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Metaplectic Spinor Fields on Phase Space: a Path-Integral Approach

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ABSTRACT

In this paper we study spinor fields on the phase-space of a generic hamiltonian system. Under linearized canonical transformations these spinors transform according to the metaplectic representation of $Sp(2N)$. We derive a path-integral for their time evolution and discuss their dynamical and geometrical properties. In particular we show that they can be interpreted as semiclassical wave-functions for the associated hamiltonian.

1. INTRODUCTION

In relativistic field theory it is well known that the definition of spinor fields on a (possibly curved) space-time manifold $\mathcal{M}_n$ involves replacing the group of local frame rotations by its covering group. Depending on the signature of $\mathcal{M}_n$, $O(n)$ is replaced by $Spin(n)$, or $O(1, n-1)$ by $Spin(1, n-1)$, respectively. It is perhaps less well known [1] that a similar construction also can be performed on the phase-space of any quantum system. The role of the Lorentz group which rotates the frames in the local tangent spaces is now taken by the symplectic group $Sp(2N)$, the group of linear canonincal transformations. Tensors fields on phase-space transform according to tensor products of the vector representation of $Sp(2N)$. They are the analogue of integer spin fields on space-time. What corresponds to fields of half-integer spin are the metaplectic spinor fields on phase-space. They form representations of the covering group of $Sp(2N)$, the metaplectic group $Mp(2N)$. In many respects the relation between $Mp(2N)$ and $Sp(2N)$ is similar to the relation between $Spin(n)$ and $O(n)$. For example, in analogy with the rule $\frac{1}{2} \otimes \frac{1}{2} = 1 \oplus 0$ for $SU(2)$ representations, say, one can combine two metaplectic spinors to form a vector. However, unlike $Spin(n)$, which possesses finite dimensional representations, those of $Mp(2N)$ are infinite dimensional. Typically the representation space is a Hilbert space on which $Mp(2N)$ is realized by unitary operators.

In the past, metaplectic representations were mostly studied in the framework of geometric quantization [2] and in the context of the semiclassical approximation [3]. More recently metaplectic spinors made their appearance in theories with Parisi-Sourlas supersymmetry [4], in the covariant quantization of the Green-Schwarz superstring [5] and in models of anyon superconductivity [6]. The purpose of the present paper is to introduce and to study a path-integral for the time-evolution of metaplectic spinor fields which can be defined on the phase-space of any hamiltonian system. The evolution equation for the spinors is specified in such a way that certain bilinears constructed out of them (analogs of $\bar{\psi}\gamma^\mu\psi$ for Dirac fields) transform as vector fields under the hamiltonian flow. The path-integral involves an integration over anticommuting "world-line spinors" η and $\bar{\eta}$ transforming in the metaplectic representation. It turns out that, on one hand, η and $\bar{\eta}$ are closely related to semiclassical wavefunctions, and, on the other hand, they are "square roots" of the Jacobi fields c^a which describe small fluctuations around classical trajectories [7]. To be precise, c^a may be written as $\bar{\eta}\gamma^a\eta$ with the metaplectic Dirac matrices γ^a. We shall see that the natural geometrical setting for metaplectic spinor fields is that of a Hilbert bundle over phase-space. In the same way as tensor fields "live" in the local (co)tangent spaces, metaplectic spinor fields assume values in (infinite dimensional) Hilbert spaces which should be visualized as fibers over the points of phase-space. The Jacobi fields are related to Grassmannian variables (ghosts) c^a which transforms in the vector representation of $Sp(2n)$; they provide a basis in the local tangent spaces. In a similar way the metaplectic ghosts provide a basis in the local Hilbert spaces.

The rest of this paper is organized as follows. In section 2 we introduce a path-integral for the hamiltonian time evolution of tensors transforming in any representation of $Sp(2N)$. In section 3 we construct its spinor representation in a language which is appropriate for our purposes, and in section 4 we discuss spinor fields on phase-space. In section 5 we investigate the properties of, what we could call, "world-line spinors" appearing in our path-integral. Finally in section 6, the world-line spinors are related to semiclassical wave-functions.

