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QUANTUM DEFORMED CANONICAL TRANSFORMATIONS, W_∞-ALGEBRAS AND UNITARY TRANSFORMATIONS

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ABSTRACT

We investigate the algebraic properties of the quantum counterpart of the classical canonical transformations using the symbol-calculus approach to quantum mechanics. In this framework we construct a set of pseudo-differential operators which act on the symbols of operators, *i.e.*, on functions defined over phase-space. They act as operatorial left- and right-multiplication and form a $W_\infty \times W_\infty$- algebra which contracts to its diagonal subalgebra in the classical limit. We also describe the Gel'fand-Naimark-Segal (GNS) construction in this language and show that the GNS representation-space (a doubled Hilbert space) is closely related to the algebra of functions over phase-space equipped with the star-product of the symbol-calculus.

1. INTRODUCTION

Quantum Mechanics (QM) has been a very successful theory from a phenomenological point of view but not so much from a more formal point of view, especially in its connection with *Classical Mechanics* (CM). What we mean is that the well known *"correspondence principle"*, on which the Dirac[1] formulation of QM is based, has often been questioned[2]. As it is nicely explained in refs.[3]and [4] the Dirac canonical quantization can be considered as a map *"D"* sending real functions defined on phase space $f_1, f_2, \cdots$ to hermitian operators $\widehat{f}_1, \widehat{f}_2 \cdots$ which act on some appropriate Hilbert space[3,4]. According to Dirac, this map should satisfy the following requirements

$$(a) \quad \mathcal{D}(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 \widehat{f}_1 + \lambda_2 \widehat{f}_2, \quad \lambda_{1,2} \in R$$

$$(b) \quad \mathcal{D}(\{f_1, f_2\}_{pb}) = \frac{1}{i\hbar}[\widehat{f}_1, \widehat{f}_2]$$

$$(c) \quad \mathcal{D}(1) = I$$

$$(d) \quad \widehat{q}, \widehat{p} \text{ are represented irreducibly.}$$

$$(1.1)$$

Here $\{\cdot, \cdot\}_{pb}$ denotes the Poisson bracket, and I is the unit operator.

Groenwald and van Hove[2] showed that it is impossible to satisfy these conditions for all functions. They can be established only for functions which are at most quadratic in p and q. This obstruction to a complete quantization implies that not all canonical transformation generated by an arbitrary real function f, *i.e.*, $\delta(\cdot) = \{\cdot, f\}_{pb}$ can be implemented as unitary transformations

$$\delta(\cdot) = \frac{1}{i\hbar}[(\cdot), \widehat{f}] \qquad (1.2)$$

on the Hilbert space. In order to avoid the consequences of this "no-go" theorem one has to relax some of the conditions of (1.1). In geometric quantization[5] condition (d) is abandoned, for instance, and in Moyal quantization[6] condition (b) is modified. Moyal quantization involves replacing the Poisson bracket in (b) by a new, $\hbar$-dependent bracket, the Moyal bracket. It is a "deformation"[7] of the Poisson bracket to which it reduces in the classical limit $\hbar \to 0$.

The formal structure underlying the Moyal quantization is the "symbol calculus"[8]. Generally speaking it deals with the representation of operators acting on some Hilbert space $\mathcal{V}$ in terms of functions defined on a suitable manifold $\mathcal{M}$. The "symbol map", *symb*, is a linear one-to-one map from the space of operators on $\mathcal{V}$ to $Fun(\mathcal{M})$, the space of

complex functions on $\mathcal{M}$. We write $f = symb(\widehat{f})$ for the function f representing the operator $\widehat{f}$. The inverse of the symbol map, the "operator map", op, associates a unique operator to any complex $f \in Fun(\mathcal{M}) : op(f) = \widehat{f}$. On $Fun(\mathcal{M})$ one introduces the so-called star product $*$ which expresses the operator product $\widehat{f_1}\widehat{f_2}$ in terms of the symbols of $\widehat{f_1}$ and $\widehat{f_2}$:

$$symb(\widehat{f_1}\widehat{f_2}) = symb(\widehat{f_1}) * symb(\widehat{f_2}) \tag{1.3}$$

Like the operator product the star product is non-commutative but associative.

Let us now return to quantum mechanics and let us assume that $\mathcal{V}$ is the state space of some quantum mechanical system with N degrees of freedom and that $\mathcal{M} \equiv \mathcal{M}_{2N}$ is its phase-space. A particularly important type of symbol is the Weyl symbol[9]. It is defined as

$$f(\phi^a) = \int \frac{d^{2N}\phi_0}{(2\pi\hbar)^N} \; exp\left[\frac{i}{\hbar}\phi_0^a \omega_{ab} \phi^b\right] Tr\left[\widehat{T}(\phi_0)\widehat{f}\right] \tag{1.4}$$

Here $\phi^a = (p^1, \cdots p^N, q^1, \cdots q^N)$, $a = 1 \cdots 2N$, are canonical coordinates on $\mathcal{M}_{2N}$, and ω_{ab} are the coefficients of the corresponding symplectic two-form. In terms of $N \times N$ blocks we have: $\omega_{pq} = -\omega_{qp} = I_N$. Furthermore, the operators $\widehat{T}(\phi_0)$ are defined as:

$$\widehat{T}(\phi_0) = exp\left[\frac{i}{\hbar}\phi_0^a \omega_{ab}\widehat{\phi}^b\right] \equiv exp\left[\frac{i}{\hbar}(p_0\widehat{q} - q_0\widehat{p})\right] \; , \quad \widehat{\phi}^a \equiv (\widehat{p}^i, \widehat{q}^i) \tag{1.5}$$

They are the translation operators on phase-space[10]. Inverting (1.4) we obtain the inverse symbol map:

$$\widehat{f} = \int \frac{d^{2N}\phi \; d^{2N}\phi_0}{(2\pi\hbar)^{2N}} \; f(\phi) \; exp\left[\frac{i}{\hbar}\phi^a \omega_{ab}\phi_0^b\right] \widehat{T}(\phi_0) \tag{1.6}$$

For real functions f the operator $\widehat{f}$ is hermitian. If $f = symb(\widehat{f})$ is complex, then the complex-conjugate function f^* is the symbol of the hermitian-adjoint operator:

$$op(f^*) = \widehat{f}^\dagger \; , \quad f^* = symb(\widehat{f}^\dagger) \tag{1.7}$$

Arbitrary (mixed) states of the system are characterized by $\varrho = symb(\widehat{\varrho})$ where $\widehat{\varrho}$ is the density operator. In particular, for pure states $\widehat{\varrho} = |\psi\rangle \langle\psi|$ one obtains the familiar Wigner[9] function

$$\varrho_\psi(p,q) = \int d^N x \; exp\left[-\frac{i}{\hbar}px\right]\psi(q + \frac{1}{2}x)\psi^*(q - \frac{1}{2}x) \tag{1.8}$$

Using the symbols of density operators $\widehat{\varrho}$ and of observables $\widehat{\mathcal{O}}$ it is possible to reformulate

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the full machinery of quantum mechanics. Expectation values in the state $|\psi\rangle$, say, read

$$<\psi|\widehat{\mathcal{O}}|\psi> = \int \frac{d^{2N}\phi}{(2\pi\hbar)^N} \varrho_\psi(\phi)\mathcal{O}(\phi) \tag{1.9}$$

The star product for the Weyl symbol is given by an infinite-order differential operator:

$$(f*g)(\phi) = f(\phi)\, exp\Big[i\frac{\hbar}{2} \overleftarrow{\partial_a}\, \omega^{ab}\, \overrightarrow{\partial_b}\Big]g(\phi)$$
$$= \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\frac{i\hbar}{2}\Big)^n \omega^{a_1 b_1}\cdots\omega^{a_n b_n}(\partial_{a_1}\cdots\partial_{a_n}f)(\partial_{b_1}\cdots\partial_{b_n}g) \tag{1.10}$$

Here ω^{ab} is the inverse of ω_{ab}: $\omega^{ab}\omega_{bc} = \delta^a_c$. Hence $\omega^{qp} = -\omega^{pq} = I_N$ so that in a slightly more standard notation

$$(f*g) = f(p,q)\, exp\Big[\frac{i\hbar}{2}\Big(\frac{\overleftarrow{\partial}}{\partial q^i}\frac{\overrightarrow{\partial}}{\partial p_i} - \frac{\overleftarrow{\partial}}{\partial p_i}\frac{\overrightarrow{\partial}}{\partial q^i}\Big)\Big]\, g(p,q) \tag{1.11}$$

In the classical limit the star-product becomes the ordinary pointwise product: $(f*g)(\phi) = f(\phi)g(\phi) + O(\hbar)$. By expressing either f or g in terms of its Fourier transform, it is easy to derive the following representation of the star product[*]

$$(f*g)(\phi) = f(\phi^a + \frac{i\hbar}{2}\omega^{ab}\frac{\partial}{\partial\widetilde{\phi}^b})\, g(\widetilde{\phi})|_{\widetilde{\phi}=\phi}$$
$$= g(\phi^a - \frac{i\hbar}{2}\omega^{ab}\frac{\partial}{\partial\widetilde{\phi}^b})\, f(\widetilde{\phi})|_{\widetilde{\phi}=\phi} \tag{1.12}$$

