ISTITUTO NAZIONALE DI FISICA NUCLEARE

Sezione di Trieste

INFN/AE-93/09

21 aprile 1993

P. Furlan, Y.S. Stanev and I.T. Todorov

EXCHANGE RELATIONS AND CORRELATION FUNCTIONS FOR A QUANTUM PARTICLE ON THE SU_2 GROUP MANIFOLD

Servizio Documentazione
dei Laboratori Nazionali di Frascati

EXCHANGE RELATIONS AND CORRELATION FUNCTIONS FOR A QUANTUM PARTICLE ON THE SU_2 GROUP MANIFOLD

PAOLO FURLAN(1)
YASSEN S. STANEV(2)* AND IVAN T. TODOROV(2)*1

(1). Dipartimento di Fisica Teorica dell' Universitá di Trieste and I.N.F.N. Sezione di Trieste, Italy
(2). The Erwin Schrödinger International Institute for Mathematical Physics (ESI), Vienna, Austria.

ABSTRACT. We consider the SO_4 invariant quantum dynamics of a point particle moving on the 3-sphere (or equivalently, of the relative motion of the spherical top). Quantum exchange relations for different times are derived with an "R matrix" depending on the time difference and on the conserved angular momentum. Their implications for correlation functions are spelled out. The chiral exchange relations of Alekseev and Faddeev [1] are also extended to different times.

INTRODUCTION

A fresh view of the dynamics of a point particle moving on a group manifold G can serve both as an introduction to G current algebra models and as a part of their study – concerning the zero modes (for a sample of references on the Hamiltonian approach to such models – see [2,3,4]). Alekseev and Faddeev [1] presented an R-matrix treatment of the phase space $\Gamma = T^*SU_2$ with an emphasis on its splitting into chiral parts which admit a natural quantum group deformation. Here we present a manifestly $SO(4)$ invariant solution of the quantum mechanical model (independent of the choice of splitting into right and left movers). It is pointed that the $SO(4)$ spectrum of the state space coincides with that of the non-relativistic hydrogen atom. The operator form of the solution

$$g(t) = e\left(-\left(\ell + \frac{3}{4}\hbar\right)t\right) g(0), \quad e(x) = e^{ix},$$

where $g \in SU_2$, $\ell = \ell_1\sigma_1 + \ell_2\sigma_2 + \ell_3\sigma_3$ is the right invariant angular momentum matrix that is a (hermitian) element of the Lie algebra su_2, allows one to compute correlation functions of $g(t)$ at different times. (We work out the example of the 4-point function.) Our main result is the derivation of generalized "exchange relations"

$$g(t_2) \otimes g(t_1) = R_{12}\, g(t_1) \otimes g(t_2) = g(t_1) \otimes g(t_2)\, \widetilde{R}_{12}$$

[1]* On leave from the Institute for Nuclear Research, Bulgarian Academy of Sciences, Tsarigradsko Chaussee 72, BG-1784 Sofia, Bulgaria.

where the "R-matrix" (or rather "$6-j$ symbol") depends on the time difference $t_{12} = t_1 - t_2$ (playing the role of a spectral parameter) and on the conserved right (or left) invariant angular momentum ℓ (or $\tilde{\ell}$). The representatives of the resulting R-matrices with operator valued entries on a set of (permuted) n-point correlation functions satify a generalized Yang-Baxter equation (Eq. (3.19) below). $\widetilde{R}_{12}$ appears to provide an example for an R-matrix depending on a spectral parameter in the framework of quasi-coassociative bialgebras (cf. [5])[1]. In Sec. 4 we review the results of ref. [1] on the chiral R-matrices of the "top-model" and extend them too to different times.

1. The phase space $\Gamma = T^*SU_2$ imbedded in $T^*\mathbf{C}^2$

Writing the SU_2 group element $g = (g_\alpha^\beta)$ as a pair of conjugate 2-spinors

$$g = \begin{pmatrix} w^1 & w^2 \\ -w_2^* & w_1^* \end{pmatrix} \tag{1.1a}$$

or

$$g_1^\alpha = w^\alpha, \quad g_2^\beta = w_\alpha^* \epsilon^{\alpha\beta}, \quad (\epsilon^{\alpha\beta}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{1.1b}$$

we find that gg^* is a multiple of the unit matrix

$$gg^* = \det(g)\cdot 1, \quad \det g = w^* w \ \ (= w_\alpha^* w^\alpha). \tag{1.2}$$

Thus, the configuration space $SU_2 \approx S^3$ appears as a real hypersurface in $\mathbf{C}^2$ given by the constraint equation

$$S^3 : ww^* - 1 = 0 \quad (w \in \mathbf{C}^2). \tag{1.3}$$

We shall derive the PB structure on Γ from the canonical PB on $T^*\mathbf{C}^2$. Let

$$p = \begin{pmatrix} \pi^{*1} & \pi^{*2} \\ -\pi_2 & \pi_1 \end{pmatrix} \tag{1.4}$$

be the canonical momentum matrix. The non-zero PB on $T^*\mathbf{C}^2$ are

$$\{w^\alpha, \pi_\beta\} = \delta_\beta^\alpha = \{w_\beta^*, \pi^{*\alpha}\}. \tag{1.5}$$

The restriction to Γ yields the secondary constraint

$$2\mu \equiv \mathrm{tr}(gp^*) = w\pi + w^*\pi^* = 0, \quad \{w^*w - 1, \mu\} = 1. \tag{1.6}$$

Rather than computing the Dirac brackets for $\pi^{(*)}$ and $w^{(*)}$ we shall single out a subalgebra $\mathcal{A}(\Gamma)$ of the algebra of functions on $T^*\mathbf{C}^2$ whose Dirac brackets

[1] We thank Volker Schomerus for this remark.

