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ON $\widehat{sl}(3)$ REDUCTION, QUANTUM GAUGE TRANSFORMATIONS, AND $\mathcal{W}-$ ALGEBRAS SINGULAR VECTORS

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Abstract

The problem of describing the singular vectors of $\mathcal{W}_3$ and $\mathcal{W}_3^{(2)}$ Verma modules is addressed, viewing these algebras as BRST quantized Drinfeld-Sokolov (DS) reductions of $A_2^{(1)}$. Singular vectors of an $A_2^{(1)}$ Verma module are mapped into $\mathcal{W}$ algebra singular vectors and are shown to differ from the latter by terms trivial in the BRST cohomology. These maps are realized by quantum versions of the highest weight DS gauge transformations.

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1. The quantum DS reduction of $\widehat{sl}(2)$ Verma modules to Virasoro Verma modules was considered by two of the authors in [1]. The main tool in that work was a kind of quantum gauge transformation which translates, in an explicit fashion, the singular vectors of $\widehat{sl}(2)$ into Virasoro singular vectors. This letter is a generalization of the method of [1] to the case of $\widehat{sl}(3)$.

There are two different reductions of $\widehat{sl}(3)$ – to the $\mathcal{W}_3$ algebra [2] of Zamolodchikov (Z) or the $\mathcal{W}_3^{(2)}$ algebra [3] of Polyakov-Bershadsky (PB), best described in the general scheme of de Boer-Tjin [4]. After setting the notation and emphasising the connection between the classical highest weight DS gauge [5], [6] and the work [4] we proceed to define the relevant quantum gauge transformations. With their help we illustrate on a few examples how the reduction of the singular vectors of Verma modules is accomplished. In particular we recover the subclass of $\mathcal{W}_3$ singular vectors obtained in [7] and its counterpart for Verma modules of $\mathcal{W}_3^{(2)}$. The details are left for a more systematic work in preparation. For general information we refer to the reviews [8], [9].

2. Here we set some notation and recall a few facts about the affine Lie algebra $A_2^{(1)}$ (see [10],[11]).

The positive roots of A_2 are the simple ones α_1, α_2 and the highest root $\alpha_3 \equiv \alpha_1 + \alpha_2$. The Cartan – Killing form on A_2 is $\langle X, Y\rangle = tr(XY)$. Denote by $\left(C^{ij}\right)_{i,j=1,2} = \langle\alpha_i, \alpha_j\rangle$ the Cartan matrix and by (C_{ij}) – its inverse . The Cartan-Weyl basis consists of e^a, f^a, $a = 1,2,3$, and $h^i = [e^i, f^i]$, $i = 1,2$, generating the subalgebras $\mathtt{n}_+$, $\mathtt{n}_-$ and $\mathtt{h}$ of $A_2 = \mathtt{n}_+ \oplus \mathtt{h} \oplus \mathtt{n}_-$ respectively.

The algebra $A_2^{(1)}$ with the derivation $\mathtt{d}$ added admits the decomposition $\mathtt{N}_+ \oplus \mathtt{H} \oplus \mathtt{N}_-$ where $\mathtt{H}$ is the Cartan subalgebra consisting of $\mathtt{d}$, $\mathtt{h}$ and the central element, which will be assumed to be a fixed complex number k, different from -3. The subalgebras $\mathtt{N}_+$ and $\mathtt{N}_-$ are generated by $\{e_0^1, e_0^2, f_1^3\}$ and $\{f_0^1, f_0^2, e_{-1}^3\}$ respectively. A $A_2^{(1)}$ Verma module is built by the action of $\mathtt{N}_-$ on a highest weight state V_λ, annihilated by the elements of $\mathtt{N}_+$. The projection of the weight λ on $\mathtt{h}^*$ is denoted by $\overline{\lambda}$; $\langle\overline{\lambda}, \alpha_1\rangle + \langle\overline{\lambda}, \alpha_2\rangle + \langle\lambda, \alpha_0\rangle = k$. Denote $M_i = \langle\lambda + \rho, \alpha_i\rangle$, $i = 1,2,3$, and $\nu^{-1} = k + 3$; $\langle\rho, \alpha_j\rangle = 1$, $j = 0,1,2$.