The Moyal bracket $\{\cdot,\cdot\}_{mb}$ is defined as the commutator with respect to star multiplication:

$$\{f,g\}_{mb} = \frac{1}{i\hbar}(f*g - g*f)$$
$$= symb\big(\frac{1}{i\hbar}[\widehat{f},\widehat{g}]\big) \tag{1.13}$$

For $\hbar \to 0$ the Moyal bracket approaches the Poisson bracket: $\{f,g\}_{mb} = \{f,g\}_{pb} + O(\hbar^2)$. In our notation the Poisson bracket reads

$$\{f,g\}_{pb} \equiv \partial_a f\, \omega^{ab}\, \partial_b g \tag{1.14}$$

From the fact that the star-multiplication is associative it follows that the Moyal bracket fulfills the Jacobi identity and that

$$\{f,g_1*g_2\}_{mb} = \{f,g_1\}_{mb}*g_2 + g_1*\{f,g_2\}_{mb} \tag{1.15}$$

This means that $\{f,\cdot\}_{mb}$ with $f \in Fun(\mathcal{M})$ fixed, is a derivation[3] on the algebra $\big(Fun(\mathcal{M}),*\big)$. It has been proven[7] that every derivation on this algebra has this structure:

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[*] For a review and previous work see ref.[8].

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given an operation D (derivation) with $D(f * g) = (Df) * g + f * (Dg)$ it always exists an $f \in Fun(\mathcal{M}_{2N})$ such that $Dg = \{f, g\}_{mb}$. This means that every derivation is an *internal* derivation. Because of (1.3) this also holds for commutators but it does not hold for Poisson brackets. This is another indication that a homomorphism between Poisson brackets and commutators, as required in (b) of eq. (1.1), cannot exist in general. On the other hand, if we replace the Poisson bracket on the LHS of (b) by the Moyal bracket, we can find a quantization map $\mathcal{D}$ which satisfies all the requirements $(a) - (d)$ of (1.1) *for all* functions $f_{1,2} \in Fun(\mathcal{M}_{2N})$, namely the operator map $op = symb^{-1}$:

$$op(\{f_1, f_2\}_{mb}) = \frac{1}{i\hbar}[\widehat{f_1}, \widehat{f_2}] \tag{1.16}$$

The unitary transformation

$$\delta_f(\cdot) = \frac{1}{i\hbar}[\widehat{f}, (\cdot)] \tag{1.17}$$

are now in a one-to-one correspondence with the following transformations acting on symbols

$$
\begin{aligned}
\delta_f(\cdot) &= \{f, \cdot\}_{mb} \\
&= \{f, \cdot\}_{pb} + \text{ higher derivatives}
\end{aligned}
\tag{1.18}
$$

The transformations (1.18) could be called "*quantum deformed canonical transformations*". They act on c-number functions, as in classical mechanics, but two consecutive transformations are composed by forming the Moyal bracket $\{f_1, f_2\}_{mb}$ instead of the Poisson bracket. In this sense, Moyal quantization is a special example of the general framework of non-commutative geometry[11] where spaces are investigated in terms of the algebra of functions defined on them. The phase-space of quantum mechanics is a "non-commutative manifold" which is obtained from the classical one by "deforming" the algebra $(Fun(\mathcal{M}_{2N}), \{\cdot, \cdot\}_{pb})$ to $(Fun(\mathcal{M}_{2N}), \{\cdot, \cdot\}_{mb})$.

The purpose of this paper is to study the covariance properties of quantum mechanics by using the notion of *quantum deformed canonical transformations*. Our emphasis is on the relation between the phase-space formulation and the standard Hilbert space formulation. In Section-2 we construct a set of pseudo-differential operators[12] on $Fun(\mathcal{M}_{2N})$ corresponding to operatorial left and right multiplication. They form a kind of $W_\infty \times W_\infty$ algebra. We give a careful discussion of its classical limit, in which it contracts to the diagonal subgroup W_∞^{diag}. In Section 3 we consider the deformed canonical transformations from the point of view of the Gel'fand-Naimark-Segal (GNS) construction[13] which is at the heart of

Thermo Field Dynamics (TFD)[14], for instance. In this construction the doubled Hilbert space $\mathcal{V} \otimes \mathcal{V}$ plays a central role. We show explicitly that $\mathcal{V} \otimes \mathcal{V}$ may be identified with the space of symbols, $Fun(\mathcal{M}_{2N})$. In Section 4 we realize the $W_\infty \times W_\infty$ algebra in the recently proposed[15] extended Moyal formalism which was used to formulate a tensor calculus on quantum phase-space.

2. THE $W_\infty \times W_\infty$ GENERATORS

For each symbol $f = symb(\widehat{f})$ we define the operators L_f and R_f acting on the space of symbols as

$$
\begin{aligned}
L_f g &= f * g = symb(\widehat{f}\,\widehat{g}) \\
R_f g &= g * f = symb(\widehat{g}\,\widehat{f})
\end{aligned}
\tag{2.1}
$$

for all $g = symb(\widehat{g})$. The operators L_f and R_f implement on $Fun(\mathcal{M}_{2N})$ the left and right multiplication with $\widehat{f}$, respectively. By employing the form (1.11) of the star-product, we can represent L_f and R_f by the following pseudo-differential operators

$$
\begin{aligned}
L_f &= \; :f(\phi^a + \frac{i\hbar}{2}\omega^{ab}\partial_b): \\
R_f &= \; :f(\phi^a - \frac{i\hbar}{2}\omega^{ab}\partial_b):
\end{aligned}
\tag{2.2}
$$

Here the normal ordering symbol $:(\cdots):$ means that all derivatives ∂_a should be placed to the right of all ϕ's. This makes sure that

$$
:f(\phi^a \pm \frac{i\hbar}{2}\omega^{ab}\partial_b): \; g(\phi) = f(\phi^a \pm \frac{i\hbar}{2}\omega^{ab}\frac{\partial}{\partial\widetilde{\phi}^b})g(\widetilde{\phi})|_{\widetilde{\phi}=\phi}
\tag{2.3}
$$

The normal ordering can be performed explicitly in terms of an integral involving the Fourier transform

$$
\widetilde{f}(l) = \int \frac{d^{2N}l}{(2\pi)^{2N}} \; exp\,(-il_a\phi^a)f(\phi)
\tag{2.4}
$$

Since $:exp(il_a\phi a \pm \frac{\hbar}{2}l_a\omega^{ab}\partial_b) := exp(il_a\phi^a)\,exp(\pm\frac{\hbar}{2}l_a\omega^{ab}\partial_b)$ one obtains

$$
\begin{aligned}
L_f &= \int d^{2N}l \; \widetilde{f}(l) \; exp(il_a\phi^a) \; exp\big[-\frac{\hbar}{2}l_a\omega^{ab}\partial_b\big] \\
R_f &= \int d^{2N}l \; \widetilde{f}(l) \; exp(il_a\phi^a) \; exp\big[+\frac{\hbar}{2}l_a\omega^{ab}\partial_b\big]
\end{aligned}
\tag{2.5}
$$

Using the definition (2.1) and the fact that the star-product is associative, it is easy to work

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out the commutator algebra of the L's and the R's. For instance,

$$
\begin{aligned}
[L_{f_1}, L_{f_2}]g &= L_{f_1}L_{f_2}g - L_{f_2}L_{f_1}g \\
&= f_1 * (f_2 * g) - f_2 * (f_1 * g) \\
&= (f_1 * f_2 - f_2 * f_1) * g \\
&= i\hbar L_{\{f_1,f_2\}_{mb}}g
\end{aligned}
\tag{2.6}
$$

and similarly for the other brackets. The result is

$$
\begin{aligned}
\big[\, L_{f_1}, L_{f_2}\big] &= +i\hbar L_{\{f_1,f_2\}_{mb}} \\
\big[\, R_{f_1}, R_{f_2}\big] &= -i\hbar R_{\{f_1,f_2\}_{mb}} \\
\big[\, L_{f_1}, R_{f_2}\big] &= 0
\end{aligned}
\tag{2.7}
$$

We observe that the pseudo-differential operators L_f and $-R_f$ generate two commuting copies $\mathcal{A}_L$ and $\mathcal{A}_R$ of the same algebra $\mathcal{A}$ which is isomorphic to the Moyal bracket algebra on $Fun(\mathcal{M}_{2N})$. In order to analyze the classical limit of the algebra (2.7) it is useful to define the following linear combinations

$$
\begin{aligned}
V_f &= \frac{1}{i\hbar}(L_f - R_f) \\
A_f &= \frac{1}{i\hbar}(L_f + R_f)
\end{aligned}
\tag{2.8}
$$

It is easy to see that they satisfy the commutation relations

$$
\begin{aligned}
\big[V_{f_1}, V_{f_2}\big] &= V_{\{f_1,f_2\}_{mb}} & (a) \\
\big[V_{f_1}, A_{f_2}\big] &= A_{\{f_1,f_2\}_{mb}} & (b) \\
\big[A_{f_1}, A_{f_2}\big] &= V_{\{f_1,f_2\}_{mb}} & (c)
\end{aligned}
\tag{2.9}
$$