coincide with the original PB. To this end we introduce the right invariant angular momentum

$$\ell = \begin{pmatrix} \ell_3 & \ell_- \\ \ell_+ & -\ell_3 \end{pmatrix} = i\,pg^* \quad \text{or} \quad \ell_a = \frac{i}{2}\operatorname{tr}(pg^*\sigma_a) \quad a = 1,2,3, \tag{1.7a}$$

i.e.

$$i\ell_+ = w^*\epsilon\pi, \quad i\ell_- = \pi^*\epsilon w, \quad 2i\ell_3 = w\pi - w^*\pi^* \tag{1.7b}$$

and its left invariant counterpart

$$\tilde{\ell} = -ig^*p \;\; (= ip^*g) = -g^*\ell g \;\; (gg^* = 1) \tag{1.8a}$$

or

$$\tilde{\ell}_a = \frac{1}{2i}\operatorname{tr}(g^*p\sigma_a), \quad \tilde{\ell}_+ = i(w^1\pi_2 - w_2^*\pi^{*1}) \;\; \text{etc.} \tag{1.8b}$$

ℓ and $\tilde{\ell}$ generate left and right infinitesimal SU_2 shifts:

$$i\{\ell_a, g\} = -\frac{1}{2}\sigma_a g, \quad i\left\{\tilde{\ell}_a, g\right\} = \frac{1}{2}g\sigma_a \tag{1.9}$$

(and similar formulae for the PB with p). They imply that the angular momenta have vanishing PB with the constraints,

$$\{\ell_a, w^*w\} = 0 = \{\ell_a, \mu\}, \tag{1.10}$$

(and similar relations for $\tilde{\ell}$), thus appearing as gauge invariant observables corresponding to vector fields tangent to Γ. They span among themselves the $SU_2 \times SU_2$ PB Lie algebra:

$$\{\ell_a, \ell_b\} = \epsilon_{abc}\,\ell_c, \quad \left\{\tilde{\ell}_a, \tilde{\ell}_b\right\} = \epsilon_{abc}\,\tilde{\ell}_c, \tag{1.11}$$

$$\left\{\ell_a, \tilde{\ell}_b\right\} = 0. \tag{1.12}$$

The similarity relation (1.8a) between $-\ell$ and $\tilde{\ell}$ implies that left and right angular momentum squares coincide:

$$\frac{1}{2}\operatorname{tr}\ell^2 = L^2\,(= \ell_a\ell^a) = \frac{1}{2}\operatorname{tr}\tilde{\ell}^2. \tag{1.13}$$

The subalgebra $\mathcal{A}(\Gamma)$ with the above mentioned property is generated by ℓ, $\tilde{\ell}$ and g subject to the relations (1.9) (1.11-13) and

$$\{g \overset{\otimes}{,} g\} = 0. \tag{1.14}$$

(The variable μ that serves to define the secondary constraint (1.6) is excluded from $\mathcal{A}(\Gamma)$.)

A general $SU_2 \times SU_2$ invariant Hamiltonian is a function of L^2 and of the constraint (1.3). For an appropriate choice of the time parameter we can write

$$H = L^2 + \lambda(w^* w - 1). \tag{1.15}$$

This implies conservation of angular momentum

$$\dot{\ell} = 0 = \dot{\tilde{\ell}} \tag{1.16}$$

and yields the Lagrangean

$$\begin{aligned} \mathcal{L} &= i\,\mathrm{tr}(\ell \dot{g} g^*) - H = -i\,\mathrm{tr}(\tilde{\ell} g^* \dot{g}) - H = \\ &= \mathrm{tr}\left\{ i\ell \dot{g} g^* - \frac{1}{2}L^2 - \frac{\lambda}{2}(gg^* - 1)\right\} = \\ &= \frac{1}{2}\,\mathrm{tr}(\dot{g}\dot{g}^* - \lambda(gg^* - 1)) = \\ &= \dot{w}\dot{w}^* - \lambda(ww^* - 1) \end{aligned} \tag{1.17}$$

which allows to identify the linear momentum (1.4) with $\dot{g}$ and the angular momenta ℓ and $\tilde{\ell}$ with (the traceless parts of) $i\dot{g}g^*$ and $-ig^*\dot{g}$.

2. Quantum $\mathcal{A}(\Gamma)$: equations of motion and their solution; vacuum representation

We define the quantum algebra $\mathcal{A}_\hbar = \mathcal{A}_\hbar(\Gamma)$ as the algebra generated by ℓ, $\tilde{\ell}$ and g with the PB (1.9) (1.11) (1.12) and (1.14) represented by commutators according to the rule

$$i\hbar\{\ ,\ \} \to [\ ,\ ]. \tag{2.1}$$

The deformation parameter $\hbar$ also appears in the matrix counterpart of (1.13),

$$H = \ell^2 + \hbar\ell = (\ell_1^2 + \ell_2^2 + \ell_3^2)\mathbf{1} = L(L+\hbar)\mathbf{1} = \tilde{\ell}^2 + \hbar\tilde{\ell}, \tag{2.2}$$

and in the equations of motion:

$$i\dot{g}(t) = \hbar^{-1}[g(t), H] = -\frac{1}{2}\left[g(t), \tilde{\ell}_a\right]_+ \sigma_a = -g(t)\left(\tilde{\ell} + \frac{3}{4}\hbar\right) = \left(\ell + \frac{3}{4}\hbar\right) g(t). \tag{2.3}$$

While the quantum matrix equations $\ell = i\dot{g}g^*$, $\tilde{\ell} = -ig^*\dot{g}$ require renormalization (subtraction of the trace in the right hand side) the counterpart of (1.8b) remains unchanged:

$$\tilde{\ell}_a = \frac{1}{2i}\,\mathrm{tr}(g^*\dot{g}\sigma_a), \quad \ell_a = \frac{i}{2}\,\mathrm{tr}(\dot{g}g^*\sigma_a). \tag{2.4}$$