If some of the conditions

$$M_i = \begin{cases} \pm\left(m_i - \frac{n_i}{\nu}\right) & m_i, n_i \in I\!N \\ m_i & m_i \in I\!N \end{cases} \tag{1}$$

$i = 1,2,3$, hold, then the Verma module of highest weight λ is reducible. The highest weights of the embedded modules are obtained by the shifted action of the affine Weyl group on λ (to be denoted by $w_\alpha \cdot \lambda$) . In particular the simplest series of singular vectors correspond to the simple roots α_1, α_2, α_0 (i.e., $\langle\lambda + \rho, \alpha\rangle = m_1$, m_2, or, $1/\nu - M_3$, respectively, are positive integers). Explicitly the corresponding singular vectors are given by

$$V_{w_{\alpha_i}\cdot\lambda} = (f_0^i)^{m_i}\, V_\lambda, \quad i = 1,2, \qquad V_{w_{\alpha_0}\cdot\lambda} = (e_{-1}^3)^{\frac{1}{\nu} - M_3}\, V_\lambda\,. \tag{2}$$

In general decomposing the relevant element of the affine Weyl group into simple reflections w_0, w_1, w_2, ($w_j \equiv w_{\alpha_j}$, $j = 0,1,2$), one can write down expressions for the singular vectors [11]. As an illustration consider the weights for which $M_1 = m - \frac{1}{\nu}$, $m \in I\!N$. Then there is a singular vector corresponding to the root $\alpha = \alpha_0 + \alpha_3 + \alpha_1$

$$V_{w_\alpha\cdot\lambda} = V_{w_1 w_0 w_2 w_0 w_1\cdot\lambda} = (f_0^1)^{m+\frac{1}{\nu}}\,(e_{-1}^3)^{m-\frac{1}{\nu}+M_2}\,(f_0^2)^m\,(e_{-1}^3)^{\frac{1}{\nu}-M_2}\,(f_0^1)^{m-\frac{1}{\nu}}\,V_\lambda\,. \tag{3}$$

The above formula is a monomial of the generators raised, in general, to complex powers acting on V_λ. Nevertheless these monomials can be rewritten as ordinary (integer power) polynomials of the elements of the subalgebra $\mathtt{N}_-$. Note that formally the first (counted from the right), the first two, the first three, etc., factors of these monomials also give singular vectors, but only formally, since the corresponding operators are not elements of the universal enveloping algebra.

For the elements of the chiral algebras we will use both the notation in terms of modes A_n and currents $A(z) = \sum_{n\in\mathbb{Z}} A_n\, z^{-n-\Delta_A}$. We assume that all Kac-Moody currents $X(z)$ have $\Delta_X = 1$. The (anti)commutation relations of the modes are equivalent to the singular part of the operator product expansion (OPE). The normal product of two fields $(A\,B)(z)$ will be defined as the zero order term in the expansion of A and B. We have used the computer program of [12] for the more tedious OPE computations.

3. We briefly recall the classical situation [5], [6]. One considers the matrix operator $\kappa\,\partial_z + A(z)$ with $A(z) = f^a(z)\,t_+^a + e^a(z)\,t_-^a + C_{ij}\,h^i(z)\,t_0^j$, where $t_\pm^a$, $a=1,2,3$ and t_0^i, $i=1,2$ is the Cartan-Weyl basis of $sl(3)$ and summation over repeated indices is assumed. The functions $e^a(z)$, $f^a(z)$, and $h^i(z)$ together with the constant κ are coordinates on the dual $\widehat{sl}(3)^*$ of $\widehat{sl}(3)$ and they close a classical (Poisson bracket) KM algebra. The first step in the Hamiltonian reduction of the classical KM algebra is to impose a set of first class constraints on the e's, corresponding to a subalgebra $\mathfrak{n}^0$ of the nilpotent algebra $\mathfrak{n}_+$. The second step is to factor out the gauge group generated by these constraints. The result is a new phase space with coordinates described by gauge invariant functions on the constrained space and Poisson bracket inherited from that on $\widehat{sl}(3)^*$.

The action of the group is given by the coadjoint representation $A_g(z) = g^{-1}(z)\,A(z)\,g(z) + \kappa g^{-1}(z)\,\partial g(z)$, $g(z) = 1 + \sum b^a(z)\,t_+^a$, where the sum is over $t_+^a \in \mathfrak{n}^0$. The factorisation is done by fixing the gauge, i.e., performing a gauge fixing transformation with parameters $b^a(z)$ being proper functions of the unconstrained currents and their derivatives.

Let us recall the two ways of reducing $\widehat{sl}(3)^*$, corresponding to the two inequivalent embeddings of $sl(2)$ into $sl(3)$. In the case of the principal embedding the constraints are

$$e^1(z) = e^2(z) = 1, \qquad e^3(z) = 0\,. \tag{4}$$

There are three gauge fixing conditions and hence two surviving gauge invariant functions. This corresponds to the splitting of the adjoint representation of $sl(3)$ into a spin 1 and a spin 2 representation of $sl(2)$. Choosing the highest weight gauge one has $h^i \to 0$, $f^1 - f^2 \to 0$, $f^1 + f^2 \to u_1$, $f^3 \to u_2$, with the gauge invariant functions $u_i = u_i(f^a, h^j)$ being

