ISTITUTO NAZIONALE DI FISICA NUCLEARE

Sezione di Trieste

INFN/AE-93/18

8 ottobre 1993

E. Gozzi

COUNTING PERIODIC TRAJECTORIES VIA TOPOLOGICAL CLASSICAL MECHANICS

Servizio Documentazione
dei Laboratori Nazionali di Frascati

Counting periodic trajectories via topological classical mechanics

E.Gozzi

Dipartimento di Fisica Teorica, Università di Trieste,
Strada Costiera 11, P.O.Box 586, Trieste, Italy
and INFN sezione di Trieste

ABSTRACT

We prove that the number of periodic trajectories of arbitrary period T on the flow tangent to periodic trajectories in phase space of the same period T, is equal to the Euler number of the underlying phase-space. This results holds for systems with compact phase-space and isolated periodic orbits.

In the last few years we[1] have developed a path-integral approach to classical Hamiltonian mechanics (CPI). This is nothing else than the path-integral version of the operatorial formulation of classical mechanics of Koopman and J.von Neumann[2]. One advantage of our formulation is that it brings to light the well-known geometry[3] of classical mechanics, and it does so using the modern language of BRS and Supersymmetry invariances on the same lines of modern topological field theories[4][5]. Koopman and von Neumann[2] put forward an *operatorial* approach to classical mechanics (CM) in 1931 in order to better compare it with quantum mechanics (QM). We will review now this formulation[2] for the reader not familiar with it. Let us start from Hamilton's equations:

$$\dot{\phi}^a(t) = \omega^{ab}\partial_b H\big(\phi(t)\big) \tag{1}$$

where $\phi^a \equiv \big(q^1,\cdots,q^n,p_1,\cdots,p_n\big)$, $a = 1,\cdots,2n$, are coordinate on a $2n$-dimensional phase-space $\mathcal{M}_{2n}$, H is the Hamiltonian and $\omega^{ab} = -\omega^{ba}$ is the standard symplectic matrix. Any probability density function $\varrho(\phi^a,t)$ on phase space has a time-evolution of the form

$$\frac{\partial}{\partial t}\varrho(\phi^a,t) = -\{\varrho,H\} \equiv -\hat{L}\varrho(\phi^a,t) \tag{2}$$

where $\hat{L} = -\partial_a H\omega^{ab}\partial_b$ is the well-known Liouville operator which is the central element of the operatorial approach to classical mechanics[2]. Equation (2) can be formally solved as

$$\varrho(\phi,t) = e^{-\hat{L}t}\varrho(\phi,0) \tag{3}$$

This is basically the operatorial version of Classical Hamiltonian mechanics. As the reader may guess, as there is an operatorial approach there must exist also a corresponding path-integral formulation like it happens in quantm mechanics. The simplest idea one can envision is to write a classical generating functional of the form:

$$\mathrm{Z}_{cm}[j] = \int \mathcal{D}\phi^a\widetilde{\delta}[\phi^a - \phi^a_{cl}]\; exp\int j\phi\; dt \tag{4}$$

where ϕ^a_{cl} are the classical solutions of Hamilton's equations. The delta-functional is forcing the system to lie on its classical trajectories and it can be rewritten as:

$$\widetilde{\delta}\big[\phi^a - \phi^a_{cl}\big] = \widetilde{\delta}\big[\dot{\phi}^a - \omega^{ab}\partial_b H\big]\, det\Big(\partial_t\delta^a_b - \omega^{ac}\partial_c\partial_b H\Big) \tag{5}$$

Fourier transforming the delta function on the RHS of (5), using an auxiliary field λ_a , and exponentiating the determinant via anticommuting variables c^a,$\bar{c}_a$, we can re-write (4) as:

$$Z_{cm} = \int \mathcal{D}\phi^a \mathcal{D}\lambda_a \mathcal{D}c^a \mathcal{D}\bar{c}_a \, exp \; i \int dt \, \{\widetilde{\mathcal{L}} + j\phi\} \tag{6}$$

with the Lagrangian*

$$\widetilde{\mathcal{L}} = \lambda_a \left[\dot{\phi}^a - \omega^{ab}\partial_b H(\phi)\right] + i\bar{c}_a\left[\partial_t \delta^a_b - \omega^{ac}\partial_c\partial_b H(\phi)\right] c^b \tag{7}$$