If one uses the "row spinors" (1.1) then the left and right invariant momenta $\tilde{\ell}$ and ℓ assume a rather different role. While $\tilde{\ell}$ displays the spinorial character of w,

$$\left[w, \tilde{\ell}_a\right] = -\frac{\hbar}{2} w \sigma_a, \qquad \left[w^*, \tilde{\ell}_a\right] = \frac{\hbar}{2} \sigma_a w^*, \tag{2.5}$$

$2\ell_3$ plays the role of a charge operator (such that w^* and w appear as positively respectively negatively charged):

$$2[\ell_3, w] = -\hbar w, \qquad 2[\ell_3, w^*] = \hbar w^*. \tag{2.6}$$

$\mathcal{A}_\hbar(\Gamma)$ admits an antilinear involution ϑ, "the TCP symmetry", such that

$$\vartheta(w(t)) = w^*(-t), \qquad \vartheta(w^*(t)) = w(-t), \tag{2.7a}$$

$$\vartheta(\ell_a) = -\ell_a, \qquad \vartheta(\tilde{\ell}_a) = -\tilde{\ell}_a. \tag{2.7b}$$

We shall view the elements of $\mathcal{A}_\hbar(\Gamma)$ as operators in the vacuum Hilbert space $\mathcal{H}$ with a unique $SU_2 \times SU_2$ invariant state $\langle\, 0 \mid \ \mid 0\,\rangle$ such that $\mid 0\,\rangle$ is a cyclic vector with respect to $\mathcal{A}_\hbar(\Gamma)$ and

$$\ell_a \mid 0\,\rangle = 0 = \tilde{\ell}_a \mid 0\,\rangle. \tag{2.8}$$

The involution $\vartheta(A)$ is then implemented by an antiunitary operator Θ such that

$$\Theta \mid 0\,\rangle = \mid 0\,\rangle, \qquad \Theta A\, \Theta^{-1} = \vartheta(A), \qquad \Theta^2 = 1. \tag{2.9}$$

Charge conservation implies that only even point correlation functions with an equal number of w and w^* can be nonvanishing. Θ invariance and antiunitarity

$$\langle\, \Theta\Phi \mid \Theta\Psi\,\rangle = \overline{\langle\, \Phi \mid \Psi\,\rangle} = \langle\, \Psi \mid \Phi\,\rangle, \tag{2.10}$$

allow to relate correlation functions with opposite order of factors, e.g.

$$\begin{aligned} &\langle\, 0 \mid \overset{1}{w}(t_1)\, (\overset{2}{w})^*(t_2) \cdots \overset{2n-1}{w}(t_{2n-1})\, (\overset{2n}{w})^*(t_{2n}) \mid 0\,\rangle = \\ &= \langle\, 0 \mid (\overset{2n}{w})^*(-t_{2n})\ \overset{2n-1}{w}(-t_{2n-1}) \cdots (\overset{2}{w})^*(-t_2)\ \overset{1}{w}(-t_1) \mid 0\,\rangle. \end{aligned} \tag{2.11}$$

($\overset{i}{w}{}^{(*)}$ stands, as usual, for the ith factor in a 2n-fold tensor product.)

Using the solution of the equations of motion (2.3),

$$g(t) = g\, e\left(\left(\tilde{\ell} + \frac{3}{4}\hbar\right) t\right) \ \left(= e\left(-\left(\ell + \frac{3}{4}\hbar\right) t\right) g\right), \qquad g \equiv g(0) \tag{2.12a}$$

or

$$w(t) = w\, e\left(\left(\tilde{\ell} + \frac{3}{4}\hbar\right) t\right), \qquad w^*(t) = e\left(-\left(\tilde{\ell} + \frac{3}{4}\hbar\right) t\right) w^*, \tag{2.12b}$$

where

$$e(x) = e^{ix} \Rightarrow e\left(\left(\tilde{\ell} + \frac{1}{2}\hbar\right) t\right) = \cos\left(\frac{N}{2}\hbar t\right) + i\frac{2\ell + \hbar}{N\hbar}\sin\left(\frac{N}{2}\hbar t\right), \quad (2.13a)$$

$$N^2\hbar^2 = (2L + \hbar)^2, \quad (2.13b)$$

Eq. (2.12b) and Θ and time translation invariance allow us to write

$$\langle\, 0 \,|\, w^{\alpha}(t_1) w^*_{\beta}(t_2) \,|\, 0 \,\rangle = \langle\, 0 \,|\, w^*_{\beta}(t_1) w^{\alpha}(t_2) \,|\, 0 \,\rangle = \frac{1}{2}\, e\Big(-\frac{3}{4}\hbar t_{12}\Big)\, \delta^{\alpha}_{\beta}, \quad t_{12} = t_1 - t_2. \quad (2.14)$$

The normalization is dictated by the (quantum) constraint (1.3). One can also rewrite (2.14) in terms of g:

$$\langle\, 0 \,|\, g^{\beta_1}_{\alpha_1}(t_1)\, g^{\beta_2}_{\alpha_2}(t_2) \,|\, 0 \,\rangle = \frac{1}{2}\, \epsilon_{\alpha_1\alpha_2}\, \epsilon^{\beta_1\beta_2}\, e\Big(-\frac{3}{4}\hbar t_{12}\Big), \quad (2.15)$$

where ϵ is the unit antisymmetric tensor (1.1b). The 2n-point function is uniquely determined from the initial (equal time) condition

$$\langle\, 0 \,|\, g^{\beta_1}_{\alpha_1} \ldots g^{\beta_{2n}}_{\alpha_{2n}} \,|\, 0 \,\rangle = \frac{1}{(n+1)!} \sum \prod_{i<j} \epsilon_{\alpha_i\alpha_j}\, \epsilon^{\beta_i\beta_j}; \quad (2.16a)$$

in terms of $w^{(*)}$ the non-vanishing equal time expectation values are expressed in terms of the Euler beta-function:

$$\Big\langle\, 0 \,\Big|\, (w^1 w^*_1)^m\, (w^2 w^*_2)^n \Big|\, 0 \,\Big\rangle = B(m+1, n+1) = \frac{m!\, n!}{(m+n+1)!}; \quad (2.16b)$$