By a lengthy calculation it can be checked explicitly that the operators (2.5) indeed satisfy this algebra. This has been done in ref.[16] in a similar context. In our approach this is not necessary because we have related the operators (2.5) to left and right multiplication. For later use we also write down the exponential form of the generators:

$$
\begin{aligned}
op\left(e^{L_f}g\right) &= exp[\widehat{f}]\,\widehat{g} \\
op\left(e^{R_f}g\right) &= \widehat{g}\,exp[\widehat{f}] \\
op\left(e^{V_f}g\right) &= exp[-\frac{i}{\hbar}\widehat{f}]\,\widehat{g}\,exp[\frac{i}{\hbar}\widehat{f}] \\
op\left(e^{A_f}g\right) &= exp[-\frac{i}{\hbar}\widehat{f}]\,\widehat{g}\,exp[-\frac{i}{\hbar}\widehat{f}]
\end{aligned}
\tag{2.10}
$$

For real f, the operators V_f generate unitary transformations $\widehat{g} \to U\widehat{g}U^{-1}$, while the A_f's map $\widehat{g} \to U\widehat{g}U$. If, say, $\widehat{g} = \widehat{\varrho} = |\psi\rangle\langle\psi|$ is the density matrix associated to the pure state $|\psi\rangle$, then clearly V_f maps pure states on pure ones, while A_f does not.

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Going now back to the operators L_f and R_f, the interesting point about them is that they provide a link between the Hilbert space formulation of QM and the geometry of phase-space. As is well known, CM was formulated in the language of phase-spaces and their classical differential geometry while QM was framed in the language of Hilbert spaces and operators. Thanks to the symbol calculus, we are now in a position to study both CM and QM within the same setting, namely as operations on $Fun(\mathcal{M}_{2N})$. The quantum deformed canonical transformations have a dual interpretation therefore: in the Hilbert space formalism they are unitary transformations, in the phase- space formalism they are a kind "distorted" volume preserving diffeomorphisms on $\mathcal{M}_{2N}$. It is interesting to investigate the novel geometric structures which the Hilbert space formalism of QM induces on phase-space. Let us first look at the situation in CM. There any real function $f \in Fun(\mathcal{M}_{2N})$ gives rise to a hamiltonian vector field

$$h_f^a = \omega^{ab} \partial_b f(\phi) \tag{2.11}$$

which generates a canonical transformation. This is a transformation

$$\phi'^a = \phi^a + \delta\phi^a \quad , \quad \delta\phi^a = -h^a(\phi) \tag{2.12}$$

which preserves the symplectic two form:

$$\frac{\partial \phi'^a}{\partial \phi^c} \frac{\partial \phi'^b}{\partial \phi^d} \omega_{ab} = \omega_{cd} \tag{2.13}$$

Scalar functions transform as

$$\delta_f g = \{f, g\}_{pb} = -h_f^a \partial_a g \tag{2.14}$$

In particular, the time evolution of classical probability densities $\varrho_{cl}(\phi, t)$ is a special canonical transformation generated by $h^a \equiv h_H^a = \omega^{ab} \partial_b H$, where H is the Hamiltonian. Therefore

$$\partial_t \varrho_{cl} = \{H, \varrho\}_{pb} = -h^a \partial_a \varrho_{cl} \tag{2.15}$$

In QM the density operator evolves according to von Neumann's equation

$$i\hbar \, \partial_t \widehat{\varrho} = \left[\widehat{H}, \widehat{\varrho}\right] \tag{2.16}$$

which, in the symbol language, becomes

$$\partial_t \varrho = \{H, \varrho\}_{mb} \tag{2.17}$$

Here $\varrho = symb(\widehat{\varrho})$ is similar to the classical density ϱ_{cl} in (2.15) , but it is not positive definite. Therefore it is referred to as a pseudodensity. A special example of a pseudodensity

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is the Wigner function (1.8). Using (1.13), (2.1) and (2.8), eq. (2.17) may be rewritten as

$$\partial_t \varrho = \frac{1}{i\hbar} \left(L_H - R_H \right) \varrho = V_H \varrho \qquad (2.18)$$

More generally, any transformation of the form

$$\delta_f \widehat{g} = \frac{1}{i\hbar} [\widehat{f}, \widehat{g}] \qquad (2.19)$$

can be written as

$$\delta_f g = V_f g \qquad (2.20)$$

For f real, $op(f) = \widehat{f}$ is hermitian so that (2.19) is indeed an infinitesimal unitary transformation $\widehat{g} \mapsto \widehat{U}\widehat{g}\widehat{U}^{-1}$. Comparing (2.14) to (2.20) we see that the pseudo-differential operator $-V_f$ is the quantum deformed version of the hamiltonian vector field $h_f^a \partial_a$. A natural question to ask is which one, if any, is the role of A_f in classical geometry.

Let us look at the classical limit $\hbar \to 0$ of the above pseudo-differential operators in more detail. From eq. (2.2) we obtain

$$
\begin{aligned}
L_f &= f(\phi) - \frac{i\hbar}{2} h^a \partial_a + O(\hbar^2) \\
R_f &= f(\phi) + \frac{i\hbar}{2} h^a \partial_a + O(\hbar^2) \\
V_f &= - h^a \partial_a + O(\hbar) \\
A_f &= \frac{2}{i\hbar} f(\phi) + O(\hbar)
\end{aligned}
\qquad (2.21)
$$

We see that L_f, R_f, and V_f start with a term of zeroth order in $\hbar$: both L_f and R_f are the usual pointwise multiplication with $f(\phi)$ and V_f is a first order operator to lowest order. The situation is different for A_f which starts with a term of order $\frac{1}{\hbar}$. Therefore matrix elements of the form

$$< g_1 | A_f | g_2 > = \int d^{2N} \phi \, g_1^*(\phi) \, A_f \, g_2(\phi) \qquad (2.22)$$

blow up in the classical limit, and A_f is not a well defined map $Fun(\mathcal{M}_{2N}) \to Fun(\mathcal{M}_{2N})$ anymore. We conclude that only the transformations generated by V_f, but not those of A_f, possess a classical limit. As a consequence the algebra $\mathcal{A}_L \times \mathcal{A}_R$ of eq. (2.7) or, equivalently, of eq. (2.9) is "contracted" in the classical limit to its diagonal subalgebra $\mathcal{A}_{diag}$ generated

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by V_f: $\mathcal{A}_L \times \mathcal{A}_R \longrightarrow \mathcal{A}_{diag}$. The remaining commutation relations

$$\left[V_{f_1}^{cl}, V_{f_2}^{cl}\right] = V_{\{f_1,f_2\}_{pb}}^{cl} \tag{2.23}$$

have the Poisson brackets appearing on the RHS now. Eq. (2.23) is the same as the Lie bracket of the hamiltonian vector fields:

$$\left[h_{f_1}^a \partial_a, h_{f_2}^b \partial_b\right] = -h_{\{f_1,f_2\}_{pb}}^a \partial_a \tag{2.24}$$

Again we see that it is very natural to consider V_f as the *quantum deformed hamiltonian vector field*. For $\hbar \to 0$ it becomes a first order operator, *i.e.*, a generic vector field with a Lie bracket algebra. For $\hbar \neq 0$ this algebra is deformed to the algebra (a) of (2.9) which involves the Moyal bracket as "structure constants". The deformed algebra cannot be represented by differential operators of finite order but only by pseudo-differential operators, *i.e.*, V_f contains terms with an arbitrary number of derivatives*

$$V_f = \omega^{ab} \partial_a f \, \partial_b - \frac{\hbar^2}{24} \omega^{a_1 b_1} \omega^{a_2 b_2} \omega^{a_3 b_3} \, \partial_{a_1} \partial_{a_2} \partial_{a_3} f \, \partial_{b_1} \partial_{b_2} \partial_{b_3} + \cdots \tag{2.25}$$

Let us now come back to the algebra $\mathcal{A}$ generated by L_f or $-R_f$. For the special case of a two-dimensional phase-space $\mathcal{M}_2$, $\mathcal{A}$ can be identified with a version of the W_∞-algebra found[16] as the $N \to \infty$ limit of the W_N-algebras which arose in the studies of conformal field theory models[17]. Recently W_∞-algebras also appeared in the theory of planar fermion systems in the context of the fractional quantum Hall effect and of matrix models[18]. An explicit form of our algebra $\mathcal{A}$ which displays its structure constants is obtained by choosing a basis on $Fun(\mathcal{M}_2)$ and working out the Moyal brackets among the basis elements. We briefly discuss the case of a phase-space with the topology of a torus[16][19]: $\mathcal{M}_2 = S^1 \times S^1$. The basis elements can be chosen as[16]

$$f_{\vec{m}}(\phi) = -exp(\vec{m} \cdot \vec{\phi}) \equiv -exp(im_a \phi^a) \tag{2.26}$$

where $\vec{m} \equiv (m_1, m_2) \in \mathbf{Z}^2$ and $0 \leq \phi^a \leq 2\pi$. The Moyal brackets of the $f_{\vec{m}}$'s are easily

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* This is an indication of the non-local nature of quantum mechanics.