$$u_1 = f^1 + f^2 + u_1^{(\mathrm{ff})} = f^1 + f^2 + \frac{C_{ij}}{2} h^i h^j + \kappa\,\partial h^3\,,$$

$$u_2 = f^3 + \frac{\kappa}{2}\partial(f^1 - f^2) + C_{2j}h^j\, f^1 - C_{1j}h^j\, f^2 + u_2^{(\mathrm{ff})}\,, \tag{5}$$

$$u_2^{(\mathrm{ff})} = C_{1i}C_{2j}(C_{1l} - C_{2l})h^l\, h^i\, h^j\; + \frac{\kappa}{2}\left(C_{1i}\, h^i\,\partial h^1 - C_{2i}\, h^i\,\partial h^2\right) + \frac{\kappa^2}{6}\partial^2(h^1 - h^2)\,,$$

thus providing the generating functions of the classical analogue of the Zamolodchikov $\mathcal{W}_3$ algebra (the (Z) case for short). The gauge fixing transformation is

$$g(z) = 1 + \sum_{a=1}^{3} b_a(z)\,t_+^a\,, \qquad \text{with} \quad b_i(z) = C_{ij}\,h^j(z)\,, \quad i = 1,2\,, \tag{6}$$

and $b_3(z)$ is a function of $f^i, h^i, \partial h^i, i = 1,2$, which we will not specify.

The other embedding gives the classical analogue of the $\mathcal{W}_3^{(2)}$ algebra (the (PB) case). The constraints are

$$e^3(z) = 1, \quad e^2(z) = 0\,. \tag{7}$$

Applying the highest weight gauge transformation

$$g^{(\mathrm{PB})}(z) = 1 + e^1(z)\, t_+^2 + \frac{h^3(z)}{2}\, t_+^3\,, \tag{8}$$

one gets $e^1 \to 0$, $h^3 \to 0$, and

$$\begin{aligned} h^1 - h^2 \quad &\to \quad u_0 = h^1 - h^2\,, \qquad & f^2 \quad &\to \quad u_{1/2}^- = f^2 + e^1 h^2 + \kappa \partial e^1\,, \\ f^1 \quad &\to \quad u_{1/2}^+ = f^1\,, \qquad & f^3 \quad &\to \quad u_1^{(\mathrm{PB})} = f^3 + e^1 f^1 + \tfrac{1}{2} C_{ij} h^i h^j + \tfrac{\kappa}{2} \partial h^3\,. \end{aligned} \tag{9}$$

The four gauge invariant polynomials u_i (corresponding to the four $sl(2)$ representations – of spin 0, $\frac{1}{2}$, $\frac{1}{2}$, and 1, in the adjoint representation of $sl(3)$) generate the classical (Poisson) algebra $\mathcal{W}_3^{(2)}$.

4. In the BRST formalism one needs a pair of fermionic ghost fields b^a, c^a for each constraint. They have OPEs $b^a(z)\, c^b(w) = \frac{\delta_{a,b}}{z-w} + \ldots$ and $\Delta_{b^a} + \Delta_{c^a} = 1$. We choose $\Delta_{b^a} = 0$ and $\Delta_{c^a} = 1$.

First we will consider the reduction leading to the $\mathcal{W}_3$ algebra of Zamolodchikov (Z) [2] associated to the principal embedding. The BRST charge is

$$Q = \sum_{a=1}^{3} (e^a\, c^a)_{-1} - (b^3 (c^1\, c^2))_{-1} - c_0^1 - c_0^2\,. \tag{10}$$

Following the general scheme of [13], [4] let us introduce the "hatted" currents $\hat{X}^a(z) = X^a(z) + f_\beta^{a\alpha}\, (b^\beta\, c_\alpha)(z)$, where the summation indices α, β correspond to the constrained generators e^α, $\alpha = 1, 2, 3$. Explicitly

$$\begin{aligned} &\hat{f}^1 = f^1 + (b^2\, c^3)\,, \qquad \hat{f}^2 = f^2 - (b^1\, c^3)\,, \qquad \hat{f}^3 = f^3\,, \\ &\hat{e}^1 = e^1 + (b^3\, c^2)\,, \qquad \hat{e}^2 = e^2 - (b^3\, c^1)\,, \qquad \hat{e}^3 = e^3\,, \\ &\hat{h}^1 = h^1 + 2(b^1 c^1) - (b^2 c^2) + (b^3 c^3)\,, \qquad \hat{h}^2 = h^2 + 2(b^2 c^2) - (b^1 c^1) + (b^3 c^3)\,. \end{aligned} \tag{11}$$

The OPEs among the fields $\hat{f}, \hat{h}$ are the same as among the corresponding unhatted ones with the only difference that k is shifted to $k+3$.