(the sum in (2.16a) is spread over $(2n-1)!!$ different products of $2n$ ϵ factors each). For instance, the 4-point function has the form

$$\langle\, 0 \,|\, \overset{1}{g}(t_1)\, \overset{2}{g}(t_2)\, \overset{3}{g}(t_3)\, \overset{4}{g}(t_4) |\, 0 \,\rangle = \langle\, 0 \,|\, \overset{1}{g}\overset{2}{g}\overset{3}{g}\overset{4}{g} \,|\, 0 \,\rangle\, e\left(-\frac{\hbar}{4}(3t_{14} + (1 + 4P_{34})t_{23})\right) \quad (2.17)$$

where P_{34} is the operator permuting the indices $\beta_3\beta_4$ such that

$$(P_{34})^2 = 1 \quad \text{and} \quad \langle\, 0 \,|\, \overset{1}{g}\overset{2}{g}\overset{3}{g}\overset{4}{g} \,|\, 0 \,\rangle (P_{12} - P_{34}) = 0 \quad (2.18)$$

for

$$\langle\, 0 \,|\, \overset{1}{g}\overset{2}{g}\overset{3}{g}\overset{4}{g} \,|\, 0 \,\rangle =$$

$$= \frac{1}{6}\left(\epsilon_{\alpha_1\alpha_2}\epsilon_{\alpha_3\alpha_4}\epsilon^{\beta_1\beta_2}\epsilon^{\beta_3\beta_4} + \epsilon_{\alpha_1\alpha_3}\epsilon_{\alpha_2\alpha_4}\epsilon^{\beta_1\beta_3}\epsilon^{\beta_2\beta_4} + \epsilon_{\alpha_1\alpha_4}\epsilon_{\alpha_2\alpha_3}\epsilon^{\beta_1\beta_4}\epsilon^{\beta_2\beta_3}\right). \quad (2.19)$$

Due to the identity

$$\epsilon^{\beta_1\beta_3}\epsilon^{\beta_2\beta_4} = \epsilon^{\beta_1\beta_2}\epsilon^{\beta_3\beta_4} + \epsilon^{\beta_1\beta_4}\epsilon^{\beta_2\beta_3} \tag{2.20}$$

only two out of the three structures appearing in (2.19) are independent.

The spectrum of our Hamiltonian (1.15), the angular momentum square, is $s(s+1)\hbar^2$, where $s = 0, \frac{1}{2}, 1, \dots$ so that the spectrum of the operator N introduced in (2.13) runs over the natural numbers. The integrability of the model is reflected in the existence of three commuting integrals of motion, L^2, ℓ_3, and $\tilde{\ell}_3$. The state space $\mathcal{H}$ is spanned by the canonical basis

$$|\, N\, s_3\, \tilde{s}_3 \,\rangle, \qquad (\ell_3 - \hbar s_3)\,|\, N\, s_3\, \tilde{s}_3 \,\rangle = 0 = (\tilde{\ell}_3 - \hbar \tilde{s}_3)\,|\, N\, s_3\, \tilde{s}_3 \,\rangle \tag{2.21}$$

where

$$\frac{1-N}{2} \leq \overset{(\sim)}{s}_3 \leq \frac{N-1}{2}, \qquad N = 1, 2, \dots, \tag{2.22}$$

the eigenvalue N being thus N^2 fold degenerate. We observe that $\mathcal{H}$ has the $SO(4)$ structure of the space of bound states of the non-relativistic hydrogen atom. In fact, the energy spectrum of the hydrogen atom is reproduced by the Hamiltonian

$$H_{\rm hydr} = Mc^2 - \frac{m\alpha^2}{2N^2}. \tag{2.23}$$

3. A quantum R-matrix with operator valued entries

The operators $g(t)$ satisfy for different times a generalized exchange relation of the type

$$\overset{2}{g}(t_2)\; \overset{1}{g}(t_1) = R_{12}\; \overset{1}{g}(t_1)\; \overset{2}{g}(t_2) = \overset{1}{g}(t_1)\; \overset{2}{g}(t_2)\, \widetilde{R}_{12}, \tag{3.1}$$

where R_{12} and $\widetilde{R}_{12}$ depend on the time difference t_{12} and on the conserved right and left invariant angular momenta ℓ_a and $\tilde{\ell}_a$, respectively. To construct $\widetilde{R}_{12}$ and thus derive (3.1) we express $\overset{1}{g}(t_1)$ in terms of $\overset{1}{g}(t_2)$ and use the commutatativity of g at equal times:

$$\begin{aligned} &\overset{2}{g}(t_2)\; \overset{1}{g}(t_1) = \overset{1}{g}(t_2)\; \overset{2}{g}(t_2)\, e\left(\left(\overset{1}{\tilde{\ell}} + \frac{3}{4}\hbar\right) t_{12}\right) = \\ &= \overset{1}{g}(t_1)\, e\left(-\overset{1}{\tilde{\ell}}\, t_{12}\right) \overset{2}{g}(t_2)\, e\left(\overset{1}{\tilde{\ell}}\, t_{12}\right) = \overset{1}{g}(t_1)\; \overset{2}{g}(t_2)\, U_{12}\, e\left(\overset{1}{\tilde{\ell}}\, t_{12}\right) \end{aligned}$$

where U_{12} is a unitary operator defined by

$$e\left(-\overset{1}{\tilde{\ell}}\, t_{12}\right) \overset{2}{g}(t_2) = \overset{2}{g}(t_2)\, U_{12}$$

for

$$\overset{1}{\tilde{\ell}}\overset{2}{g} = \overset{2}{g}\left(\overset{1}{\tilde{\ell}} + \hbar\left(P - \frac{1}{2}\right)\right), \tag{3.2}$$

where $P = P_{12}$ is a permutation operator related to the su_2 Casimir invariant in the tensor product space:

$$\overset{1}{\underline{\sigma}} \times \overset{2}{\underline{\sigma}} = 2P - \mathbf{1}. \tag{3.3}$$