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calculated, and the algebra (2.7) becomes (with $L_{\vec{m}} \equiv L_{f_{\vec{m}}}$ etc. and $\vec{m}\omega\vec{n} \equiv m_a \omega^{ab} n_b$)

$$
\begin{aligned}
\left[L_{\vec{m}}, L_{\vec{n}}\right] &= +2i\sin\left(\frac{\hbar}{2}\vec{m}\omega\vec{n}\right) L_{\vec{m}+\vec{n}} \\
\left[R_{\vec{m}}, R_{\vec{n}}\right] &= -2i\sin\left(\frac{\hbar}{2}\vec{m}\omega\vec{n}\right) R_{\vec{m}+\vec{n}} \\
\left[L_{\vec{m}}, R_{\vec{n}}\right] &= 0
\end{aligned}
\tag{2.27}
$$

and similarly for (2.9):

$$
\begin{aligned}
\left[V_{\vec{m}}, V_{\vec{n}}\right] &= \frac{2}{\hbar}\sin\left(\frac{\hbar}{2}\vec{m}\omega\vec{n}\right) V_{\vec{m}+\vec{n}} \\
\left[V_{\vec{m}}, A_{\vec{n}}\right] &= \frac{2}{\hbar}\sin\left(\frac{\hbar}{2}\vec{m}\omega\vec{n}\right) A_{\vec{m}+\vec{n}} \\
\left[A_{\vec{m}}, A_{\vec{n}}\right] &= \frac{2}{\hbar}\sin\left(\frac{\hbar}{2}\vec{m}\omega\vec{n}\right) V_{\vec{m}+\vec{n}}
\end{aligned}
\tag{2.28}
$$

In the classical limit $\hbar \to 0$ the first of these equations becomes

$$
\left[V_{\vec{m}}, V_{\vec{n}}\right] = (\vec{m}\omega\vec{n}) V_{\vec{m}+\vec{n}}
\tag{2.29}
$$

This is the algebra of $sdiff(T^2)$ of (classical) area preserving diffeomorphisms on the torus, sometimes referred to as the w_∞-algebra. The VV-relations of (2.28) are a deformation of it: the W_∞-algebra. Note that the rescaled generators $-\frac{i}{\hbar}L_{\vec{m}}$ and $\frac{i}{\hbar}R_{\vec{m}}$ satisfy the same commutation relations. Therefore, at least for $\hbar \neq 0$, we may also identify $\mathcal{A}_L$ and $\mathcal{A}_R$ with W_∞. In a slight abuse of language, we shall call $\mathcal{A}$ a W_∞- algebra even for dimensions $2N > 2$. (See also ref.[20] for $2N$-dimensional lattice algebras.)

3. THE GNS CONSTRUCTION

Roughly speaking, the GNS construction[13,14] is a procedure to go from trace-type averages to bra/ket-type averages†. Let us consider an arbitrary operator $\widehat{B} \in \Omega(\mathcal{V})$ where $\Omega(\mathcal{V})$ denotes the space of operators on the Hilbert space $\mathcal{V}$. By choosing an orthonormal basis $\{|\alpha\rangle\}$ on $\mathcal{V}$ we can represent $\widehat{B}$ as

$$
\widehat{B} = \sum_{\alpha,\beta} B_{\alpha\beta} |\alpha\rangle \langle\beta| \quad , \quad B_{\alpha\beta} \equiv \langle\alpha|\widehat{B}|\beta\rangle
\tag{3.1}
$$

The GNS construction (at least in the version used in Thermo Field Dynamics[14]) associates

---

† Our discussion of the GNS construction is informal and it has no pretension of mathematical rigorosity. For its much more general and abstract form, see refs.[13]. Our presentation is close to the one of Ojima[14] for Thermo Field Theory.

to this operator $\widehat{B}$ a vector $\|\widehat{B}\gg \in \mathcal{V}\otimes\mathcal{V}$ in the tensor product of $\mathcal{V}$ with itself:

$$\|\widehat{B}\gg = \sum_{\alpha,\beta} B_{\alpha\beta}\,|\alpha\rangle\otimes|\beta\rangle \qquad (3.2)$$

We denote the "vector" map relating $\|\widehat{B}\gg$ to $\widehat{B}$ as

$$vec\ :\quad \Omega(\mathcal{V})\to\mathcal{V}\otimes\mathcal{V}\ ,\quad \widehat{B}\mapsto\|\widehat{B}\gg = vec(\widehat{B}) \qquad (3.3)$$

This defines a one-to-one correspondence between the operators on $\mathcal{V}$ and the elements of the doubled Hilbert space $\mathcal{V}\otimes\mathcal{V}$. On the other hand, the symbol map

$$symb\ :\quad \Omega(\mathcal{V})\to\ Fun(\mathcal{M}_{2N})\ ,\quad \widehat{B}\mapsto symb(\widehat{B}) \qquad (3.4)$$

represents operators on $\mathcal{V}$ by functions on $\mathcal{M}_{2N}$. As a consequence, the composite map $vec\circ symb^{-1}\equiv vec\circ op$ provides a one-to-one correspondence between the functions on phase-space and the doubled Hilbert space:

$$Fun(\mathcal{M}_{2N})\cong\ \mathcal{V}\otimes\mathcal{V} \qquad (3.5)$$

The rest of this section is devoted to a detailed study of this relation between functions on phase-space and states in the doubled Hilbert space. As we shall see, this connection is rather intriguing from many points of view.

The original motivation[14] for the introduction of $\mathcal{V}\otimes\mathcal{V}$ was that it allows to represent *statistical* averages in a Hilbert space language. In this way a state of a quantum system can be described by a vector in a Hilbert space, $\mathcal{V}\otimes\mathcal{V}$, even if this state, from the point of view of $\mathcal{V}$, is not a pure one. This approach is particularly useful in quantum field theory at finite temperature because, via the GNS construction, we can use all the techniques we know from zero-temperature field theory (perturbation theory, etc.) and apply them to the doubled theory "living" on $\mathcal{V}\otimes\mathcal{V}$. This approach is known as the Thermo Field Dynamics[14]. In this framework states of the type $|\psi\rangle\otimes|0\rangle$ and $|0\rangle\otimes|\psi\rangle$ are called "particles" and "tilde-particles", respectively. (Here $|0\rangle$ is the vacuum state.) The usual attitude is that the tilde-particles are only auxiliary quantitites without too much physical significance. In the light of the identification of $Fun(\mathcal{M}_{2N})$ with $\mathcal{V}\otimes\mathcal{V}$ we see that the "tilde particles" are nothing unnatural, but rather something very important because they allow for a close link between quantum mechanical structures and classical ones.

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Quite trivially we can associate to any operator $\widehat{A} \in \Omega(\mathcal{V})$ a new operator acting on $\mathcal{V} \otimes \mathcal{V}$, namely $\widehat{A} \otimes I$, where I is the identity operator. Let us calculate the expectation value of $\widehat{A} \otimes I$ in the state $\| \widehat{B} \gg$ of (3.2). With the dual vector defined as

$$\ll \widehat{B} \| = \sum_{\alpha, \beta} B^*_{\alpha\beta} \ \langle \alpha | \otimes \langle \beta | \tag{3.6}$$

and using $< \alpha | \beta > = \delta_{\alpha\beta}$, one obtains easily

$$
\begin{aligned}
\ll \widehat{B} \| \ \widehat{A} \otimes I \ \| \widehat{B} \gg &= \sum_{\alpha, \cdots, \delta} B^*_{\gamma\delta} \ B_{\alpha\beta} \ \langle \gamma | \otimes \langle \delta | \ (\widehat{A} \otimes I) \ |\alpha\rangle \otimes |\beta\rangle \\
&= \sum_{\alpha, \cdots, \delta} B^*_{\gamma\delta} B_{\alpha\beta} < \gamma | \widehat{A} | \alpha > < \delta | \beta > \\
&= \sum_{\alpha} \langle \alpha | \widehat{A} \widehat{B} \widehat{B}^\dagger | \alpha \rangle \\
&\equiv Tr[\widehat{A} \widehat{B} \widehat{B}^\dagger]
\end{aligned}
\tag{3.7}
$$

Let us assume that we are given a quantum system on $\mathcal{V}$ with a hermitian density operator which does not necessarily correspond to a pure state ($\widehat{\varrho}^2 \neq \widehat{\varrho}$). The eigenvalues p_α of $\widehat{\varrho}$ give the probability to find the system in the associated eigenstates $|\alpha\rangle$. Then the trace averages

$$\left\langle \widehat{A} \right\rangle = Tr[\widehat{A} \ \widehat{\varrho}] \tag{3.8}$$

can be re-written as a bra-ket expectation value of the type (3.7) once we have found an operator $\widehat{B} \equiv \widehat{\varrho}^{\frac{1}{2}}$ such that $\widehat{B}\widehat{B}^\dagger = \widehat{\varrho}$. This operator is easily constructed in the diagonal basis of $\widehat{\varrho}$ in which the diagonal elements are $B_\alpha = (p_\alpha)^{\frac{1}{2}}$. Therefore $\widehat{B}$ can be chosen hermitian. Thus

$$\left\langle \widehat{A} \right\rangle = \ll \widehat{\varrho}^{\frac{1}{2}} \| \ \widehat{A} \otimes I \ \| \widehat{\varrho}^{\frac{1}{2}} \gg \tag{3.9}$$

with

$$\| \widehat{\varrho}^{\frac{1}{2}} \gg = \sum_{\alpha} (p_\alpha)^{\frac{1}{2}} \ |\alpha\rangle \otimes |\alpha\rangle \tag{3.10}$$

in the eigenbasis of $\widehat{\varrho}$. Even though we are dealing with mixed states, we have now succeeded in describing them by a vector in a Hilbert space, rather than by a density operator, but the prize we have to pay is the doubling of the Hilbert space.