The reduced currents $T(z)\,, W(z)$, commuting with Q can be obtained as a "quantization" of the gauge invariant differential polynomials (5). Namely, substitute all generators in (5) by their hatted counterparts (11), normal order the products (i.e., replace $h^j(z) f^1(z)$ by $(\hat{h}^j \hat{f}^1)(z)$, etc.), and identify $\kappa = \frac{1}{\nu} - 1 = k+2$. In modes this gives

$$\begin{aligned} \frac{1}{\nu}\, L_n &= \hat{f}_{n+1}^1 + \hat{f}_{n+1}^2 + \frac{1}{\nu}\, L_n^{(\mathrm{ff})} = \hat{f}_{n+1}^1 + \hat{f}_{n+1}^2 + \frac{C_{ij}}{2} (\hat{h}^i \hat{h}^j)_n + (\frac{1}{\nu} - 1)(\partial \hat{h}^3)_n\,, \\ a_w\, W_n &= \hat{f}_{n+2}^3 + \frac{1}{2}(\frac{1}{\nu} - 1)\, (\partial \hat{f}^1 - \partial \hat{f}^2)_{n+1} + C_{2i}\, (\hat{h}^i\, \hat{f}^1)_{n+1} - C_{1i}\, (\hat{h}^i\, \hat{f}^2)_{n+1} + a_w\, W_n^{(\mathrm{ff})}\,, \\ a_w W_n^{(\mathrm{ff})} &= C_{1i} C_{2j} (C_{1l} - C_{2l})\, (\hat{h}^i\, (\hat{h}^j\, \hat{h}^l))_n \\ &\quad + \frac{1}{2}\, (\frac{1}{\nu} - 1)\, \Big(C_{1i}\, (\hat{h}^i\, \partial \hat{h}^1) - C_{2i}\, (\hat{h}^i\, \partial \hat{h}^2)\Big)_n + \frac{1}{6}\, (\frac{1}{\nu} - 1)^2\, (\partial^2 \hat{h}^1 - \partial^2 \hat{h}^2)_n\,. \end{aligned} \tag{12}$$

The standard normalization [14] of the W current is recovered choosing $a_w = -\nu^{-3/2}\sqrt{\frac{5c_\nu+22}{48}}$ for the overall constant.

After appropriate identifications $T^{(\mathrm{ff})}$ and $W^{(\mathrm{ff})}$ reproduce the free field realization of [14]. Both (12) and their (ff) parts have the commutation relations of the Zamolodchikov $\mathcal{W}_3$ algebra with conformal anomaly

$$c_\nu = 50 - 24\left(\nu + \frac{1}{\nu}\right). \tag{13}$$

The expressions for the reduced currents were computed in [4] by directly solving the cohomological "tic-tac-toe" set of equations and they coincide (up to some numerical misprints) with the so defined (12).

The BRST operator implementing the constraints (7) is

$$Q^{(\mathrm{PB})} = (e^3\, c^3)_{-1} + (e^2\, c^2)_{-1} - c_0^3\,. \tag{14}$$

The hatted quantities now read

$$\begin{aligned}
&\hat{e}^1 = e^1 + (b^3\, c^2)\,, \qquad \hat{e}^2 = e^2\,, \qquad \hat{e}^3 = e^3\,,\\
&\hat{f}^1 = f^1 + (b^2\, c^3)\,, \qquad \hat{f}^2 = f^2\,, \qquad \hat{f}^3 = f^3\,,\\
&\hat{h}^1 = h^1 - (b^2\, c^2) + (b^3\, c^3)\,, \quad \hat{h}^2 = h^2 + 2(b^2 c^2) + (b^3\, c^3)\,.
\end{aligned} \tag{15}$$

The reduced quantum generators computed in [4] are again recovered according to the rules of quantisation of the classical expressions (9), this time identifying the parameter κ with $\kappa^{(\mathrm{PB})} \equiv \frac{1}{\nu} - 2 = k+1$. Using for simplicity in the expansions of the fields the dimensions inherited from the KM algebra (i.e, $\triangle_{G^-} = 2 = 2\triangle_{G^+}$, while the standard half-integer modes are recovered by a simple redefinition) one has

$$\begin{aligned}
&H_n = \frac{1}{3}(\hat{h}_n^2 - \hat{h}_n^1)\,, \quad G_n^+ = \hat{f}_n^1\,, \quad G_n^- = \hat{f}_{n+1}^2 + (\hat{e}^1\,\hat{h}^2)_n + (\partial\hat{e}^1)_n\,,\\
&\frac{1}{\nu}L_n = \hat{f}_{n+1}^3 + (\hat{e}^1\,\hat{f}^1)_n + \frac{C_{ij}}{2}(\hat{h}^i\,\hat{h}^j)_n + (\frac{1}{\nu} - 2)(\frac{\partial\hat{h}^3}{2})_n\,.
\end{aligned} \tag{16}$$