Noting that P is involutive and interchanges particle momenta,

$$P^2 = \mathbf{1}, \qquad P\overset{1}{\tilde{\ell}} = \overset{2}{\tilde{\ell}}P \tag{3.4}$$

and using the identities

$$\left(\overset{1}{\tilde{\ell}} + \overset{2}{\tilde{\ell}}\right)(P - \mathbf{1}) = 0, \tag{3.5a}$$

which implies $\left[\overset{1}{\tilde{\ell}}\overset{2}{\tilde{\ell}} + \overset{2}{\tilde{\ell}}\overset{1}{\tilde{\ell}} + 2L(L+\hbar)\right](P - \mathbf{1}) = 0$, and

$$\hbar\left[P, \overset{1}{\tilde{\ell}}\right] - \left[\overset{1}{\tilde{\ell}}, \overset{2}{\tilde{\ell}}\right] = \overset{2}{\tilde{\ell}}\overset{1}{\tilde{\ell}} - \overset{1}{\tilde{\ell}}\overset{2}{\tilde{\ell}} + \hbar\left(\overset{2}{\tilde{\ell}} - \overset{1}{\tilde{\ell}}\right)P = 0, \tag{3.5b}$$

we deduce from (2.13) and the above calculation that $\tilde{R}_{12}$ can be written in the form

$$e\left(-\frac{\hbar}{2}t_{12}\right)\tilde{R}_{12} = F_1 + F_2P + F_3\left(\overset{1}{\tilde{\ell}} + \overset{2}{\tilde{\ell}}\right) + F_4\overset{2}{\tilde{\ell}}P + F_5\overset{2}{\tilde{\ell}}\overset{1}{\tilde{\ell}} \equiv \mathcal{F}_{12} \tag{3.6a}$$

where the exponential has been factored out for later convenience and

$$F_i = F_i(t_{12}, N^2), \qquad N^2\hbar^2 = (2L + \hbar)^2. \tag{3.6b}$$

To compute F_i we differentiate (3.1) with respect to t_1 finding

$$\overset{1}{g}(t_1)\overset{2}{g}(t_2)\left\{\dot{\mathcal{F}}_{12} + i\left[\overset{1}{\tilde{\ell}}, \mathcal{F}_{12}\right] + i\hbar P\mathcal{F}_{12}\right\} = 0. \tag{3.7}$$

To see that the structures in (3.6a) are reproduced we use the identity

$$\left[\overset{1}{\tilde{\ell}}, \overset{2}{\tilde{\ell}}\right] = i\hbar\overset{1}{\underline{\sigma}} \times \overset{2}{\underline{\sigma}}\tilde{\underline{\ell}} = \hbar\left(\overset{2}{\tilde{\ell}} - \overset{1}{\tilde{\ell}}\right)P. \tag{3.8}$$

The resulting system of ordinary differential equations for F_i with the initial condition

$$\widetilde{R}_{12}(0) = \mathbf{1} \Leftrightarrow F_i(0, N^2) = \delta_{i1}, \tag{3.9}$$

has a unique solution given by

$$\begin{aligned} \hbar^2 F_5 &= \frac{2}{N^2-1}\left(e^{-i\hbar t} - \frac{1}{N^2}\cos N\hbar t + \frac{i}{N}\sin N\hbar t\right) - \frac{2}{N^2} = \\ &= -\frac{i}{3}\hbar^3 t^3 - \frac{\hbar^4}{12}t^4 + i\,\frac{N^2+1}{60}\,\hbar^5 t^5 + \ldots, \end{aligned} \tag{3.10a}$$

$$\begin{aligned} \hbar F_4 = i\hbar \dot{F}_5 &= \frac{2}{N^2-1}\left(-\cos N\hbar t + \frac{i}{N}\sin N\hbar t + e^{-i\hbar t}\right) = \\ &= \hbar^2 t^2 - i\frac{\hbar^3}{3}t^3 - \frac{N^2+1}{12}\,\hbar^4 t^4 + \ldots, \end{aligned} \tag{3.10b}$$

$$\hbar F_3 = \frac{\cos N\hbar t - 1}{N^2} = -\frac{1}{2}\hbar^2 t^2 + \frac{N^2}{24}\,\hbar^4 t^4 + \ldots, \tag{3.10c}$$

$$F_2 = -i\,\frac{\sin N\hbar t}{N} = -i\hbar t + i\frac{N^2}{6}\,\hbar^3 t^3 + \ldots, \tag{3.10d}$$

$$\begin{aligned} F_1 &= 1 + \frac{\cos N\hbar t - 1}{N^2} + \hbar^2 F_5 \frac{N^2-1}{4} = \\ &= 1 - \frac{1}{2}\hbar^2 t^2 - i\frac{N^2-1}{12}\,\hbar^3 t^3 + \ldots. \end{aligned} \tag{3.10e}$$

The operator $\mathcal{F}_{12}$ (just as well as R_{12}) goes into its inverse if we interchange the labels 1 and 2

$$\mathcal{F}_{12}(t_{12})\,\mathcal{F}_{21}(t_{21}) = \mathbf{1} = \mathcal{F}_{12}(t_{12})\,(\mathcal{F}_{12}(t_{12}))^*, \tag{3.11}$$

in accord with the involutivity of (3.1).