The associated Hamilton function is

$$\widetilde{\mathcal{H}} = \lambda_a \omega^{ab}\partial_b H + i\bar{c}_a \omega^{ac}\partial_c\partial_b H c^b \tag{8}$$

Using standard techniques[6] we can easily compute, from the path-integral (6), the equal-time (anti-) commutators of ϕ^a,λ_a, c^a and $\bar{c}_a$ and ,we find that

$$\langle [\phi^a, \lambda_b] \rangle = i \, \delta^a_b, \langle [\bar{c}_b, c^a] \rangle = \delta^a_b \tag{9}$$

All other commutators vanish†. In particular, ϕ^a and ϕ^b commute for all values of the indices a and b. In terms of the q's and p's (which were combined into ϕ^a) this means $\langle [q^i, p_j] \rangle = 0$ for all i and j. This shows very clearly that we are doing classical mechanics and not quantum mechanics. The operator algebra (9) can be realized by differential operators

$$\lambda_a = -i\frac{\partial}{\partial\phi^a} \, , \; \bar{c}_a = \frac{\partial}{\partial c^a} \tag{10}$$

and multiplicative operators ϕ^a and c^a acting on functions $\widetilde{\varrho}(\phi^a, c^a, t)$. Inserting the above operators into $\widetilde{\mathcal{H}}$ we obtain:

$$\widetilde{\mathcal{H}} = -i\omega^{ab}\partial_b H\partial_a + i\frac{\partial}{\partial c^a}\omega^{ac}\partial_c\partial_b H c^b \tag{11}$$

It is clear from (11) that the Grassmannian part of $\widetilde{\mathcal{H}}$ gives zero if applied on distribution $\widetilde{\varrho}$ that do not contain anticommuting variables, while the bosonic part is $(-i)$ times the Liouvillian: $\widetilde{\mathcal{H}}|_{(c=0)} = -i\hat{L}$.

⋆ For more details, expecially regarding the extra currents which we add in (6), see ref. [1]

† Note that these are really commutators[1] and not Poisson brackets.

This confirms that our path- integral is the right one to reproduce the operatorial approach of Koopman and von Neumann or, stated in a another way, we can say that the measure in the path-integral that produces the Liouville operator is just a Dirac delta on the classical paths. We did, somehow, the analog of what Feynman did for the Schroedinger operator: he asked himself which was the weight in a path-integral that produces the Schroedinger kernel, and he found that it was $exp\ i\ S$; for the Liouvillian, instead, it is just a Dirac delta.

The reader at this point may wonder why, to the get the Liouvillan, we had to cut off the Grassmannian part in (11). To understand that we have to explain which is the meaning of the full $\widetilde{\mathcal{H}}$ of eqn.(11). This is explained in detail in ref.[1] and we refer the reader to that paper for details. There it is shown that the Grassmannian variables c^a can be interpreted as "forms", i.e., $c^a = d\phi^a$, in the cotangent space to phase space, while the $\bar{c}_a$ are a basis in the tangent space to phase-space (*i.e.*, they are a basis in the vector-field-space). The whole Cartan calculus on phase-space (exterior derivative, inner products, etc) has been translated in ref.[1], into a calculus based on these Grassmannian variables. It is then easy to prove[1] that $\widetilde{\mathcal{H}}$ is nothing else than the Lie-derivative $l_{(dH)^\sharp}$ of the Hamiltonian flow‡, precisely:

$$\widetilde{\mathcal{H}} = -il_{(dH)^\sharp}$$

This Lie-derivative generates the time-evolution not only of distributions in phase-space $\varrho(\phi^a)$ but also of general distributions $\widetilde{\varrho}(\phi^a, c^a)$ which are forms in phase-space. When we restrict this Lie-derivative to act only on $\varrho(\phi^a)$, then it becomes just the Liouvillian.

In this paper we will exploit those features of this formulation of classical mechanics which are more similar to those of the so called Topological Field Theories[4](TFT). The TFT are field theories whose only non-zero observables are topological invariants of the space of fields on which they are defined. The main feature of these theories is that they are invariant under deformation of some metric defined in field space .This is reflected in the fact that the Hamiltonian is a pure BRS variation, and the measure in the path-integral is BRS-invariant. The BRS we talk about is the BRS associated to the local symmetry of deformation of the metric we mentioned above. We will not enter into more details of these theories but refer the reader to ref.[4]. Also our formulation of CM has the feature that the

‡ Here we have used the notation[3] $(dH)^\sharp \equiv \omega^{ab}\partial_b H \partial_a$ for the Hamiltonian vector field generated by the gradient of H, and l_v denotes the Lie-derivative along some vector field v .