One readily checks that (16) generate the $\mathcal{W}_3^{(2)}$ algebra with conformal anomaly

$$c_\nu = 25 - \frac{6}{\nu} - 24\,\nu\,. \tag{17}$$

5. We will consider modules Ω_λ that are tensor products of a $A_2^{(1)}$ Verma module with highest weight vector V_λ and ghost Fock module. To simplify notation we will avoid indicating explicitly tensor products, thus assuming that V_λ is annihilated by the positive modes of all $b^i(z)$ and the nonnegative modes of $c^i(z)$. Clearly any singular vector in the module of $A_2^{(1)}$ built on V_λ is a singular vector in Ω_λ and furthermore we can use equivalently the hatted counterparts of the three generating elements of $\mathtt{N}_-$ to build these vectors, since $(b^i\, c^j)_0\, V_\lambda = 0$. The BRST charge Q annihilates all singular vectors. It is immediate that this is also true for the positive modes of the reduced generators (12) or (16), as well as for the zero mode G_0^-. Let now $V_{w\cdot\lambda}$ be some singular vector of weight $w \cdot \lambda$ (including also the vacuum state). For the zero modes in the (Z) case we have

$$L_0 V_{w\cdot\lambda} = L_0^{(\mathrm{ff})} V_{w\cdot\lambda} = h_{w\cdot\lambda}^{(2)} V_{w\cdot\lambda} , \qquad W_0 V_{w\cdot\lambda} = W_0^{(\mathrm{ff})} V_{w\cdot\lambda} = h_{w\cdot\lambda}^{(3)} V_{w\cdot\lambda} , \tag{18}$$

where

$$\begin{aligned} h_\lambda^{(2)} &= \frac{\nu}{2} C_{ij}\, \lambda(h_0^i)\, (\lambda - 2\kappa\rho)(h_0^j) = \frac{\nu}{2} \langle \overline{\lambda}, \overline{\lambda} - 2\kappa\rho \rangle , \qquad \kappa = \frac{1}{\nu} - 1 , \\ a_w h_\lambda^{(3)} &= C_{1i}\, C_{2j}\, (C_{1l} - C_{2l})\, \langle \overline{\lambda} - \kappa\rho, \alpha_i \rangle \langle \overline{\lambda} - \kappa\rho, \alpha_j \rangle \langle \overline{\lambda}, \alpha_l \rangle , \end{aligned} \tag{19}$$

and they are invariant under the shifted action on λ of the finite group generated by $w_0 w_1 w_0$ and $w_0 w_2 w_0$. This is equivalent to the well known invariance under a κ-shifted action of the finite Weyl group on the projected weights, i.e., if $\overline{\lambda}' - \kappa\bar{\rho} = w(\overline{\lambda} - \kappa\bar{\rho})$ then $h_\lambda^{(p)} = h_{\lambda'}^{(p)}$, $p = 2, 3$ and w is a word made of w_1 and w_2.

In the (PB) case we have

$$L_0^{(\mathrm{PB})} V_{w\cdot\lambda} = h_{w\cdot\lambda}^{(\mathrm{PB})} V_{w\cdot\lambda} , \qquad H_0 V_{w\cdot\lambda} = q_{w\cdot\lambda} V_{w\cdot\lambda} , \tag{20}$$

where

$$h_\lambda^{(\mathrm{PB})} = \frac{\nu}{2} \langle \overline{\lambda}, \overline{\lambda} - \kappa^{(\mathrm{PB})} \rho \rangle , \qquad q_\lambda = (C_{2j} - C_{1j})\, \langle \lambda, \alpha_j \rangle , \qquad \kappa^{(\mathrm{PB})} = \frac{1}{\nu} - 2 . \tag{21}$$

Now (21) are invariant under the shifted action of w_0 on λ or equivalently – under a $\frac{1}{2}\kappa^{(\mathrm{PB})}$-shifted action on the projected weights $\overline{\lambda}$ of the reflection in the α_3 direction, i.e., if $\overline{\lambda}' - \frac{1}{2}\kappa^{(\mathrm{PB})}\bar{\rho} = w_{\alpha_3}(\overline{\lambda} - \frac{1}{2}\kappa^{(\mathrm{PB})}\bar{\rho})$, then $h_\lambda^{(\mathrm{PB})} = h_{\lambda'}^{(\mathrm{PB})}$ and $q_\lambda = q_{\lambda'}$. In particular we can identify V_λ with the highest weight state $|h_\lambda^{(2)}, h_\lambda^{(3)}\rangle$, or $|q_\lambda, h_\lambda^{(\mathrm{PB})}\rangle$, of a $\mathcal{W}_3$, or a $\mathcal{W}_3^{(2)}$ Verma module.