2. THE PATH-INTEGRAL

Let $\mathcal{M}_n$ be an n-dimensional manifold with local coordinates ϕ^a, and let $Diff(\mathcal{M}_n)$ be the group of diffeomorphisms on $\mathcal{M}_n$. Furthermore, let χ be an arbitrary tensor or spinor field on $\mathcal{M}_n$. Under the action of an element of $Diff(\mathcal{M}_n)$ that drags the field through an infinitesimal displacement $\delta\phi^a = -h^a(\phi)$ (where $h \equiv h^a\partial_a$ is some vector field on $\mathcal{M}_n$) the tensor changes by an amount $\delta\chi = l_h\chi$ where

$$l_h = h^a\partial_a - \partial_b h^a G^b{}_a \tag{2.1}$$

denotes the Lie-derivative[8]. Here $G^b{}_a$ are the generators of $GL(n, R)$ in the representation to which χ belongs. If $\chi \equiv (\chi^\alpha)$ is arranged as a column vector, the generators $G^b{}_a \equiv (G^b{}_a{}^\beta{}_\alpha)$ form a matrix basis of the corresponding Lie algebra. The indices $\alpha, \beta \cdots$ could be coordinates indices $a, b \cdots$, frame indices or spinor indices. Coordinate indices can be converted to frame indices by contraction with appropriate vielbein fields. In the resulting local frame basis χ is a scalar under diffeomorphism, but under local frame rotations it may transform in any representation of $O(n)$ or $O(1, n-1)$, respectively. This includes spinor representations provided we go over to the covering group $Spin(n)$ or $Spin(1, n-1)$, respectively.

In the present paper the manifold $\mathcal{M}_n \equiv \mathcal{M}_{2N}$ under consideration is not a space-time manifold, but rather the phase-space of an arbitrary hamiltonian system. It is a symplectic manifold[9] implying that its dimensionality is even $(n = 2N)$ and that it carries a closed, non-degenerate two-form $\omega = \frac{1}{2}\omega_{ab} d\phi^a \wedge d\phi^b$. For the simplicity of the presentation we assume that canonical coordinates $\phi^a = (p^i, q^i), i = 1 \cdots N, a = 1 \cdots 2N$, can be introduced globally such that $\omega_{ab} = -\omega_{ba}$ becomes a constant matrix with entries ± 1. Its inverse is denoted ω^{ab}: $\omega_{ac}\omega^{cb} = \delta^c_a$. The only nonvanishing components of ω^{ab} are $\omega^{N+i,j} = -\omega^{j,N+i} = \delta^{ij}$, $i, j = 1 \cdots N$. We specify a scalar function H on $\mathcal{M}_{2N}$ which will serve as a Hamiltonian. It gives rise to the hamiltonian vector field[8] $(\partial_a \equiv \frac{\partial}{\partial \phi^a})$

$$h^a(\phi) = \omega^{ab}\partial_b H(\phi) \tag{2.2}$$

so that Hamilton's equation reads $\dot{\phi}^a = h^a(\phi)$. For any two scalar functions f and g on $\mathcal{M}_{2N}$ we define the Poisson bracket

$$\{f, g\}_{pb} = \partial_a f \omega^{ab} \partial_b g \tag{2.3}$$

In classical hamiltonian dynamics the time evolution of densities $\varrho(\phi, t)$ on phase-space is governed by the equation

$$-\partial_t \varrho = \{\varrho, H\}_{pb} = l_h \varrho \tag{2.4}$$

The RHS of eq.(2.4) is exactly the Lie derivative of ϱ along the hamiltonian vector field h. Because ϱ is a scalar, l_h has the simple form $l_h = h^a \partial_a$. Tensors of higher rank evolve under the hamiltonian flow in an analogous fashion:

$$\begin{aligned} -\partial_t \chi^{b_1 b_2 \cdots}_{a_1 a_2 \cdots} &= l_h \chi^{b_1 b_2 \cdots}_{a_1 a_2 \cdots} \\ &= h^c \partial_c \chi^{b_1 b_2 \cdots}_{a_1 a_2 \cdots} + \partial_{a_1} h^c \chi^{b_1 b_2 \cdots}_{c a_2 \cdots} - \partial_c h^{b_1} \chi^{c b_2 \cdots}_{a_1 a_2 \cdots} + - \cdots \end{aligned} \tag{2.5}$$

In the following we shall consider eq.(2.1) only for the hamiltonian vector fields h of (2.2) which generate canonical transformations or symplectic diffeomorphisms, *i.e.*, those diffeomorphisms which keep invariant the symplectic 2-form ω. As the formalism we shall set up is covariant only under those transformations, the generators ${G^b}_a$ generate the symplectic group $Sp(2N)$ rather than the full $GL(2N; R)$. We shall see that this group plays a role similar to the Lorentz group which rotates local frames in spacetime.