$\widetilde{\mathcal{H}}$ is a pure BRS variation, in fact:

$$\widetilde{\mathcal{H}} = -i\big[Q, [\bar{Q}, h]\big] \tag{12}$$

where $Q = ic^a\lambda_a$ is a BRS charge, $\bar{Q} = i\bar{c}_a\omega^{ab}\lambda_b$ an anti-BRS charge and $h = (dH)^\sharp$ the Hamiltonian vector field. The commutators we use are those defined in (9). Besides this feature anyhow we would need, to make our path-integral a TFT, that the measure in the path-integral (6) is BRS invariant. This is so if we choose, for example* periodic boundary conditions (pbc) *i.e.*:

$$\phi^a(T) = \phi^a(0) \equiv \phi_0^a \;\;,\;\; c^a(T) = c^a(0) \equiv c_0^a \tag{13}$$

where T is an interval of time to which we restrict the path-integral (6). So the path-integral which defines our theory is

$$Z_{cm}^{pbc}[j] = \int\limits_{pbc} \mathcal{D}\phi^a\mathcal{D}\lambda_a\mathcal{D}c^a\mathcal{D}\bar{c}_a \; exp\; i\int\limits_0^T dt\; \widetilde{\mathcal{L}} + j\phi \tag{14}$$

Now, with these b.c., our theory is truly similar to a TFT. Of course *this is not generic classical mechanics* which is defined by (6) and which, differently from any TFT, has also observables which are not topological invariants of the theory. This is guaranteed by the measure not being BRS invariant. The theory instead defined by (14) has, as observables, only topological invariants of the phase-space manifold and we called it in the title "*topological classical mechanics*"(TCM). To learn more about this TCM, we refer the reader to ref.[5] where the analogy between TFT and TCM have been worked out in more details. In particular in ref.[5] we show what is the analog for TCM of the deformation of the metric in TFT, and why all observables from (14) are topological invariants. We do not want anyhow to repeat all that material here.

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* There are other choices of BRS invariant boundary conditions, for example:

$$\phi^a(0) = k_1 \;\; \phi^a(T) = k_2 \;\;,\;\; c^a(0) = c^a(T) = 0$$

with k_1, k_2 constants. Onether choice is

$$\phi^a(0) = e^{i\alpha}\phi^a(T) \;\;, c^a(0) = e^{i\alpha}c^a(0)$$

The first topological invariant which we can calculate from (14) is $Z^{pbc}_{cm}[0]$ *i.e.*, the generating functional with the current put to zero, *i.e.* the partition function for TCM. This was already done in ref.[5] but we will repeat it here with more details because it is the heart of the present paper. It is easy to see[1,5] that $Z^{pbc}_{cm}[0]$ can be written as

$$\begin{aligned} Z^{pbc}_{cm}[0] &= \int_{pbc} \mathcal{D}\phi^a \mathcal{D}\lambda_a \mathcal{D}c^a \mathcal{D}\bar{c}_a \, exp \, i \int_0^T dt \, \widetilde{\mathcal{L}} \\ &= \int d^{2n}\phi_0 \, d^{2n}c_0 \, K(\phi^a_0, c^a_0, T|\phi^a_0, c^a_0, 0) \end{aligned} \tag{15}$$

where ϕ_0, c_0 are the initial and final points (coincident) and

$$K(\phi^a_f, c^a_f, T|\phi^a_i, c^a_i, 0) = \delta^{(2n)}(\phi^a_f - \phi^a_{cl}(t_f, \phi_i))\delta^{(2n)}(c^a_f - C^a_{cl}(t_f, c_i, [\phi])) \tag{16}$$

is the transition-probability between an initial configuration ϕ_i, c_i and a final one ϕ_f, c_f. In (16) ϕ^a_{cl} and C^a_{cl} devote classical solutions of the equations of motion derivable from $\widetilde{\mathcal{L}}$ of (7), that are

$$\begin{aligned} \dot{\phi}^a - \omega^{ab}\partial_b H &= 0 \\ [\partial_t \delta^a_b - \omega^{ac}\partial_c\partial_b H]c^b &= 0 \end{aligned} \tag{17}$$