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Let us now return to the Moyal formalism. So far we have the chain of maps relating symbols to operators and states in $\mathcal{V} \otimes \mathcal{V}$:

$$Fun(\mathcal{M}_{2N}) \underset{symb}{\overset{op}{\rightleftarrows}} \Omega(\mathcal{V}) \underset{vec^{-1}}{\overset{vec}{\rightleftarrows}} \mathcal{V} \otimes \mathcal{V} \qquad (3.11)$$

It will be convenient to define the map

$$\Theta \equiv vec \circ op : \quad Fun(\mathcal{M}_{2N}) \rightarrow \mathcal{V} \otimes \mathcal{V} , \quad B \rightarrow \Theta(B) = \| \widehat{B} \gg \qquad (3.12)$$

which takes us directly from the symbols to the vectors in $\mathcal{V} \otimes \mathcal{V}$. By their very definition, density operators have a non-negative spectrum. They are represented by a subspace of $Fun(\mathcal{M}_{2N})$ which consists of symbols (pseudo-densities) ϱ for which $op(\varrho)$ is non-negative. Unfortunately it is not possible to characterize this subspace very explicitly[21]. It follows from (1.7) and the hermiticity of $\widehat{\varrho}$ that ϱ is real. Similarly the symbols of observables $\mathcal{O} = \mathcal{O}^{\dagger} \in \Omega(\mathcal{V})$ are real, but not subject to further restrictions. Nonhermitian operators on $\mathcal{V}$ have complex symbols in general. In order to investigate the meaning of complex conjugation on $Fun(\mathcal{M}_{2N})$ we introduce the following map $\mathcal{J}$ on any of the three spaces listed in (3.11). $\mathcal{J}$ is defined as the *antilinear* extension of the following operations:

$$\mathcal{J}(f) = f^* , \quad \forall f \in Fun(\mathcal{M}_{2N}) \qquad (3.13)$$

$$\mathcal{J}(\widehat{B}) = \widehat{B}^{\dagger} , \quad \forall \widehat{B} \in \Omega(\mathcal{V}) \qquad (3.14)$$

$$\mathcal{J}(|\alpha\rangle \otimes |\beta\rangle) = |\beta\rangle \otimes |\alpha\rangle , \quad \forall |\alpha\rangle \otimes |\beta\rangle \in \mathcal{V} \otimes \mathcal{V} \qquad (3.15)$$

In the literature[13,14] $\mathcal{J}$ is referred to as the modular conjugation operator. It plays an important role in the GNS construction because, at the level of $\mathcal{V} \otimes \mathcal{V}$, it interchanges the first with the second Hilbert space, see eq. (3.15). For our purposes it is natural to introduce its action on symbols as ordinary complex conjugation and on operators as hermitian conjugation because then $\mathcal{J}$ commutes with op and vec:

$$\mathcal{J}(vec(B)) = \mathcal{J}\| \widehat{B} \gg = vec(\mathcal{J}(\widehat{B})) = vec(\widehat{B}^{\dagger}) = \| \widehat{B}^{\dagger} \gg \qquad (3.16)$$

$$\mathcal{J}(op(B)) = \mathcal{J}(\widehat{B}) = op(\mathcal{J}(B)) = op(B^*) \qquad (3.17)$$

Eq. (3.16) follows easily from the definitions (3.2), (3.15) and in eq. (3.17) we used (1.7). Clearly $\mathcal{J}$ is an involution, $\mathcal{J}^2 = 1$. Because hermitian conjugation of operators changes

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the order of products, a similar property must hold for the symbols. In fact, from (1.11) it follows that

$$\mathcal{J}(f * g) = \mathcal{J}(g) * \mathcal{J}(f) \tag{3.18}$$

The modular conjugation $\mathcal{J}$ is useful in pushing forward to the space $\mathcal{V} \otimes \mathcal{V}$ the usual operator multiplication which takes place in the space $\mathcal{V}$. An elementary calculation shows that for any $\widehat{f}, \widehat{B} \in \Omega(\mathcal{V})$

$$\|\widehat{f}\widehat{B} \gg = (\widehat{f} \otimes I)\|\widehat{B} \gg \tag{3.19}$$

$$\|\widehat{B}\widehat{f} \gg = \mathcal{J}(\widehat{f}^\dagger \otimes I)\mathcal{J}\|\widehat{B} \gg \tag{3.20}$$

The two $\mathcal{J}$'s on the RHS of (3.20) have the following origin: in going from $\widehat{B}$ in (3.1) to $\|\widehat{B} \gg$ in (3.2), we have turned the "bra" $\langle \alpha|$ into a "ket" $|\alpha\rangle$, therefore the *right* multiplication of $\widehat{B}$ by $\widehat{f}$ means that $\widehat{f}$ acts on the second factor of $\mathcal{V} \otimes \mathcal{V}$. In (3.20) this is achieved by first interchanging the two factors of $\mathcal{V} \otimes \mathcal{V}$, then acting with $\widehat{f}^\dagger$ on the first factor, and then interchanging the factors once more to restore the original order. In matrix language we could write $\mathcal{J}(\widehat{f}^\dagger \otimes I)\mathcal{J} = I \otimes \widehat{f}^T$ where $(\cdot)^T$ denotes transposition.

We shall now return to the W_∞-generators L_f and R_f and use them in order to write

$$op(L_f B) = \widehat{f}\widehat{B}$$
$$op(R_f B) = \widehat{B}\widehat{f} \tag{3.21}$$

Applying the *vec*-map to (3.21) and using (3.19), (3.20) yields for $\Theta \equiv vec \circ op$:

$$\Theta(L_f B) = (\widehat{f} \otimes I)\,\Theta(B)$$
$$\Theta(R_f B) = \mathcal{J}(\widehat{f}^\dagger \otimes I)\mathcal{J}\,\Theta(B) \tag{3.22}$$

These are the relations we wanted to derive. They show that, at the level of symbols, the operators L_f and R_f induce transformations on the first and second factor of the doubled Hilbert space $\mathcal{V} \otimes \mathcal{V}$, respectively. In practice B could be the symbol of some positive operators $\widehat{B} \equiv \widehat{\varrho}^{\frac{1}{2}}$ representing a (mixed) state, say, and $\widehat{f} \equiv A$ some hermitian observable for instance. Then, in the language of Thermo Field Dynamics[14], the operator $A \equiv A \otimes I$ would refer to the "particles" while the "tilde operator"

$$\widetilde{A} \equiv \mathcal{J}A\mathcal{J} = \mathcal{J}(A \otimes I)\mathcal{J} \tag{3.23}$$

would play the corresponding role for the "tilde particles" living in the second factor of $\mathcal{V} \otimes \mathcal{V}$. Thus, for f real, L_f and R_f generate unitary transformations on the Hilbert spaces

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of "particles" and "tilde particles", respectively. Eq. (3.22) yields

$$\Theta\left(e^{iL_f}B\right) = \left(e^{i\widehat{f}}\otimes I\right)\Theta(B)$$
$$\Theta\left(e^{iR_f}B\right) = \mathcal{J}\left(e^{-i\widehat{f}}\otimes I\right)\mathcal{J}\,\Theta(B)$$

(3.24)

In section 2 we saw that, at the quantum level, the canonical transformations had a direct product structure $\mathcal{A}_L \times \mathcal{A}_R \sim W_\infty \times W_\infty$. In the GNS language the transformations correspond to independent unitary transformations on the first and the second factor of the representation space, respectively. We have argued that in the classical limit the algebra $\mathcal{A}_L \times \mathcal{A}_R$ is contracted to its diagonal subalgebra $\mathcal{A}_{diag}$ generated by $V_f = \frac{1}{i\hbar}\left(L_f - R_f\right)$. These transformations act on the first and the second factor of $\mathcal{V} \otimes \mathcal{V}$ with the same unitary transformation. In the case of the linear combination A_f, which decouples for $\hbar \to 0$, "particles" and "tilde particles" would be transformed differently. Therefore only those quantum canonical transformations survive the classical limit $\hbar \to 0$ which treat particles and tilde particles on an equal footing. Thus $\mathcal{A}_f$ has no analogue in the classical differential geometry of phase space.