One can derive a similar relation for $R_{12} = R_{12}(t_{12}; \ell_a)$ (of (3.1)); the result is

$$\overset{2}{g}(t_2)\ \overset{1}{g}(t_1) = R_{12}\ \overset{1}{g}(t_1)\ \overset{2}{g}(t_2), \quad R_{12}(t_{12}; \ell) = \widetilde{R}_{12}(t_{12}; -\tilde{\ell}). \tag{3.12}$$

Acting on the right and left vacuum R_{12} and $\widetilde{R}_{12}$ reduce to the same numerical matrix:

$$\begin{aligned} \widetilde{R}_{12}|\,0\,\rangle &= e^{-i\hbar(P-\frac{1}{2})t_{12}}|\,0\,\rangle, \\ \langle\,0\,|R_{12} &= \langle\,0\,|e^{-i\hbar(P-\frac{1}{2})t_{12}}, \end{aligned} \tag{3.13a}$$

the vacuum value of $\mathcal{F}_{12}$ in both cases being

$$\langle\,\mathcal{F}_{12}(t)\,\rangle = e^{-i\hbar Pt} = \cos\hbar t - iP\sin\hbar t. \tag{3.13b}$$

More generally, we can compute the action of $R_{i\,i+1}$ on correlation functions. The above analysis tells us that the resulting expressions for $\mathcal{F}_{12}$ and $\mathcal{F}_{2n-1\,2n}$ are given by (3.13b) with $P = P_{12}$ and $P = P_{2n-1\,2n}$ (and $t = t_{12}$, $t = t_{2n-1\,2n}$), respectively. It is instructive to compute as a less trivial example the middle R-matrix, R_{23}, for the 4-point function (2.17), (2.19).

We first note, using (2.19) and (2.20), that each three permutations P_{ij}, for either $i, j = 2, 3, 4$ or $i, j = 1, 2, 3$, acting on the equal time 4-point function are linearly dependent:

$$\langle\,0\,|\,\overset{1}{g}\overset{2}{g}\overset{3}{g}\overset{4}{g}\,|\,0\,\rangle(P_{23}+P_{34}+P_{24}) = 0 = \langle\,0\,|\,\overset{1}{g}\overset{2}{g}\overset{3}{g}\overset{4}{g}\,|\,0\,\rangle(P_{12}+P_{23}+P_{13}). \tag{3.14a}$$

Indeed, this is true for any one of the three terms in the right hand side of (2.19). Denoting the matrix $\left(\epsilon^{\beta_i\beta_j}\right)$ by $\epsilon(ij)$ we can write

$$\epsilon(1i)\,\epsilon(jk)\,(P_{23} + P_{34} + P_{24}) = 0 = \epsilon(ij)\,\epsilon(k4)\,(P_{12} + P_{23} + P_{13}) \tag{3.14b}$$

for any permutation (i, j, k) of (2,3,4) (respectively (1,2,3)). Furthermore, we note the relations

$$(N-2)\,g\,|\,0\,\rangle = 0 = \left(\overset{i}{\tilde{\ell}}\overset{4}{g} - \hbar\,\overset{4}{g}\left(P_{i4} - \frac{1}{2}\right)\right)\,|\,0\,\rangle, \quad i = 2, 3. \tag{3.15}$$

As a result we find the following equivalent realizations of $\mathcal{F}_{23}$ in the space of 4-point functions:

$$\langle\,\widetilde{\mathcal{F}}_{23}\,\rangle_4 = e(\hbar\,t_{23}\,P_{34})\,e(\hbar\,t_{23}\,P_{24}), \tag{3.16a}$$

$$\langle\,\mathcal{F}_{23}\,\rangle_4 = e(\hbar\,t_{23}\,P_{12})\,e(\hbar\,t_{23}\,P_{13}) \tag{3.16b}$$

(the first acting from the right, the second from the left of (2.17)). The matrix R_{13} defined as the ratio of corresponding 4-point functions (2.17),

$$R_{13} = e(\frac{\hbar}{2}\,t_{13})\mathcal{F}_{13}, \tag{3.17a}$$

where we omit the brackets indicating matrix elements and set

$$\mathcal{F}_{13} = e(\frac{\hbar}{2}\,t_{12}\,P_{23})\,e(\frac{\hbar}{2}\,t_{23}\,P_{12}), \tag{3.17b}$$

depends on the middle point t_2 (rather than just on the time difference t_{13}). The R_{ij} (and $\mathcal{F}_{ij}$) so defined are verified to satisfy the relations

$$R_{i\,i+1}(t)\, R_{i\,i+1}(-t) = \mathbf{1}, \tag{3.18}$$

$$R_{12}^{(123)}(t_{12})\, R_{13}^{(213)}(t_{13})\, R_{23}^{(231)}(t_{23}) = R_{13}^{(123)} = R_{23}^{(123)}(t_{23})\, R_{13}^{(132)}(t_{13})\, R_{12}^{(312)}(t_{12}), \tag{3.19}$$

where the upper indices (ijk) stand for the order of $\overset{i}{g}(t_i)$ to which R is applied.

We note that such generalized Yang-Baxter equations that reflect the operator dependence of R are reminiscent to the relations found by Mack and Schomerus in their study of quasi co-associative quantum symmetries [5].

The "quantum R matrix" allows to obtain its classical counterpart. Setting

$$\left\{\overset{1}{g}(t_1),\, \overset{2}{g}(t_2)\right\} = \lim_{\hbar\to 0} \frac{i}{\hbar}\left[\overset{2}{g}(t_2),\, \overset{1}{g}(t_1)\right] \tag{3.20}$$

we find

$$\left\{\overset{1}{g}(t_1),\, \overset{2}{g}(t_2)\right\} = \overset{1}{g}(t_1)\ \overset{2}{g}(t_2)\, \tilde{r}(t_{12}, \tilde{\ell}), \tag{3.21}$$

where

$$\tilde{r}(t, \tilde{\ell}) = tP - \frac{i}{2} t^2 \left(\overset{1}{\tilde{\ell}} + \overset{2}{\tilde{\ell}}\right) + i t^2\, \overset{2}{\tilde{\ell}}\, P + \frac{1}{3} t^3\, \overset{2}{\tilde{\ell}}\overset{1}{\tilde{\ell}}, \tag{3.22}$$

or a similar expression with $r(t, \ell) = \tilde{r}(t, -\ell)$ acting on the left. We can also write a simpler looking (linear in t) expression for the PB on the price of having an operator acting on both sides of $\overset{1}{g}\overset{2}{g}$:

$$\begin{aligned} \left\{\overset{1}{g}(t_1),\, \overset{2}{g}(t_2)\right\} = {} & \frac{t_{12}}{2}\ \overset{1}{g}(t_1)\ \overset{2}{g}(t_2)\, L^{-2}\, \overset{1}{\tilde{\ell}}\overset{2}{\tilde{\ell}} - \\ & - \frac{i}{4L^2}\left[P,\, \overset{1}{g}(t_1)\, \overset{2}{g}(t_2)\left(\overset{1}{\tilde{\ell}} - \overset{2}{\tilde{\ell}}\right)\right]_+ \end{aligned} \tag{3.23}$$