Note that the solution C_{cl} functionally depends on the path $\phi_{cl}(t)$, as it clear in solving the second of equations (17). Also note[1] that the variables c^a have the same equation of motion as the Jacobi fields, $\delta\phi(t)$, which are defined as being the first variation around the classical trajectories. In fact, doing the first variation of the first of eqs. (17), we get

$$[\partial_t \delta^a_b - \omega^{ac}\partial_c\partial_b H]\delta\phi^b = 0 \tag{18}$$

which is the same equation as that of the c^a. So we can identify $c^a(t) \approx \delta\phi^a(t)$ and we can say that the motion of the c^a is the *tangent flow* to the classical motion of the $\phi_{cl}(t)$. The reader may be puzzled that in eq. (16) there are two Dirac deltas while in eq. (4) we only have one over the commuting variables. This puzzle is easily solved if one notes that in (15), once we integrate over c_0, only the Dirac delta over ϕ is left as in (4). Another puzzle which may worry the reader is in explaing how the Dirac delta over the c^a managed to appear in (16). Let us look at the $c^a, \bar{c}_a$ piece in (7). It is a first order fermionic action very similar to the usual Dirac action. It is well-known[7] that for action of this type the quantum mechanical transition amplitude is given by a δ-function containing a solution of the classical equation of motion. This is due to the fact that for any Lagrangian of the form $\bar{\psi}D\psi$, where D is some first order differential operator, the integral over $\bar{\psi}$ gives rise to a δ-function $\delta[D\psi]$. Thus only ψ-paths with $D\psi = 0$,*i.e.*, solutions of the equation of motion, contribute to the path-integral. This is what gives rise to the second Dirac delta in (16). A third puzzle for the reader may arise in explaining how the determinant which appeared in (5) magically disappeared in (16). The reason is very simple: in doing the λ_a integration in the first line of (15), we obtained the Dirac delta of the equation of motion for ϕ^a that is the Dirac delta appearing on the RHS of eq. (5) without determinant. The same thing we got for the variable c^a by integrating away the $\bar{c}_a$. In going then from the Dirac deltas of the eqs. of motion to the Dirac deltas of the solutions, we had to divide by the determinant for the bosonic variable, and multiply it for the anticommuting ones. These two determinants are equal and so they cancel each other producing eq. (16).

Having now clarified the interpretation of the $c^a(t)$ as describing the *tangent flow* to the classical motion of the $\phi^a_{cl}(t)$, let us read off the meaning of the $\mathrm{Z}^{pbc}_{cm}[0]$ of eq. (15) with the help of eq. (16). Basically, being the $K(\cdot|\cdot)$ a product of Dirac deltas, $\mathrm{Z}^{pbc}_{cm}[0]$ counts how many periodic trajectories of period T there are in $c^a(t)$,*i.e.* in the *tangent flow*; trajectories anyhow that are *tangent* not to generic trajectories in $\phi^a_{cl}(t)$ but only to those periodic with the same period T. So, to summarize, we can say that $\mathrm{Z}^{pbc}_{cm}[0]$ *counts the number of periodic trajectories of period* T *in the flow tangent to the periodic trajectories in phase-space of the same period T.* Operatively what one does is the following: first one builds all periodic trajectories of period T in phase-space, $\phi^a(t)$, then constructs all tangent trajectories to these and select only those which are periodic of the same period T. This number is $\mathrm{Z}^{pbc}_{cm}[0]$.

Let us now proceed to calculate $\mathrm{Z}^{pbc}_{cm}[0]$. We shall first compute $\mathrm{Z}^{pbc}_{cm}[0]$ in the limit $T \to 0$ and we shall later show that $\mathrm{Z}^{pbc}_{cm}[0]$ does not depend on T. The proof which we present here is the same we did in ref.[5]. We do it again here for completness. Let us first expand the solutions of eq. (17) for small T:

$$\begin{aligned} \phi^a_{cl}(T) &= \phi^a_0 + h^a(\phi_0)T + O(T^2) \\ C^a_{cl} &= c^a_0 + \partial_b h^a(\phi_0)c^b_0 T + O(T^2) \end{aligned} \tag{19}$$

where ϕ^a_0, c^a_0 are the initial conditions and $h^a = \omega^{ab}\partial_b H$ the components of the Hamiltonian vector field. Inserting the expansion (19) into the $K(\cdot|\cdot)$ of eq.(15), we get:

$$\begin{aligned} K(\phi_0^a, c_0^a, T|\phi_0^a, c_0^a, 0) &= \delta^{(2n)}(h^a(\phi_0)T)\ \delta^{(2n)}(\partial_b h^a(\phi_0)c^b T) \\ &= \delta^{(2n)}(h^a(\phi_0))\ \delta^{(2n)}(\partial_b h^a(\phi_0)c_0^b) \\ &= \delta^{(2n)}(h^a(\phi_0))\ det[\partial_b h^a(\phi_0)]\ \delta^{(2n)}(c_0^a) \end{aligned} \tag{20}$$