Looking now at the action of the modular conjugation $\mathcal{J}$ on L_f, R_f (for f real) we see that they are are interchanged:

$$\mathcal{J}R_f\mathcal{J} = L_f$$
$$\mathcal{J}L_f\mathcal{J} = R_f$$
$$\mathcal{J}V_f\mathcal{J} = V_f$$
$$\mathcal{J}A_f\mathcal{J} = -A_f$$

(3.25)

This is most easily seen in eqs.(2.2). Eqs.(3.25) and the algebra (2.9) suggest an amusing analogy with current algebra in QCD, say: R_f and L_f are right- and left-handed currents which are interchanged by the "parity operator" $\mathcal{J}$. The linear combinations V_f and A_f correspond to vector and axial vector currents which are even and odd under parity, respectively. The contraction to the vector-subgroup in the classical limit, $\mathcal{A}_L \times \mathcal{A}_R \to \mathcal{A}_{diag}$, is reminiscent of chiral symmetry breaking in this language.

In CM we are used to the fact that a real function $f(\phi)$ gives rise to one generator of a canonical transformation, namely to the hamiltonian vector field $h_f^a = \omega^{ab}\partial_b f$. In QM instead we have two deformed "vector fields", L_f and R_f, so one might suspect that we are dealing with a reducible representation of the canonical transformations. This is actually not the case, however, because L_f and R_f act on different Hilbert spaces and in a different

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manner. Representing both $\mathcal{A}_L$ and $\mathcal{A}_R$ on the space of states allows also $\mathcal{J}$ to be represented on $\mathcal{V} \otimes \mathcal{V}$ and on the space of symbols, respectively. This is not possible in the standard pure-state QM dealing with one Hilbert space $\mathcal{V}$ only. In this case $\mathcal{J}$ would map vectors $|\psi\rangle \in \mathcal{V}$ into dual vectors $\langle\psi| \in \mathcal{V}^*$. In the GNS construction the second factor of $\mathcal{V} \otimes \mathcal{V}$ plays the role of the dual space $\mathcal{V}^*$, so that $\mathcal{J}$ does not carry the states out of the space.

It is instructive to compare the present approach to the one of geometric quantization[5]. There one has the problem of representing canonical transformations on the prequantum Hilbert space of functions $\psi(p, q)$. This space is remarkably similar to the space of symbols but one finds[5] that the canonical transformations cannot all be represented *irreducibly* on the prequantum Hilbert space. As a way out, one introduces a *polarization*, which, roughly speaking, reduces the space of functions $\psi(p, q)$ to a subset (for example the Schrödinger polarization reduces it to the functions $\psi(q)$). The resulting wave functions form the state space $\mathcal{V}$. On $\mathcal{V}$ the canonical transformations are represented irreducibly but, as discussed above, $\mathcal{J}$ cannot be represented on this space. In a pictorial way, we can summarize all this by saying that geometric quantization solved the problem of the irreducibilty of the representations of canonical transformations by *halving* the number of arguments of ψ, while the GNS solved the same problem by *doubling* the Hilbert space.

Let us finally look at the time evolution of states in $\mathcal{V} \otimes \mathcal{V}$ in more detail. In operatorial language the time evolution of density operators $\widehat{\varrho}$ is governed by von Neumann's equation

$$i\hbar \, \partial_t \widehat{\varrho} = \left[\widehat{H}, \widehat{\varrho} \right] \tag{3.26}$$

In the symbol language it becomes

$$\partial_t \varrho = \left\{ H, \varrho \right\}_{mb} \tag{3.27}$$

The *vec*-map would associate to $\widehat{\varrho}$ the vector $\| \widehat{\varrho} \gg \in \mathcal{V} \otimes \mathcal{V}$, which is not what we want however. Actually it is the bracket

$$\ll \widehat{\varrho}^{\frac{1}{2}} \| \widehat{A} \otimes I \| \widehat{\varrho}^{\frac{1}{2}} \gg = Tr \left[\widehat{\varrho} \, \widehat{A} \right] \tag{3.28}$$

which gives the correct statistical average of $\widehat{A}$. This means that we have to distinguish the

symbol of the density operator itself

$$\varrho = symb(\widehat{\varrho}) \qquad (3.29)$$

from the symbol representing its square root:

$$\sigma \equiv symb(\widehat{\varrho}^{\frac{1}{2}}) \qquad (3.30)$$

Applying $symb$ to $\widehat{\varrho}^{\frac{1}{2}}\widehat{\varrho}^{\frac{1}{2}} = \widehat{\varrho}$ we see that the symbol $\sigma(\phi)$ is the "star-square root" of $\varrho(\phi)$:

$$\sigma(\phi) * \sigma(\phi) = \varrho(\phi) \qquad (3.31)$$

(In the limit $\hbar \to 0$, $\sigma_{cl}(\phi) = \varrho_{cl}(\phi)^{\frac{1}{2}}$ is the ordinary square root of ϱ_{cl}.) Thus

$$\Theta(\sigma) = vec(op(\sigma)) = \|\widehat{\varrho}^{\frac{1}{2}} \gg \qquad (3.32)$$

In order to reproduce the time-evolution of ϱ we require that

$$\partial_t \sigma = \{H, \sigma\}_{mb}$$
$$= \frac{1}{i\hbar}(L_H - R_H)\sigma = V_H\sigma \qquad (3.33)$$

because then eq. (1.15) implies for $\varrho = \sigma * \sigma$:

$$\partial_t \varrho = \partial_t \sigma * \sigma + \sigma * \partial_t \sigma$$
$$= \{H, \sigma\}_{mb} * \sigma + \sigma * \{H, \sigma\}_{mb}$$
$$= \{H, \sigma * \sigma\}_{mb} \qquad (3.34)$$
$$= \{H, \varrho\}_{mb}$$

Applying $\Theta = vec \circ op$ to (3.33) we obtain the correponding equation for $\|\widehat{\varrho}^{\frac{1}{2}} \gg$:

$$i\hbar \, \partial_t \|\widehat{\varrho}^{\frac{1}{2}} \gg = \Theta(i\hbar \, \partial_t \sigma)$$
$$= \Theta(L_H\sigma) - \Theta(R_H\sigma)$$
$$= \left[(\widehat{H} \otimes I) - \mathcal{J}(\widehat{H} \otimes I)\mathcal{J}\right]\|\widehat{\varrho}^{\frac{1}{2}} \gg \qquad (3.35)$$
$$= \bar{H}\|\widehat{\varrho}^{\frac{1}{2}} \gg$$

In the last line of eq. (3.35) we used (3.22). In eq. (3.35) we recognize[14] the Hamiltonian for the time evolution of GNS states:

$$\bar{H} \equiv \widehat{H} \otimes I - \mathcal{J}(\widehat{H} \otimes I)\mathcal{J} \qquad (3.36)$$

Of course we could have derived (3.35) in the standard manner by starting directly from the equation for the operator $\widehat{\varrho}$. Here instead we wanted to emphasize that it is the symbol

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σ (rather than ϱ) which represents the GNS states in $Fun(\mathcal{M}_{2N})$. At the level of σ, the two parts of the Hamiltonian $\bar{H}$ are the W_∞-generators L_H and R_H:

$$
\begin{aligned}
H &\equiv \widehat{H} \otimes I \longleftrightarrow L_H \\
\widetilde{H} &\equiv \mathcal{J}\left(\widehat{H} \otimes I\right)\mathcal{J} \longleftrightarrow R_H \\
\bar{H} &\equiv H - \widetilde{H} \longleftrightarrow i\hbar V_H
\end{aligned}
\tag{3.37}
$$

At this point one could ask why, even at the quantum level, the evolution of the symbols σ or ϱ, respectively, involves only the operator V_H, but not A_H:

$$
\partial_t \varrho(\phi, t) = V_H \varrho(\phi, t)
\tag{3.38}
$$

Clearly the V_H-transformation corresponds to the usual unitary time-evolution $\widehat{\varrho}(t) = U(t)\widehat{\varrho}(0)U^{-1}(t)$, but the A_H transformation (remembering eq. (2.10)) would lead to $\widehat{\varrho}(t) = U(t)\widehat{\varrho}U(t)$. Now let us assume that, at $t = 0$, $\widehat{\varrho}$ describes a pure state, i.e., that $\widehat{\varrho}(0)^2 = \widehat{\varrho}(0)$. This condition is preserved under the transformation generated by V_H, but it is not preserved by A_H. This means that V_H maps pure states on pure states, but A_H maps pure states into mixed ones. It is important to see that in the present formalism the evolution of pure states to pure states is protected by a *symmetry*, namely modular conjugation $\mathcal{J}$. In fact, let us assume that we modify eq. (3.38) by adding a piece containing an A-type generator:

$$
\partial_t \varrho(\phi, t) = \left[V_H + \epsilon A_G\right] \varrho(\phi, t)
\tag{3.39}
$$

Here ϵ is a (small) real parameter and G a second Hamiltonian. Applying $\mathcal{J}$ to eq. (3.39), the operator inside the square brackets changes to $V_H - \epsilon A_G$ according to (3.25), so that eq. (3.39) violates modular conjugation symmetry.