It is convenient to define

$$\begin{aligned} K_{ab}(\phi) &\equiv \partial_a \partial_b H(\phi) \\ \Sigma^{ab} &\equiv i\left(G^a{}_c \omega^{cb} + G^b{}_c \omega^{ca}\right) \end{aligned} \tag{2.6}$$

so that

$$l_h = h^a \partial_a + \frac{i}{2} K_{ab} \Sigma^{ab} \tag{2.7}$$

with $K_{ab} = K_{ba}$ and $\Sigma_{ab} = \Sigma_{ba}$. If χ^α stands for a vector field on $\mathcal{M}_{2N}$, $\chi^\alpha(\phi) \to v^a(\phi)$, say, then the generators are

$$\left(\Sigma^{ab}_{vec}\right)^c{}_d = -i\left(\delta^a_d \omega^{bc} + \delta^b_d \omega^{ac}\right) \tag{2.8}$$

and we recover the standard form of the Lie derivative

$$l_h v^a = h^b \partial_b v^a - \partial_b h^a v^b \tag{2.9}$$

Similarly, for one-forms, $\chi^\alpha(\phi) \to F_a(\phi)$, the generators are

$$\left(\Sigma^{ab}_{form}\right)_c{}^d = i\left(\delta^a_c \omega^{bd} + \delta^b_c \omega^{ad}\right) \tag{2.10}$$

thus reproducing the usual equation

$$l_h F_a = h^b \partial_b F_a + \partial_a h^b F_b \tag{2.11}$$

It is not difficult to check that the matrices (2.8) generate the group $Sp(2N)$ in the vector representations. The group elements $S^a{}_b$, with infinitesimal parameters $\kappa_{ab} = \kappa_{ba}$,

$$S^a{}_b = \left(1 - \frac{1}{2}\kappa_{cd} \Sigma^{cd}_{vec}\right)^a{}_b = \delta^a{}_b + \omega^{ac} \kappa_{cb} \tag{2.12}$$

satisfy the relation

$$\omega_{ab} S^a{}_c S^b{}_d = \omega_{cd} \tag{2.13}$$

i.e., they preserve the symplectic matrix ω. Hence they are elements of $Sp(2N)$. The converse is also true[3]: any symplectic matrix infinitesimally close to the identity is of the form (2.12). The group elements in the representation dual to (2.12) are

$$\begin{aligned} S_a{}^b &= \left(1 - \frac{i}{2}\kappa_{cd} \Sigma^{cd}_{form}\right)_a{}^b \\ &= \delta_a{}^b - \omega^{bc} \kappa_{ca} \end{aligned} \tag{2.14}$$

satisfying

$$S_a{}^c S_b{}^d \omega_{cd} = \omega_{ab} \tag{2.15}$$

In an analogous fashion one could write down the generators Σ^{ab} in more complicated representations of $Sp(2N)$, *e.g.* , tensor products of (2.8) and (2.10), or in the metaplectic representation to be introduced in the next section. For the time being let us consider a set of fields $\chi_\alpha(\phi, t)$ which transforms under $Sp(2N)$ in any representation. We assume that the time evolution of $\chi_\alpha(\phi, t)$ is governed by the equation:

$$\partial_t \chi_\alpha(\phi, t) = -l_h \chi_\alpha(\phi, t) = -\left[\delta_\alpha^{\ \beta} h^a \partial_a + \frac{i}{2} K_{ab}(\phi) \left(\Sigma^{ab}\right)_\alpha^{\ \beta}\right] \chi_\beta(\phi, t) \tag{2.16}$$

with Σ^{ab} in the approriate representation. Eq.(2.16) generalizes eq.(2.4) : for scalar functions $\chi \equiv \rho$ the RHS of (2.16) is a conventional Poisson bracket. Next we show that also for tensors or spinors χ_α the RHS of (2.16) may be interpreted as a Poisson bracket provided one works on an extended phase-space $\mathcal{M}_{ext}$ rather than the usual one, $\mathcal{M}_{2N}$. The extended phase space[7] $\mathcal{M}_{ext}$ is a supermanifold[10] coordinatized by $(\phi^a, \lambda_a, \eta^\alpha, \bar{\eta}_\alpha)$ where ϕ^a, $a = 1 \cdots 2N$, are the usual coordinates on $\mathcal{M}_{2N}$, λ_a is a set of $2N$ bosonic auxiliary variables conjugate to ϕ^a, while η^α and $\bar{\eta}_\alpha$ are anticommuting Grassmannian variables. We define on $\mathcal{M}_{ext}$ the extended Poisson bracket (epb) structure:

$$\begin{aligned} \{\phi^a, \lambda_b\}_{epb} = \delta^a_b \qquad &, \qquad \{\phi^a, \phi^b\}_{epb} = 0 = \{\lambda_a, \lambda_b\}_{epb} \\ \{\bar{\eta}_\beta, \eta^\alpha\}_{epb} = -i\delta^\alpha_\beta \qquad &, \qquad \{\eta^\alpha, \eta^\beta\}_{epb} = 0 = \{\bar{\eta}_\alpha, \bar{\eta}_\beta\}_{epb} \end{aligned} \tag{2.17}$$

such that the λ_a's become the "momenta" conjugate to ϕ^a. As was extensively discussed in ref.[7], the variables λ_a form a basis in the cotangent space to $\mathcal{M}_{2N}$: the $4N$ variables (ϕ^a, λ_a) can be considered coordinates on the cotangent bundle over $\mathcal{M}_{2N}$. Under diffeomorphisms on $\mathcal{M}_{2N}$, λ_a transforms like the derivatives ∂_a, $\bar{\eta}_\alpha$ in the same representation of $Sp(2N)$ as χ_α and η^α in the representation dual to it. The motivation for introducing the extended Poisson structure is that antisymmetric tensor fields on $\mathcal{M}_{2N}$ are equivalent to scalar fields on $\mathcal{M}_{ext}$. (In this context "tensor" means any object $\chi_{\alpha\beta\gamma\ldots}$ where Σ^{ab} is not necessarily in the vector representation.) The equivalence mentioned above is established by representing the field $\chi^{\beta_1 \cdots \beta_q}_{\alpha_1 \cdots \alpha_p}(\phi)$ by the following function on $\mathcal{M}_{ext}$:

$$\widehat{\chi}(\phi, \eta, \bar{\eta}) = \frac{1}{p!q!} \chi^{\beta_1 \cdots \beta_q}_{\alpha_1 \cdots \alpha_p}(\phi)\, \bar{\eta}_{\beta_1} \cdots \bar{\eta}_{\beta_q} \eta^{\alpha_1} \cdots \eta^{\alpha_p} \tag{2.18}$$

If we now introduce the "super-Hamiltonian"

$$\widetilde{\mathcal{H}} = h^a(\phi)\lambda_a + \frac{1}{2}\bar{\eta}_\alpha K_{ab}(\phi)\left(\Sigma^{ab}\right)^\alpha_{\ \beta} \eta^\beta \tag{2.19}$$

It is easy to show that the Lie-derivative of $\chi^{\beta_1 \cdots \beta_q}_{\alpha_1 \cdots \alpha_p}$ is realized as the extended Poisson bracket with $\widetilde{\mathcal{H}}$:

$$\{\widetilde{\mathcal{H}}, \widehat{\chi}\}_{epb} = -(l_h\chi)^{\wedge} \tag{2.20}$$

Here $(l_h\chi)^{\wedge}$ is constructed from $l_h\chi^{\beta_1\cdots\beta_q}_{\alpha_1\cdots\alpha_p}$ in the same way as $\widehat{\chi}$ from χ in (2.18). Eq.(2.16), for instance, is now equivalent to

$$\partial_t\widehat{\chi} = \{\widetilde{\mathcal{H}}, \widehat{\chi}\}_{epb} = -(l_h\chi)^{\wedge} \tag{2.21}$$

with $\widehat{\chi} \equiv \chi_\alpha\eta^\alpha$. For a given representation of Σ^{ab}, eq.(2.21) with $\widehat{\chi}$ given by (2.18) generalizes the time evolution (2.16) in the sense that now χ is allowed to carry any number of lower indices on which $(\Sigma^{ab})_\alpha{}^\beta$ acts from the left and upper indices on which it acts from the right. Because the variables η^α and $\bar{\eta}_\alpha$ anticommute, $\chi^{\beta_1\cdots\beta_q}_{\alpha_1\cdots\alpha_p}$ has to be antisymmetric in all upper and lower indices.