6. Now we introduce quantum analogues of the highest weight gauge fixing transformations, which will be used as a tool to transform KM singular vectors into $\mathcal{W}$ algebra ones. Starting with the two simplest vectors in (2), corresponding to the simple roots α_1, α_2 of A_2, it is clear that it is sufficient to have projections of the "full" quantum gauge transformation along "simple root directions". Thus in the (Z) case the two (projected) transformations are a straighforward generalization of the $A_1^{(1)}$ case [1], i.e.,

$$\mathcal{R}^{(i)} \equiv \mathcal{R}^{(i)}(\hat{e}_0^i) \qquad i = 1, 2 . \tag{22}$$

where

$$\mathcal{R}^{(i)}(u) = \,{}^\circ_\circ \exp \Phi_i(u) {}^\circ_\circ\, , \qquad \Phi_i(u) = C_{ij} \int_0^u du'\, \hat{h}^j_{(-)}(-u') , \tag{23}$$

the subscript $(-)$ denotes the holomorphic part of the field and the ${}^\circ_\circ\ {}^\circ_\circ$ indicate that in the expansion of the exponent the generators of the gauge transformation $u (= \hat{e}_0^i)$ should come to the right of the modes of Φ, i.e., more explicitly,

$$\begin{aligned} \mathcal{R}^{(i)}(u) &= 1 + C_{ij}\hat{h}^j_{-1}\, u + \frac{1}{2}\left((C_{ij}\hat{h}^j_{-1})^2 - C_{ij}\hat{h}^j_{-2} \right) u^2 + \ldots \\ &= \sum_{k=0} \mathcal{R}^{(i)}_{-k}\, u^k , \quad \text{where} \quad k\, \mathcal{R}^{(i)}_{-k} = \sum_{l=0}^{k-1} (-1)^{k+l-1}\, \mathcal{R}^{(i)}_{-l}\, C_{ij}\hat{h}^j_{-k+l} ,\ \mathcal{R}^{(i)}_0 = 1 . \end{aligned} \tag{24}$$

In the (PB) case the transformation in the α_1 direction is the identity (obvious since $G^+(z) = \hat{f}^1(z)$) while in the α_2 direction it is

$$\mathcal{R}^{(2)} = \,\circ\!\!\!\!\circ\, \exp \hat{e}^1_{-1}\, e^2_0 \,\circ\!\!\!\!\circ\, \equiv \sum_{k=0} \frac{1}{k!} (\hat{e}^1_{-1})^k (e^2_0)^k \,. \tag{25}$$

The quantum transformations have the following properties:

$\mathcal{R}^{(i)}$ keeps all KM singular vectors invariant and maps the states $V_t^{(i)} = (f_0^i)^t V_\lambda$, into the kernel of the BRST operator. $\mathcal{R}^{(i)}$ intertwines KM and $\mathcal{W}$ algebra generators.

More precisely the last property in the (Z) case takes the form (no summation in i)

$$\mathcal{R}^{(i)} \left(h_0^3 + 2 - \frac{1}{\nu}\right) f_0^i\, V_t^{(i)} = \frac{1}{\nu} \sum_{p=1} \Bigg(\Big(\varepsilon^i\, a_w\, \nu W_{-p} - \frac{1}{2}\big(1 + \frac{1}{\nu}(2C_{ii} - 1)\big)\, (\partial L)_{-p} \tag{26}$$

$$- \frac{C_{ii}}{\nu}\, L_{-p} \Big)\, \mathcal{R}^{(i)} + L_{-p}\, \mathcal{R}^{(i)}\, C_{ij}\, h_0^j \Bigg)\, (-e_0^i)^{p-1}\, V_t^{(i)} \,,$$

where $\varepsilon^1 = 1\,, \varepsilon^2 = -1$. Its proof is rather lengthy, though straighforward, and will be given in the detailed account of this work. The idea of the proof is roughly the same as in [1] – moving the gauge generator $u (= \hat{e}_0^i)$ to the right produces the free field part (ff) of the reduced generators while the remaining parts arise when we move to the right the "gauge parameters" (the modes of $\mathcal{R}$ in the expansion in u).

In the (PB) case the intertwining property is

$$\mathcal{R}^{(2)}\, f_0^2 V_t^{(2)} = G^-_{-1}\, \mathcal{R}^{(2)}\, V_t^{(2)} \tag{27}$$

proved by straightforward computation.

Comparing (6) and (8) with (24) and (25) respectively, one sees that the latter can be viewed as some "quantizations" of the projections of the classical gauge transformations (taken in an arbitrary representation). As in [1] one can consider alternatively the operators (23) with u identified with an auxiliary $sl(3)$ generator t_+^i instead of $\hat{e}_0^i$.