We see that QM has a universal symmetry, modular conjugation, which forbids pure states to evolve into mixed ones. It cannot be represented by the standard formulation of QM as an automorphism of $\mathcal{V}$, but this is possible in the space $\mathcal{V} \otimes \mathcal{V}$ which translates into $Fun(\mathcal{M}_{2N})$ in the Moyal approach formulated here. One of the reasons of why this formulation is very attractive is that, as $\mathcal{J}$ is now represented on the space of states, it is on the same logical footing as the other (discrete) symmetries we are familiar with. One could now try to find mechanisms that break this symmetry and allow for the transition of pure states into mixed ones. Such processes might occur under extreme conditions in quantum gravity, in the late stage of black hole evaporation, for instance[22]. If, by some novel

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mechanism, a term ϵA_G would be induced, then the resulting modified form of quantum mechanics would not have a classical limit $\hbar \to 0$ at fixed $\epsilon \neq 0$ because A_G diverges in this limit. For a classical limit to exist we need that ϵ vanishes at least proportional to $\hbar$. This possibility may arise if the A_G-term is generated as an anomaly, on the same line as the parity anomaly in odd-dimensional spacetimes[23]. Work is in progress in this direction[24].

4. THE EXTENDED MOYAL FORMALISM

In this section we identify the $\mathcal{A}_L \times \mathcal{A}_R$ algebra in the extended Moyal formalism proposed recently[15]. It was introduced in order to formulate a quantum deformed exterior calculus on phase-space. The basic idea is to work on a $8N$-dimensional supermanifold $\mathcal{M}_{8N}$, the extended phase-space, which is closely related to the tangent and cotangent bundle over the standard phase-space $\mathcal{M}_{2N}$. The coordinates on $\mathcal{M}_{8N}$ are the $8N$-tuples $(\phi^a, \lambda_a, c^a, \bar{c}_a)$ where ϕ^a are the usual coordinates on $\mathcal{M}_{2N}$, λ_a are commuting auxiliary variables, and $c^a, \bar{c}_a$ are anticommuting coordinates. Standard phase-space is identified with the $(\phi^a, 0, 0, 0)$-hypersurface in $\mathcal{M}_{8N}$. For functions on the extended phase-space, $A, B \in Fun(\mathcal{M}_{8N})$ one can introduce the extended star product $*_e$:

$$A *_e B = A \, exp\Big[\frac{i}{2}\Big(\frac{\overleftarrow{\partial}}{\partial \phi^a}\frac{\overrightarrow{\partial}}{\partial \lambda_a} - \frac{\overleftarrow{\partial}}{\partial \lambda_a}\frac{\overrightarrow{\partial}}{\partial \phi^a}\Big) + \frac{\overleftarrow{\partial}}{\partial c^a}\frac{\overrightarrow{\partial}}{\partial \bar{c}_a}\Big]B \qquad (4.1)$$

and the extended Moyal bracket (emb):

$$\{A, B\}_{emb} = \frac{1}{i}\Big[A *_e B - (-)^{[A][B]}B *_e A\Big] \qquad (4.2)$$

Here $[A] = 0$ or 1 depending on the Grassmann parity of A. Apart from obvious grading factors, the extended Moyal bracket enjoys the same algebraic properties as the ordinary Moyal bracket. The fundamental brackets are

$$\{\phi^a, \lambda_b\}_{emb} = \delta^a_b \;\; , \;\; \{\phi^a, \phi^b\}_{emb} = 0 = \{\lambda_a, \lambda_b\}_{emb} \qquad (4.3)$$

$$i\{c^a, \bar{c}_a\}_{emb} = \delta^a_b, \;\; all \; others = 0 \qquad (4.4)$$

The first of eqs. (4.3) shows that λ_a plays the role of a "momentum" conjugate to ϕ^a. Differently from the standard Moyal bracket for which $\{\phi^a, \phi^b\} = \omega^{ab}$, the ϕ^a's have vanishing em-brackets among themselves. They behave like $2N$-position coordinates on

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a $4N$-dimensional (bosonic) phase-space $\mathcal{M}_{4N} \equiv \{(\lambda_a, \phi^a)\}$. In fact, the bosonic piece of (4.1) follows from (1.11) by replacing $q^i \to \phi^a$, $p_i \to \lambda_a$. Under diffeomorphisms on $\mathcal{M}_{2N}$, λ_a and $\bar{c}_a$ transform like derivatives ∂_a and c^a like the differentials $d\phi^a$. This allows us to represent differential forms on $\mathcal{M}_{2N}$ in terms of scalar functions on $\mathcal{M}_{8N}$. A p-form $F = \frac{1}{p!} F_{a_1 \cdots a_p}(\phi) \, d\phi^{a_1} \wedge \cdots \wedge d\phi^{a_p}$, say, is replaced by

$$\widehat{F} = \frac{1}{p!} \, F_{a_1 \cdots a_p}(\phi) \, c^{a_1} \cdots c^{a_p} \in Fun(\mathcal{M}_{8N}) \qquad (4.5)$$

The anticommutativity of the c^a's mimics the wedge product now. In the quantum deformed tensor calculus proposed in ref.[15], operations such as the exterior derivative, the contraction or the Lie derivative of F were expressed as the extended Moyal bracket of $\widehat{F}$ with appropriate functions on $\mathcal{M}_{8N}$. The Lie derivative[*] along the Hamiltonian vector field h, say, is given by

$$L_h \widehat{F} = \mathcal{P}\{\widehat{F}, \widetilde{\mathcal{H}}\}_{emb} \qquad (4.6)$$

Here $\widetilde{\mathcal{H}}$ is a certain "super-Hamiltonian"[25] on the extended phase-space which is constructed from h. Furthermore $\mathcal{P}$ is the projection operator on the $(\lambda_a = 0, \bar{c}_a = 0)$-hypersurface in $\mathcal{M}_{8N}$. This type of exterior calculus has the property that "physical" fields live in (ϕ, c)-space, but the tensor manipulations are realized by Moyal brackets on the larger space $\mathcal{M}_{8N}$. This is why we need the projector $\mathcal{P}$ in eq. (4.6). We are not going into further details here; the interested reader is referred to refs.[15]. Instead we shall give a self-contained description of the $W_\infty \times W_\infty$ generators on the λ, ϕ-space, $\mathcal{M}_{4N}$. We shall also identify the modular transformations $\mathcal{J}$ and the complex structure with respect to which it is a conjugation. This is interesting in itself and we shall not consider the fermionic variables c^a and $\bar{c}_a$ here.

We consider the algebra of functions $Fun(\mathcal{M}_{4N})$ equipped with the extended star-product (4.1) with the Grassmannian piece omitted. Loosely speaking, we can identify the space $\mathcal{M}_{4N}$ with the total space of the tangent bundle over standard phase-space, $T\mathcal{M}_{2N}$[15]. The reason is that for any function f on $\mathcal{M}_{2N}$ we have $\{f(\phi), \lambda_a\}_{emb} = \partial_a f(\phi)$. This means that in the Hamiltonian formalism, λ_a plays the role of the derivatives ∂_a, i.e., of a basis in tangent space. In a sense, what we are considering here is the "lift" of the Moyal formalism from phase-space to its tangent bundle: ϕ^a are coordinates in the base manifold $\mathcal{M}_{2N}$ and λ_a are coordinates in the fibers. In order to

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[*] For the definition of the classical analog of this and similar operations, we refer the reader to refs.[3] and [22].

find the relevant $\mathcal{A}_L \times \mathcal{A}_R$ generators, it is advantageous to perform a change of variables which mixes the ϕ's with the λ's:

$$
\begin{aligned}
Z_+^a &= \phi^a + \frac{\hbar}{2}\omega^{ab}\lambda_b \\
Z_-^a &= \phi^a - \frac{\hbar}{2}\omega^{ab}\lambda_b
\end{aligned}
\tag{4.7}
$$

When expressed in terms of these new variables the extended star-product becomes

$$
A *_e B = A \, exp\left[\frac{i\hbar}{2}\Big(\frac{\overleftarrow{\partial}}{\partial Z_-^a}\omega^{ab}\frac{\overrightarrow{\partial}}{\partial Z_-^b} - \frac{\overleftarrow{\partial}}{\partial Z_+^a}\omega^{ab}\frac{\overrightarrow{\partial}}{\partial Z_+^b}\Big)\right] B
\tag{4.8}
$$

where A and B are functions of both Z_+^a and Z_-^a. Let us look in particular at functions which depend on Z_-^a or Z_+^a only; we shall refer to them, with a slight abuse of language, as "holomorphic" and "anti-holomorphic", respectively. Comparing eqs. (4.8) and (1.10) we see that for (anti-) holomorphic functions the extended star product has the same form as the usual one. In fact, we can write

$$
\begin{aligned}
A(Z_-) *_e B(Z_-) &= A(Z_-) \, exp\left[\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial Z_-^a}\omega^{ab}\frac{\overrightarrow{\partial}}{\partial Z_-^b}\right] B(Z_-) \\
&= A(\phi) * B(\phi)|_{\phi=Z_-} \\
A(Z_+) *_e B(Z_+) &= A(Z_+) \, exp\left[-\frac{i\hbar}{2}\frac{\overleftarrow{\partial}}{\partial Z_+^a}\omega^{ab}\frac{\overrightarrow{\partial}}{\partial Z_+^b}\right] B(Z_+) \\
&= B(\phi) * A(\phi)|_{\phi=Z_+}
\end{aligned}
\tag{4.9}
$$