One immediately sees that, for closed paths and short times T, the Kernel $K(\cdot|\cdot)$ receives contributions only from the points where the vector field $h^a \equiv \omega^{ab}\partial_b H$ vanishes,*i.e.*, the points in which $\partial H = 0$, as $det\ \omega^{ab} \neq 0$, These are what are called, in Morse theory,[8] the *critical points* of H . Let us indicate them with $\phi^a_{(p)}$ and let us suppose for simplicity that H has only *isolated* and non-degenerate critical points. We can then rewrite (20) as:

$$K(\phi_0^a, c_0^a, T|\phi_0^a, c_0^a, 0) = \delta^{(2n)}(c_0^a)\sum_{(p)}\frac{det[\partial_b h^a(\phi_{(p)})]}{|det[\partial_b h^a(\phi_{(p)})]|}\delta^{(2n)}(\phi_0^a - \phi^a_{(p)}) \tag{21}$$

where the determinant in the denominator has appeared in transforming the Dirac delta $\delta^{(2n)}(h^a(\phi_0))$ appearing in (20) into the Dirac delta $\delta^{(2n)}(\phi_0^a - \phi^a_{(p)})$ appearing in (21). Inserting (21) in (15) and performing the integrations, we obtain

$$\begin{aligned} Z^{pbc}_{cm}[0] &= \sum_{(p)}\frac{det[\partial_b h^a(\phi_{(p)})]}{|det[\partial_b h^a(\phi_{(p)})]|} \\ &= \sum_{(p)}\frac{det[\partial_a\partial_b H(\phi_{(p)})]}{|det[\partial_a\partial_b H(\phi_{(p)})]|} \equiv \sum_{(p)}(-1)^{i_p} \end{aligned} \tag{22}$$

where i_p denotes the index[8] of the critical point (p), *i.e.*, the number of negative eigenvalues of the Hessian of H at (p) (in local coordinates this Hessian has components $\partial_a\partial_b H$). The result (22) has a well-known interpretation: it is the Morse theory representation of the Euler number χ of the phase-space manifold on which the "Morse function" H is defined. We will not expand on these technical details here but refer the reader to ref.[8] and [9]. Basically the Morse theory says that, whatever is the function defined on a manifold, the number $\sum_{(p)}(-1)^{i_p}$, built out of the features of its critical points, is independent of the function and equals the Euler number χ of the manifold. Thus we have found that

$$Z^{pbc}_{cm}[0] = \chi \tag{23}$$

Of course the Euler number is defined for compact manifolds so we have to require that our phase-space is compact.

If we remember that we had given a physical interpretation before to Z^{pbc}_{cm}, then what eq. (23) tell is that: *the number of periodic trajectories of arbitrary period T on the flow tangent to periodic trajectories in phase-space of the same period T, is equal to the Euler number of the underlying phase-space.*

Note that the result is independent of the period T provided that the period of the trajectories in the tangent flow and in the base space are the same. It is also somehow independent of the Hamiltonian we use, provided that this Hamiltonian has *separate* and non-degenerate critical points. We will expand on this point later on. Now we would like to return to the proof we presented because the reader may be puzzled by our approximation of taking $T \to 0$. We will present a discussion of this point[5] which is a little more sofisticated mathematically of the analog discussion of the "Witten index"[10]. The reason being that the supersymmetry present in our formulation of Classical mechanics is slightly different than the one present in supersymmetric quantum mechanics[10], and besides there are other subtle points of difference that the reader expert in this stuff should be able to notice. For the reader who may not be able to follow the next steps, we advise him to skip them and read only the last part of the paper.