7. Having the quantum gauge transformations now we can describe how the KM singular vectors get transformed into $\mathcal{W}$ algebra ones. Again the arguments are generalization of the ones in [1].

Let us start with the case when the singular vector corresponds to a single simple root α_1 or α_2, as in the first equality in (2). In the (PB) case we have

$$(f_0^1)^{M_1}\, V_\lambda = (G_0^+)^{M_1}\, V_\lambda \quad \text{if } M_1 \in I\!N \qquad \text{or} \qquad (f_0^2)^{M_2}\, V_\lambda = (G^-_{-1})^{M_2}\, V_\lambda \quad \text{if } M_2 \in I\!N \,, \tag{28}$$

the first being a trivial identity while the second is obtained by applying repeatedly the relation (27) and using that $\mathcal{R}^{(2)}$ leaves the KM singular vectors invariant. (Recall that our moding of $G^\pm$ (16) differs from the standard one.) Thus we can identify the KM singular vector $V_{w_{\alpha_i} \cdot \lambda}\,, i = 1\,,$ or $i = 2$, with a singular vector (given explicitly by the r.h. sides of (28)) in the $\mathcal{W}_3^{(2)}$ Verma module.

In the (Z) case iterating the intertwining relation (26) and using the properties of $\mathcal{R}^{(i)}$ one gets

$$(-f_0^i)^{M_i}\, V_\lambda = \mathcal{O}_\lambda^{(i)}\, V_\lambda \qquad \text{if} \quad M_i \equiv \langle \lambda + \rho, \alpha_i \rangle \in I\!N \,, \tag{29}$$

with

$$\mathcal{O}_\lambda^{(i)} = \mathcal{L}^{(i)}_{M_i,0} + \sum_{k=1}^{M_i-1} \sum_{\{p_a\}_{a=1}^k} \mathcal{L}^{(i)}_{M_i,p_k} \mathcal{L}^{(i)}_{p_k,p_{k-1}} \dots \mathcal{L}^{(i)}_{p_1,0}\,, \tag{30}$$

where the second sum is over all $\{M_i > p_k > \ldots > p_1 > 0\}$ and we have denoted

$$\mathcal{L}^{(i)}_{t,t-p} = \frac{\prod_{a=1}^{p-1}(t-a)(\langle\lambda+\rho,\alpha_i\rangle + a - t)}{\nu\,(t + \langle\lambda+\rho,\alpha_0\rangle)}\,. \\ \cdot \left(\varepsilon^i\, a_w\, \nu\, W_{-p} + \left(C_{ij}\langle\lambda,\alpha_j\rangle - t + p - \frac{2}{3\nu}\right) L_{-p} - \frac{1+3\nu}{6\nu}\,(\partial L)_{-p}\right)\,. \tag{31}$$

The $\mathcal{W}_3$ singular vectors in the r.h.s. of (29) were obtained in [7] using the method of "fusion" [15]. Iterating (26) for any $t = 1, 2, \ldots, M_i$, one actually recovers also the basis elements of the matrix system equivalent to $\mathcal{O}_\lambda^{(i)}\, V_\lambda$. Note that in our approach the proof that $\mathcal{O}_\lambda^{(i)}\, V_\lambda$ is annihilated by the positive modes of the $\mathcal{W}_3$ currents is straightforward.

A remark is in order. In iterating (26) one can encounter examples when the l.h.s. vanishes at some step. For an illustration consider the case $M^1 = M^2 = 1$, then one has to use (26) once, i.e., $t = 0$. Now if $\nu = 1$ one checks using the explicit expression (12) for W_{-1} that $W_{-1}\, V_\lambda = 0$ and furthermore both sides of (26) in this case vanish due to numerical coefficients, thus not producing a nontrivial singular vector of $\mathcal{W}_3$. This phenomenon has to do with the existence of exceptional $\mathcal{W}_3$ modules defined by indecomposable representations of $\{L_0, W_0\}$ (see e.g. [16]) and we leave their analysis within our approach to a future investigation.