(where $[\ ,\ ]_+$ stands for an anticommutator). Eq. (3.23) is verified directly: differentiating in t_1, using the differential equation for g and checking the initial conditions.

4. Splitting into left and right invariant chiral models.

The 1-form $i\,\mathrm{tr}\ell dgg^{-1}$, whose differential defines the symplectic structure of the phase space Γ, can be split into left and right invariant parts, in parallel to the splitting of the basic 2-form of the Wess-Zumino-Novikov-Witten model into chiral parts-see [6,7]. We shall briefly review the treatment of this problem in [1] supplementing it with an expression for the different time PB and for the quantum exchange relations in the chiral theory.

Let u and v be unitary 2×2 matrices that relate ℓ and $\tilde{\ell}$ to the diagonal matrix $\pm L\sigma_3$:

$$\ell = uL\sigma_3 u^*, \quad (\mathbf{1} \det u =) uu^* = \mathbf{1}, \tag{4.1a}$$

$$\tilde{\ell} = -v^* L\sigma_3 v, \quad vv^* = \mathbf{1}. \tag{4.1b}$$

This is consistent with the relation $\ell g = -g\tilde{\ell}$ for

$$g = uv. \tag{4.2}$$

The basic 1-form in Γ then splits as follows:

$$i\,\mathrm{tr}\ell dgg^{-1} = i\,\mathrm{tr}\{L\sigma_3(u^{-1}du + dvv^{-1})\}. \tag{4.3}$$

The conditions (4.1) and (4.2) do not fix u and v in a unique manner: they leave room for a (time dependent) gauge transformation

$$u \to u\mathcal{D}, \quad v \to \mathcal{D}^{-1}v, \quad \mathcal{D} = \mathcal{D}(t)$$

with $\mathcal{D}$ any non-singular diagonal matrix. At the classical level u and v can be parametrized in terms of the angular momenta ℓ and $\tilde{\ell}$ and a pair of zero mode variables ξ, $\tilde{\xi}$:

$$u = \frac{1}{\sqrt{2L(L+\ell_3)}} \begin{pmatrix} L+\ell_3 & -\ell_- \\ \ell_+ & L+\ell_3 \end{pmatrix} e^{-i\frac{\xi}{2}\sigma_3}, \tag{4.4a}$$

$$v = e^{-i\frac{\tilde{\xi}}{2}\sigma_3} \frac{1}{\sqrt{2\tilde{L}(\tilde{L}-\tilde{\ell}_3)}} \begin{pmatrix} \tilde{L}-\tilde{\ell}_3 & -\tilde{\ell}_- \\ \tilde{\ell}_+ & \tilde{L}-\tilde{\ell}_3 \end{pmatrix}. \tag{4.4b}$$

(At the quantum level these relations give rise to an ordering problem.) It is convenient to regard at this stage the variables u and v as completely decoupled. We shall take the equality $L = \tilde{L}$ into account later on.

We concentrate for the moment on the right invariant u-sector and consider the first order classical Lagrangean

$$\mathcal{L} = \mathrm{tr}\{L\sigma_3\, i\, u^* \dot{u} - \lambda(u^* u - 1)\} - \frac{1}{2}L^2 \tag{4.5}$$

From (4.5) we deduce

$$i\, u^* \dot{u} = \frac{1}{2}L\sigma_3, \tag{4.6a}$$

while (4.1a) yields

$$\ell_a = i\,\mathrm{tr}(\sigma_a \dot{u} u^*). \tag{4.6b}$$

The Lagrange multiplier is found to coincide with the Hamiltonian,

$$\lambda = \frac{1}{2}L^2 = H, \tag{4.7}$$

while the classical equation of motion

$$i\,\dot{u} = \frac{1}{2}uL\sigma_3 = \frac{1}{2}\ell u \tag{4.8}$$

indicates that u carries just a half of the time dependence of g. Indeed, according to (4.2),

$$g(t) = u(t)\,v(t) = e^{-\frac{i}{2}\ell t}u(0)\,v(0)\,e^{\frac{i}{2}\bar{\ell}t}. \tag{4.9}$$

The phase space Γ_+ of the variables u and ℓ satisying (4.1a) is 4-dimensional. Indeed the solution of the constraints (4.1a) can be parametrized by the commuting integrals of motion L and ℓ_3, or the ratio

$$\mathrm{tg}\frac{\alpha}{2} = \sqrt{\frac{L-\ell_3}{L+\ell_3}} \quad (\geq 0) \tag{4.10}$$

and a pair of angle variables φ and ξ, so that

$$u = \begin{pmatrix} \cos\frac{\alpha}{2} & -\sin\frac{\alpha}{2}\,e^{-i\varphi} \\ \sin\frac{\alpha}{2}\,e^{i\varphi} & \cos\frac{\alpha}{2} \end{pmatrix} e^{-i\frac{\sigma_3}{2}\xi}, \quad \ell_3 = L\cos\alpha, \quad \ell_\pm = L\sin\alpha\,e^{\pm i\varphi}. \tag{4.11}$$