The argument we use was presented in ref.[9] and [5]. We mentioned before that $Z^{pbc}_{cm}[0]$ can be considered like the partition function of our system. Actually it is something like the "super-partition" function[10] due to the pbc for the Grassmannian variables. We can in fact write

$$Z^{pbc}_{cm} = Tr[(-)^F e^{-iT\widetilde{\mathcal{H}}}] \tag{24}$$

here the trace "Tr" is performed over a complete set of functions (or differential forms) $\widetilde{\varrho}_p(\phi^a, c^a)$ or equivalently[1] a set of antisymmetric tensor fields $\widetilde{\varrho}_{a_1,\cdots a_p}(\phi)$ $p = 1 \cdots 2n$ which are the basis of an assigned Hilbert space. The number F which enters into (24) counts the degree of the of the respective differential form, hence $(-)^F = +1(-1)$ for p even (odd). Basically the factor $(-)^F$ enters in (24) because, due to the presence of the anticommuting variables, the basis functions $\widetilde{\varrho}$ do not necessarily commute. Next, to evaluate (24) , we need to introduce a scalar product for forms. This is done by choosing a metric g_{ab} on phase-space. To evaluate the trace we need a basis and we decide to use the eigenfunctions of the Laplacian associated to this metric $\triangle_g \equiv d\delta + \delta d$. Here $d(\delta)$ denotes the exterior (co)derivative. Let $\{\psi_i^{(p)}\}$ denote a complete set of normalized eigenfunctions, then:

$$Tr[(-)^F e^{-iT\widetilde{\mathcal{H}}}] = \sum_{p=0}^{2n}(-1)^p \sum_i < \psi_i^{(p)}|e^{-Tl_h}|\psi_i^{(p)} > \tag{25}$$

where $\triangle_g \psi_i^{(p)} = \lambda_i^{(p)}\psi_i^{(p)}$ ("i" labels the various eigenfunctions with the same degrees "(p)" as differential forms) and where the inner product $< \cdot|\cdot >$ refers to g_{ab} and l_h is the Lie-derivative of the hamiltonian flow. The crucial point [10] is that the trace (25) receives contributions only from the ψ with $\lambda = 0$. Infact for $\lambda \neq 0$ we can always build a "super-multiplet" $(\psi, \widetilde{\psi})$ of eigenfunctions with the same λ but a different $(-1)^F$ .They are defined this way:

$$\begin{aligned} \widetilde{\psi} &\equiv \frac{1}{\sqrt{\lambda}}(d+\delta)\psi \\ \psi &= \frac{1}{\sqrt{\lambda}}(d+\delta)\widetilde{\psi} \end{aligned} \tag{26}$$

Since $d+\delta$ commutes with l_h we have that

$$< \psi|exp(-Tl_h)|\psi > = < \widetilde{\psi}|exp(-Tl_h|\widetilde{\psi} >$$

and therefore all contributions to (25) with $\lambda \neq 0$ cancel pairwise because ψ and $\widetilde{\psi}$ have the form number "p" different by one unit. Thus the RHS of (25) reduce to the sum over the $\lambda = 0$ eigenfunctions which are the harmonic forms. This is equivalent [8] to the trace in the various de Rham cohomolgy groups of the phase space $\mathcal{M}_{2n}$ with values in the real numbers: $\mathrm{H}^p(\mathcal{M}_{2n}, \Re)$:

$$\mathrm{Z}_{cm}^{pbc} = \sum_{p=0}^{2n}(-1)^p Tr_{\mathrm{H}^p}[e^{-Tl_h}] \tag{27}$$

The RHS of eq. (27) has a well-known interpretation in terms of the Lfschetz coincidence theorem [8], namely the RHS of (27) is the Lefschetz number [8] $Lef[exp(-Tl_h)]$ of the mapping induced by $exp(-Tl_h)$ on phase-space. The general Lefschetz theorem [8] ,for an arbitrary map "$\mathcal{I}$" of some manifold into itself, expresses alternating sums like (27) in terms of local data of the fixed points of the mapping itself. It can be shown that $Lef[\mathcal{I}]$ is always integer and that it does not depend on the Riemannian metric chosen. Furthermore $Lef[\mathcal{I}]$ is a *homotopic invariant* of $\mathcal{I}$. This implies that, if $\mathcal{I}$ is homotopic to the identity map, *i.e.* $\mathcal{I} \sim id$, the traces of $\mathbf{H}^p$ are simply given by the Betti numbers[8] b^p which are the dimension of $\mathbf{H}^p$:

$$Tr_{\mathbf{H}^p}[\mathcal{I}] = Tr_{\mathbf{H}^p}[id] = dim\ \mathbf{H}^p(\mathcal{M}, \Re) = b^p \tag{28}$$