We also see that the $*_e$-product of a holomorphic with an anti-holomorphic function does not involve any derivatives:

$$
\begin{aligned}
A(Z_-) *_e B(Z_+) &= A(Z_-)B(Z_+) \\
A(Z_+) *_e B(Z_-) &= A(Z_+)B(Z_-)
\end{aligned}
\tag{4.10}
$$

Similarly one finds for the extended Moyal brackets of (anti-)holomorphic functions

$$
\begin{aligned}
\{A(Z_-), B(Z_-)\}_{emb} &= + \hbar\{A(\phi), B(\phi)\}_{mb}|_{\phi=Z_-} \\
\{A(Z_+), B(Z_+)\}_{emb} &= - \hbar\{A(\phi), B(\phi)\}_{mb}|_{\phi=Z_+} \\
\{A(Z_-), B(Z_+)\}_{emb} &= 0
\end{aligned}
\tag{4.11}
$$

We see that, with respect to the extended Moyal bracket, the holomorphic and anti-holomorphic functions form closed sub-algebras which are mutually commuting, and each of

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which is isomorphic to the standard Moyal bracket algebra on $\mathcal{M}_{2N}$. In particular the Z's satisfy

$$
\begin{aligned}
\left\{Z_-^a, Z_-^b\right\}_{emb} &= +\hbar\omega^{ab} \\
\left\{Z_+^a, Z_+^b\right\}_{emb} &= -\hbar\omega^{ab} \\
\left\{Z_+^a, Z_-^b\right\}_{emb} &= 0
\end{aligned}
\tag{4.12}
$$

Apart from the factors of $\pm\hbar$, we get two closed algebras similar to $\left\{\phi^a, \phi^b\right\}_{mb} = \omega^{ab}$. This means that $Fun(\mathcal{M}_{4N})$, equipped with the $\{\cdot,\cdot\}_{emb}$ is equivalent to two copies of $Fun(\mathcal{M}_{2N})$ with the ordinary Moyal bracket. As we shall see, the modular conjugation $\mathcal{J}$ interchanges the two copies.

In order to recover the $W_\infty \times W_\infty$ generators on $\mathcal{M}_{4N}$, let us fix a real function $f \in Fun(\mathcal{M}_{4N})$ and let us define the following operators on $Fun(\mathcal{M}_{4N})$:

$$
\begin{aligned}
\mathcal{L}_f &= f(Z_-) *_e \\
\mathcal{R}_f &= f(Z_+) *_e
\end{aligned}
\tag{4.13}
$$

The notation $f = f(Z_-^a)$ means that we replace ϕ^a by Z_-^a in the original $f = f(\phi)$. In this way the operators $\mathcal{L}_f$ and $\mathcal{R}_f$ have a non-trivial action on any (not necessarily holomorphic and anti-holomorphic) function of ϕ and λ. Using the associativity of the $*_e$-product and eqs. (4.11), it is easy to find their commutator algebra:

$$
\begin{aligned}
\left[\mathcal{L}_{f_1}, \mathcal{L}_{f_2}\right] &= +i\hbar\mathcal{L}_{\{f_1,f_2\}_{mb}} \\
\left[\mathcal{R}_{f_1}, \mathcal{R}_{f_2}\right] &= -i\hbar\mathcal{R}_{\{f_1,f_2\}_{mb}} \\
\left[\mathcal{L}_{f_1}, \mathcal{R}_{f_2}\right] &= 0
\end{aligned}
\tag{4.14}
$$

Clearly this is isomorphic to the algebra of L_f and R_f of eq. (2.7).

We stressed repeatedly that the hypersurface in $\mathcal{M}_{4N}$ on which $\lambda = 0$ is identified with the ordinary phase-space. Let us therefore ask what is the effect of the transformations generated by $\mathcal{L}_f$ and $\mathcal{R}_f$ on this surface. For any function $A \in Fun(\mathcal{M}_{4N})$ we define its projection $\mathcal{P}A$ by

$$
\mathcal{P}A(\lambda, \phi) = A(0, \phi)
\tag{4.15}
$$

Applying $\mathcal{P}$ to eqs. (4.9) and (4.10) we have for (anti-)holomorphic functions:

$$
\begin{aligned}
\mathcal{P}\, A(Z_-) *_e B(Z_-) &= A(\phi) * B(\phi) \\
\mathcal{P}\, A(Z_+) *_e B(Z_+) &= B(\phi) * A(\phi) \\
\mathcal{P}\, A(Z_-) *_e B(Z_+) = \mathcal{P}\, A(Z_+) *_e\, B(Z_-) &= A(\phi)B(\phi)
\end{aligned}
\tag{4.16}
$$

These equations show that, upon projection, $\mathcal{L}_f$ $(\mathcal{R}_f)$ has a very simple effect on holomrphic

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(anti-holomorphic) functions:

$$P \, \mathcal{L}_f g(Z_-) = L_f g(\phi)$$
$$P \, \mathcal{R}_f g(Z_+) = R_f g(\phi)$$

(4.17)

Interpreted in the spirit of refs.[15] these equations tell us how the operators L_f and R_f get "lifted" from $\mathcal{M}_{2N}$ to $\mathcal{M}_{4N} \sim T\mathcal{M}_{2N}$. If we go over from the ordinary Moyal formalism to the extended one, we have to replace $f(\phi)*$, say, by $f(Z_-)*_e$ and set $\lambda = 0$ after having performed the derivatives implicit in $*_e$. As we mentioned already in the discussion following eq. (4.6) , the approach of refs.[15] realizes the complicated operations of the quantum exterior calculus on $\mathcal{M}_{2N}$ as the "shadow" of simpler operations on the extended phase space. In this picture L_f and R_f are the "shadows" of $\mathcal{L}_f$ and $\mathcal{R}_f$.

By now it should also be clear how the modular conjugation $\mathcal{J}$ acts on $\mathcal{M}_{4N}$. It reverses the sign of λ_a so that Z_+^a and Z_-^a get interchanged[15]. Consequently $\mathcal{J}$ also interchanges $\mathcal{L}_f$ and $\mathcal{R}_f$:

$$\mathcal{J} \mathcal{L}_f \mathcal{J} = \mathcal{R}_f$$

(4.18)

Of course $\mathcal{L}_f$ and $\mathcal{R}_f$ form a representation of $W_\infty \times W_\infty$ on the full space $Fun(\mathcal{M}_{4N})$, not only on the (anti-)holomorphic subspace. However, the projection of $\mathcal{L}_f A(Z_-, Z_+)$ has no simple interpretation in terms of the ordinary Moyal formalism: one obtains $L_f A(\phi, \phi)$ with L_f acting on the first argument of A only.

This concludes our discussion of the quantum canonical transformations on the extended phase-space, i.e., on the tangent bundle $T\mathcal{M}_{2N}$. It is interesting to note that the algebra of functions on this space seems to be very important also in recent attempts to formulate a general theory of "W-geometry"[26].

5. CONCLUSIONS

Looking back at the previous sections in the spirit of non-commutative geometry[11], we can say that quantum phase space is a typical example of a "non-commutative manifold". It can be studied by investigating the properties of the algebra of functions defined over $\mathcal{M}_{2N}$. Deforming the ordinary pointwise product on $Fun(\mathcal{M}_{2N})$ to the star-product, we can introduce the Moyal bracket and obtain a quantum deformation of the Poisson bracket. In this context "quantization" means that we replace the infinite dimensional Lie algebra of $Fun(\mathcal{M}_{2N})$ equipped with the Poisson bracket, by the deformed algebra equipped with

the Moyal bracket. Thus the Moyal bracket algebra, like the commutator algebra but unlike the Poisson bracket algebra, has an underlying associative product, with respect to which it is defined as a commutator. All the differences between the *classical* and the *quantum geometry* of $\mathcal{M}_{2N}$ are encoded in the properties of the star- product. By invoking the isomorphism between star-multiplication and operator products, we have been able to relate $Fun(\mathcal{M}_{2N})$ to the GNS representation space $\mathcal{V} \otimes \mathcal{V}$. This seems to support the view that the quantum deformed geometry of phase-space is indeed encoded in the GNS construction and in particular in the doubled Hilbert space $\mathcal{V} \otimes \mathcal{V}$. It would be nice to study the topological and cohomological property of this space by using the quantum exterior calculus developed in ref.[15] and following the lines of ref.[27].

Another issue which is particularly intriguing is the fact that modular conjugation $\mathcal{J}$ appears on a purely geometrical basis here. It owes its existence to the very nature of non-commutative geometry: only because the star product is non-commutative there can be a difference between left and right multiplication and hence an exchange transformation $\mathcal{J}$.

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