The canonical 1-form corresponding to the first term of the Lagrangean (4.5b), $i\,\mathrm{tr}\,\ell\,d\,uu^*$, gives rise to the symplectic form

$$\begin{aligned} \omega = i\,d(\mathrm{tr}\,\ell\,d\,uu^*) &= \frac{i}{2}\,d\ell_a \wedge \mathrm{tr}\{(\sigma_a + L^{-2}\ell_a\ell)duu^*\} = \\ &= dL \wedge (d\xi - d\varphi) + d\ell_3 \wedge d\varphi. \end{aligned} \tag{4.12}$$

The associated non-zero canonical PB

$$\{\xi, L\} = 1 = \{\xi, \ell_3\} = \{\varphi, \ell_3\} \tag{4.13}$$

can be written as quadratic (equal time) PB relations for the matrix elements of u:

$$\{\overset{1}{u}, \overset{2}{u}\} = \overset{1}{u}\overset{2}{u}\,r(L), \tag{4.14a}$$

$$r(L) = \frac{i}{2L}(\overset{1}{\sigma}_-\overset{2}{\sigma}_+ - \overset{1}{\sigma}_+\overset{2}{\sigma}_-) = \frac{i}{2L}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.14b}$$

As pointed out in [1] the relation (4.14) admits a $U_q(s\ell_2)$ deformation which allows one to connect it with the exchange relations for isospin $\frac{1}{2}$ primary fields in a su_2 current algebra model.

Here we only note that the method of this paper allows to compute the PB of the chiral field u for different times with the result

$$\{\overset{1}{u}(t_1), \overset{2}{u}(t_2)\} = \left\{ 1 + \frac{2P-1}{4L} \sin Lt_{12} + iP \frac{\overset{1}{\ell} - \overset{2}{\ell}}{4L^2}(1 - \cos Lt_{12}) + \right. \\ \left. + \left(t_{12} - \frac{\sin Lt_{12}}{L} \right) \frac{\overset{1}{\ell}\overset{2}{\ell}}{4L^2} \right\} \overset{1}{u}(t_1)\, \overset{2}{u}(t_2)\, r(L). \tag{4.15}$$

The corresponding quantum exchange relation can also be written in a form similar to (3.12), (3.6) or, alternatively, as a product of the equal time quantum R matrix (computed in [1])

$$R(N) = \frac{1}{N} \begin{pmatrix} N & 0 & 0 & 0 \\ 0 & \sqrt{N^2-1} & 1 & 0 \\ 0 & -1 & \sqrt{N^2-1} & 0 \\ 0 & 0 & 0 & N \end{pmatrix}, \quad N^2\hbar^2 = (2L+\hbar)^2 \tag{4.16}$$

by a pair of diagonal matrices. The quasi-classical limit of R is obtained for $\hbar \to 0$, $N \to \infty$, $N\hbar \to 2L$, finite:

$$R\left(\frac{2L}{\hbar}\right) \approx 1 - i\hbar\, r(L) \tag{4.17}$$

with $r(L)$ given by (4.14).

Using the defining relations (4.7) and

$$[u(t), L] = \hbar\, u(t)\, \frac{\sigma_3}{2}, \tag{4.18}$$

we find

$$u(t) = u(0)\, e^{it\left(\frac{\hbar}{4} - \frac{L\sigma_3}{2}\right)}; \tag{4.19}$$

as a result

$$\overset{2}{u}(t_2)\, \overset{1}{u}(t_1) = \overset{1}{u}(t_1)\, \overset{2}{u}(t_2)\, e^{i\frac{t_{12}}{2}\left(\overset{1}{\sigma}_3 L - \frac{\hbar}{2}\overset{1}{\sigma}_3\overset{2}{\sigma}_3\right)} R(L)\, e^{-i\frac{t_{12}}{2}\overset{1}{\sigma}_3 L}. \tag{4.20}$$

It is a pleasure to thank Anton Alekseev who indicated to us the latter form of the exchange relations for the u variables. Y. S. and I. T. acknowledge the hospitality of Laboratorio Interdisciplinare per le Scienze Naturali ed Umanistiche of the International School of Advanced Studies (SISSA/ISAS) and the Istituto Nazionale di Fisica Nucleare (I.N.F.N.) in Trieste, where this work was started, and of the Erwin Schrödinger International Institute for Mathematical Physics (ESI) in Vienna where this paper was completed. The work was supported in part by the Bulgarian National Foundation for Scientific Research under contract F11.

REFERENCES

1. A. Alekseev, L.D. Faddeev, $(T^*G)_t$: A toy model for conformal field theory. Commun. Math. Phys. **141** (1991) 413-422.

2. A. Alekseev, S. Shatashvili, Quantum groups and WZNW models, Commun. Math. Phys. **133** (1990) 353-368; L.D. Faddeev, On the Exchange matrix for WZNW model, Commun. Math. Phys. **132** (1990) 131-138.

3. M. Chu, P. Goddard, I. Halliday, D. Olive, A. Schwimmer, Quantization of the Wess-Zumino-Witten model on the circle, Phys. Lett. **B 266** (1991) 71-81.

4. F. Falceto, K. Gawędzki, Quantum group symmetries of conformal field theories, XX^{th} International Conference on Differential Geometric Methods in Theoretical Physics, N.Y., June 1991, Bures-sur-Yvette preprint IHES/P/91/59; Lattice Wess-Zumino-Witten model and quantum groups, Lectures at the $XXVIII^{th}$ Karpacz Winter School of Theoretical Physics, February 1992, Bures-sur-Yvette preprint IHES/P/92/73.

5. G. Mack, V. Schomerus, Quasi Hopf quantum symmetry in quantum theory, Nucl. Phys. **B 370** (1992) 185-230.

6. K. Gawędzki, Classical origin of quantum group symmetries in Wess-Zumino-Witten conformal field theory, Commun. Math. Phys. **139** (1991) 201-213.

7. G. Papadopoulos, B. Spence, The global phase space structure of the Wess-Zumino-Witten model, Phys. Lett. **B 295** (1992) 44-50.