In our case $\mathcal{I} \equiv exp(-Tl_h)$ is generated by the continuos time evolution and so $\mathcal{I}$ is in fact homotopic to the identity (with T playing the role of the homotopy parameter). Therefore eq. (27) becomes

$$Z_{cm}^{pbc}[0] = \sum_{p=0}^{2n}(-1)^p b^p = \chi \tag{29}$$

In the last step we have used the standard definition of Euler number[8]. The result (29) coincides with the result (23). Since $\mathcal{I} \sim id$ for any value of T, the relation above shows that Z_{cm}^{pbc} does not depend on T, that is what we wanted to prove.

As we mentioned before our classical path-integral has many features in common with the partition function (with pbc) of supersymmetric quantum mechanics on a curved manifold[10]. In fact both path-integrals evaluate the "Witten index". There are anyhow also crucial differences. In supersymmetric quantum mechanics the underlying manifold is a *configuration* space,*i.e.*, to be able to "quantize" a particle in this space we need a *Riemannian* structure in order to write down a Schrödinger equation or a quadratic term $\frac{1}{2}g_{ab}\dot{\phi}^a\dot{\phi}^b$ in the action. On the other hand, from the classical mechanics point of view, $\mathcal{M}_{2n}$ is a *phase-space*, *i.e.*, we have to require a *symplectic* structure in order to write down Hamilton's equation of motion. At first sight it seems surprising that in evaluating (15) , it does not matter whether we use the classical or the quantum mechanical propagation kernel $K(\cdot|\cdot)$. In both cases $K(\cdot|\cdot)$ is given by an integral over the space of loops of lenght T, with different actions, however. Since for the particular observable that is Z_{cm}^{pbc} ,the value of T may be chosen freely, we can take $T \to 0$, in which case the loops can effectively be identified with the points of $\mathcal{M}_{2n}$. In this situation the form of the action in loop-space does not matter anymore and, in particular, quantum effects are irrelevant. This is a tipical feature of TFT[4]. We prefer anyhow the representation in terms of classical mechanics first of all because in that manner we are handling a physical system well known in the literature (the Lie-derivative of the classical Hamiltonian flow), second because the supersymmetry has a clear geometrical interpretation[11] and third because, like in the case of this paper, it allows to give a physical meaning to the Witten Index, being in this case the number of trajectories we studied above.

The curious reader may now try to apply it to simple systems and see if it works*. Let us take the following two examples:

1) Phase-space as the unit sphere: In polar coordinates θ, ϕ, let us take the canonical coordinates to be $q = \phi$ and $p = cos\theta$. With Hamiltonian $H = \frac{p^2}{2}$ the orbits are:

$$q = q_0 + p_0 T, p = p_0 = constant$$

Thus periodic orbit (q changing by $2\pi n$) require

$$p_o = \frac{2\pi n}{T}, \, q_o \; arbitrary$$

These fill a finite number ($2 \; int[\frac{T}{2\pi}]$)of latitude circles, while our argument would give $\chi = 2$.

2) Phase-space as the torus: Let us take as coordinates $q \; (0, 2\pi)$ and $p \; (-1, 1)$ and $H = (\frac{2}{\pi}) sin(\frac{\pi p}{2})$. The motion is continuos and a similar calculation gives for the periodic orbit:

$$cos(\frac{\pi p_0}{2}) = \frac{2\pi n}{T}, \; q_0 \; arbitrary$$

Again there are $2 \; int[\frac{T}{2\pi}]$ such tori, but our argument would give $\chi = 0$.

Why our argument so clearly fails? The reason is that for all integrable systems (like the two above) (almost) all periodic orbits are *non-isolated* while in the proof of our argument (see eqs. (21) (22)) the condition was that the critical points (and so the periodic orbits) be *isolated*. Systems which present these features are chaotic systems which have periodic orbits that are isolated (marginally unstable). Then, to explicitly test our argument, we need to find a *solvable* chaotic system and this may be hard to find.

Acknowledgements: I wish to thank M.Berry, G.Dell'Antonio and M.Reuter for helpfull discussions and correspondence. This work has been supported by grants from INFN, MURST and NATO.

* The analisys which follows is due to M.Berry which I gratefully thanks.

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