12}^{\omega\omega}(1^+)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} = \sqrt{2}\, T_\tau^{[\gamma}\, T_{[\rho}^{\beta]}\, L_{\sigma]}^\alpha ,$$

$$[P_{13}^{\omega\chi}(1^+)]_{\tau\rho\sigma}{}^{\alpha\beta} = -\sqrt{2}\, T_\tau^{[\alpha}\, T_{[\rho}^{\beta]}\, \hat{q}_{\sigma]} ,$$

$$[P_{21}^{\omega\omega}(1^+)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} = \sqrt{2}\, L_\tau^{[\gamma}\, T_{[\rho}^{\beta]}\, T_{\sigma]}^\alpha ,$$

$$[P_{22}^{\omega\omega}(1^+)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} = L_\tau^\alpha\, T_{[\rho}^{[\beta}\, T_{\sigma]}^{\gamma]} ,$$

$$[P_{23}^{\omega\chi}(1^+)]_{\tau\rho\sigma}{}^{\alpha\beta} = \hat{q}_\tau\, T_{[\rho}^{[\alpha}\, T_{\sigma]}^{\beta]} ,$$

$$[P_{31}^{\chi\omega}(1^+)]_{\rho\sigma}{}^{\alpha\beta\gamma} = -\sqrt{2}\, T_{[\rho}^\alpha\, T_{\sigma]}^{[\beta}\, \hat{q}^{\gamma]} ,$$

$$[P_{32}^{\chi\omega}(1^+)]_{\rho\sigma}{}^{\alpha\beta\gamma} = \hat{q}^\alpha\, T_{[\rho}^{[\beta}\, T_{\sigma]}^{\gamma]} ,$$

$$[P_{33}^{\chi\chi}(1^+)]_{\rho\sigma}{}^{\alpha\beta} = T_{[\rho}^{[\alpha}\, T_{\sigma]}^{\beta]} ,$$

$$
[P(1^-)] = \begin{bmatrix} [P_{11}^{\omega\omega}(1^-)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} & [P_{12}^{\omega\omega}(1^-)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} & [P_{13}^{\omega\varphi}(1^-)]_{\tau\rho\sigma}{}^{\alpha\beta} & [P_{14}^{\omega\chi}(1^-)]_{\tau\rho\sigma}{}^{\alpha\beta} \\ [P_{21}^{\omega\omega}(1^-)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} & [P_{22}^{\omega\omega}(1^-)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} & [P_{23}^{\omega\varphi}(1^-)]_{\tau\rho\sigma}{}^{\alpha\beta} & [P_{24}^{\omega\chi}(1^-)]_{\tau\rho\sigma}{}^{\alpha\beta} \\ [P_{31}^{\varphi\omega}(1^-)]_{\rho\sigma}{}^{\alpha\beta\gamma} & [P_{32}^{\varphi\omega}(1^-)]_{\rho\sigma}{}^{\alpha\beta\gamma} & [P_{33}^{\varphi\varphi}(1^-)]_{\rho\sigma}{}^{\alpha\beta} & [P_{34}^{\varphi\chi}(1^-)]_{\rho\sigma}{}^{\alpha\beta} \\ [P_{41}^{\chi\omega}(1^-)]_{\rho\sigma}{}^{\alpha\beta\gamma} & [P_{42}^{\varphi\omega}(1^-)]_{\rho\sigma}{}^{\alpha\beta\gamma} & [P_{43}^{\chi\varphi}(1^-)]_{\rho\sigma}{}^{\alpha\beta} & [P_{44}^{\chi\chi}(1^-)]_{\rho\sigma}{}^{\alpha\beta} \end{bmatrix},
$$

$$[P_{11}^{\omega\omega}(1^-)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} = T_{\tau[\rho}\, T_{\sigma]}^{[\gamma}\, T^{\beta]\alpha} ,$$

$$[P_{12}^{\omega\omega}(1^-)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} = \sqrt{2}\, L^{\alpha[\beta}\, T_{\sigma]}^{[\gamma}\, T_{\rho]\tau} ,$$

$$[P_{13}^{\omega\varphi}(1^-)]_{\tau\rho\sigma}{}^{\alpha\beta} = \sqrt{2}\, T_{\tau[\rho}\, T_{\sigma]}^{(\alpha}\, \hat{q}^{\beta)} ,$$

$$[P_{14}^{\omega\chi}(1^-)]_{\tau\rho\sigma}{}^{\alpha\beta} = -\sqrt{2}\, T_{\tau[\rho}\, T_{\sigma]}^{[\alpha}\, \hat{q}^{\beta]} ,$$

$$[P_{21}^{\omega\omega}(1^-)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} = \sqrt{2}\, L_{\tau[\rho}\, T_{\sigma]}^{[\gamma}\, T^{\beta]\alpha} ,$$

$$[P_{22}^{\omega\omega}(1^-)]_{\tau\rho\sigma}{}^{\alpha\beta\gamma} = 2\, L_{\tau}^{\alpha}\, L_{[\rho}^{[\beta}\, T_{\sigma]}^{\gamma]} ,$$

$$[P_{23}^{\omega\varphi}(1^-)]_{\tau\rho\sigma}{}^{\alpha\beta} = 2\, \hat{q}_{\tau}\, L_{[\rho}^{(\alpha}\, T_{\sigma]}^{\beta)} ,$$

$$[P_{24}^{\omega\chi}(1^-)]_{\tau\rho\sigma}{}^{\alpha\beta} = 2\, \hat{q}_{\tau}\, L_{[\rho}^{[\alpha}\, T_{\sigma]}^{\beta]} ,$$

$$[P_{31}^{\varphi\omega}(1^-)]_{\rho\sigma}{}^{\alpha\beta\gamma} = \sqrt{2}\, T^{\alpha[\beta}\, T_{(\rho}^{\gamma]}\, \hat{q}_{\sigma)} ,$$

$$[P_{32}^{\varphi\omega}(1^-)]_{\rho\sigma}{}^{\alpha\beta\gamma} = 2\, \hat{q}^{\alpha}\, L_{(\rho}^{[\beta}\, T_{\sigma)}^{\gamma]} ,$$

$$[P_{33}^{\varphi\varphi}(1^-)]_{\rho\sigma}{}^{\alpha\beta} = 2\, T_{(\rho}^{(\alpha}\, L_{\sigma)}^{\beta)} ,$$

$$[P_{34}^{\varphi\chi}(1^-)]_{\rho\sigma}{}^{\alpha\beta} = -2\, T_{(\rho}^{[\alpha}\, L_{\sigma)}^{\beta]} ,$$

$$[P_{41}^{\chi\omega}(1^-)]_{\rho\sigma}{}^{\alpha\beta\gamma} = -\sqrt{2}\, T^{\alpha[\beta}\, T_{[\rho}^{\gamma]}\, \hat{q}_{\sigma]} ,$$

$$[P_{42}^{\varphi\omega}(1^-)]_{\rho\sigma}{}^{\alpha\beta\gamma} = 2\, \hat{q}^{\alpha}\, L_{[\rho}^{[\beta}\, T_{\sigma]}^{\gamma]} ,$$

$$[P_{43}^{\chi\varphi}(1^-)]_{\rho\sigma}{}^{\alpha\beta} = -2\, T_{[\rho}^{(\alpha}\, L_{\sigma]}^{\beta)} ,$$

$$[P_{44}^{\chi\chi}(1^-)]_{\rho\sigma}{}^{\alpha\beta} = 2\, T_{[\rho}^{[\alpha}\, L_{\sigma]}^{\beta]} .$$

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