Next let us consider the example when both M_1 and M_2 are positive integers. In this case also composite singular vectors exist, i.e., we have that $(f_0^1)^{M_1+M_2}\,(f_0^2)^{M_2}\, V_\lambda$, $(f_0^2)^{M_1+M_2}\,(f_0^1)^{M_1}\, V_\lambda$ and $(f_0^2)^{M_1}\,(f_0^1)^{M_1+M_2}\,(f_0^2)^{M_2}\, V_\lambda$ together with the highest weight vector and the other two singular vectors discussed above form a hexagon of singular vectors. The reduction of such composite vectors proceeds by reduction of the separate factors. This is straightforward because (counting from the right) the first, the first two, etc. factors also produce singular vectors. For example in the (Z) case

$$(f_0^2)^{M_1}\,(f_0^1)^{M_1+M_2}\,(f_0^2)^{M_2}\, V_\lambda = \mathcal{O}^{(2)}_{w_1 w_2\cdot\lambda}\, \mathcal{O}^{(1)}_{w_2\cdot\lambda}\, \mathcal{O}^{(2)}_\lambda\, V_\lambda\,, \tag{32}$$

while in the (PB) case one gets

$$(f_0^2)^{M_1}\,(f_0^1)^{M_1+M_2}\,(f_0^2)^{M_2}\, V_\lambda = (G^-_{-1})^{M_1}\,(G^+_0)^{M_1+M_2}\,(G^-_{-1})^{M_2} V_\lambda\,, \tag{33}$$

i.e., again equalities of singular vectors.

The last series in (2), i.e., the vectors originating from the affine (simple) root α_0 do not survive under reduction. Indeed starting with the (PB) case one can express $e^3_{-1} = 1 + \{Q^{(\mathrm{PB})}, b_0^3\}$ and using that $Q^{(\mathrm{PB})}$ annihilates the vacuum state we have

$$(e^3_{-1})^{\frac{1}{\nu} - M^3}\; V_\lambda = V_\lambda + Q^{(\mathrm{PB})}\, \ldots V_\lambda\,, \tag{34}$$

i.e., up to terms in the image of $Q^{(\mathrm{PB})}$ we recover the vacuum highest weight state which is in agreement with the invariance of (21). (Note that since the modes H_n, T_n, G^+_n, $n > 0$, G^-_m, $m \geq 0$, annihilate the KM singular vectors V_λ, $V_{w_0\cdot\lambda}$, they also annihilate the Q - exact term in the r.h.s. of (34).) The result is even more trivial in the (Z) case, since $e^3_{-1} = \{Q, b_0^3\}$, and hence the whole singular vector is Q exact.

At the end we turn to the reduction of the general singular vectors (see (1)). Postponing the detailed analysis here we only illustrate the method on the example (3), where $M_1 = m - \frac{1}{\nu}, m \in I\!N$.

The (PB) case is in complete analogy with the reduction of $A_1^{(1)}$ [1]. One has to combine the results above for the singular vectors corresponding to the simple roots. Namely one transforms sequentially the factors of (3) starting from the left and using (28) for the (nonaffine) directions α_1 and α_2. For the affine (α_0) direction one uses (34) and recalling that the reduced generators commute with the BRST operator one gets

$$V_{w_1 w_0 w_2 w_0 w_1 \cdot \lambda} = (G_0^+)^{m+\frac{1}{\nu}} (G_{-1}^-)^m (G_0^+)^{m-\frac{1}{\nu}} V_\lambda + Q^{(\mathrm{PB})} \ldots V_\lambda . \tag{35}$$

The first term in the r.h.s. provides a singular vector in the $\mathcal{W}_3^{(2)}$ Verma module. One can make sense of the above monomial with G's raised to complex powers as one does in the KM case (e.g., for the vector in (3)), namely the structure is such that the middle generator is to an integer power while each successive pair (starting from the middle and going outwards) of surrounding generators have powers adding to an integer – commuting repeatedly from the middle outwards we get an ordinary polynomial.

The (Z) case is more subtle. To be able to carry rigourously the analysis one has to make sense of (30) when M_i is noninteger. For the analogous operator in the Virasoro case this was done in [17]. Let us assume that it is possible to make an analytic continuation of (30) to complex powers of the nonaffine generators. Furthermore for the reduction in the affine direction we cannot use directly the argument above because raising the constraint $e_{-1}^3 = \hat{e}_{-1}^3 = \{Q, b_0^3\}$ to a complex power is now ill defined. On the other hand the invariance of (19) indicates that we should consider the group of generators corresponding to $w_0 w_2 w_0$ in the middle of (3) instead of considering the powers of e_{-1}^3 – which correspond to w_0. Indeed casting the middle triple into an integer powers form and accounting for the properties of Q produces a sum of powers of the generator e_{-1}^1 and of proper bilinear ghost combinations which add up to recover powers of its hatted counterpart $\hat{e}_{-1}^1 = 1 + \{Q, b_0^1\}$ (cf. (11)). Thus we get again $1 + \{Q, \ldots\}$ (up to a numerical coefficient) and hence for the full vector

$$V_{w_1 w_0 w_2 w_0 w_1 \cdot \lambda} = N \mathcal{O}^{(1)}_{w_0 w_2 w_0 w_1 \cdot \lambda} \mathcal{O}^{(1)}_\lambda V_\lambda + Q \ldots V_\lambda , \tag{36}$$

where N is a constant.

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