From the extended Poisson formalism it is a small step[7] to the path-integral which provides a formal solution of eq.(2.16) and its generalizations. Let us consider the following phase-space path-integral over paths on $\mathcal{M}_{ext}$:

$$\mathbf{Z}(\phi_2, \eta_2, t_2; \phi_1, \eta_1, t_1) \equiv \int \mathcal{D}\phi\ \mathcal{D}\lambda\ \mathcal{D}\eta\ \mathcal{D}\bar{\eta}\ exp\Big[i\int_{t_1}^{t_2} dt\{\lambda_a\dot{\phi}^a + i\bar{\eta}_\alpha\dot{\eta}^\alpha - \widetilde{\mathcal{H}}\}\Big] \tag{2.22}$$

with the boundary conditions $\phi^a(t_{1,2}) = \phi^a_{1,2}$ and $\eta^\alpha(t_{1,2}) = \eta^\alpha_{1,2}$. From the terms in the exponential containing time-derivatives one concludes that the operatorial formalism equivalent to the path-integral (2.22) is based upon the commutation relations

$$\begin{aligned} [\,\phi^a, \lambda_b] &= i\delta^a_b \quad , \quad [\phi^a, \phi^b] = 0 = [\lambda_a, \lambda_b] \\ [\,\bar{\eta}_\beta, \eta^\alpha] &= \delta^\alpha_\beta \quad , \quad [\bar{\eta}_\alpha, \bar{\eta}_\beta] = 0 = [\eta^\alpha, \eta^\beta] \end{aligned} \tag{2.23}$$

(Here $[\cdot,\cdot]$ denotes the Z_2-graded commutator.) In a Schrödinger-type representation of the algebra (2.23) we can represent the "coordinates" ϕ^a and η^α as multiplication operators and the "momenta" λ_a and $\bar{\eta}_\alpha$ as derivatives:

$$\lambda_a = -i\frac{\partial}{\partial\phi^a} \quad , \quad \bar{\eta}_\alpha = \frac{\partial}{\partial\eta^\alpha} \tag{2.24}$$

The representation space is spanned by functions of the form

$$\chi(\phi, \eta) = \sum_p \frac{1}{p!}\ \chi_{\alpha_1\alpha_2\cdots\alpha_p}(\phi)\ \eta^{\alpha_1}\eta^{\alpha_2}\cdots\eta^{\alpha_p} \tag{2.25}$$

If we are dealing with a finite dimensional representation of Σ^{ab}, the indices α_i run over finitely many values only and the summation over p terminates at some point due to the antisymmetry of the η's. If, for instance, $\Sigma^{ab} = \Sigma^{ab}_{form}$, then $\chi_{\alpha_1 \cdots \alpha_p} \rightarrow \chi_{a_1 \cdots a_p}$ is a p-form and the expansion of a generic function $\chi(\phi, \eta)$ involves monomials with at most $2N$ anticommuting variables. In this case $\eta^\alpha \rightarrow \eta^a$ transforms as a vector under $Sp(2N)$.

For the "wave-function" (2.25) we postulate the Schrödinger-type equation

$$i\partial_t \chi(\phi, \eta, t) = \widetilde{\mathcal{H}}(\phi, \lambda = -i\frac{\partial}{\partial \phi}, \eta, \bar{\eta} = \frac{\partial}{\partial \eta})\chi(\phi, \eta, t) \tag{2.26}$$

with $\widetilde{\mathcal{H}}$ given by eq. (2.19). It is then obvious that the path-integral (2.22) provides the solution of eq.(2.26):

$$\chi(\phi, \eta, t) = \int d\phi_0 \; d\eta_0 \; K(\phi, \eta, t | \phi_0, \eta_0, t_0) \chi(\phi_0, \eta_0, t_0) \tag{2.27}$$

Inserting (2.25) into (2.26) it is easy to convince oneself that the time-derivative of the coefficient functions $\chi_{\alpha_1 \cdots \alpha_p}(\phi, t)$ is again given by their Lie derivative

$$\partial_t \chi_{\alpha_1 \cdots \alpha_p}(\phi, t) = -l_h \chi_{\alpha_1 \cdots \alpha_p}(\phi, t) \tag{2.28}$$

This shows that (for tensors with lower indices only) the "classical" equation (2.21) is completely equivalent to the "Schrödinger-type" equation (2.26). The reason for this equivalence of the classical and the "quantum-type" formulation can be traced back to the fact that the Lagrangian entering (2.22)

$$\begin{aligned} \widetilde{\mathcal{L}} &= \lambda_a \dot{\phi}^a + i\bar{\eta}_\alpha \dot{\eta}^\alpha - \widetilde{\mathcal{H}} \\ &= \lambda_a \big(\dot{\phi}^a - h^a(\phi)\big) + i\bar{\eta}_\alpha \big[\delta^\alpha{}_\beta \partial_t + \frac{i}{2} K_{ab}(\phi) (\Sigma^{ab})^\alpha{}_\beta\big] \eta^\beta \end{aligned} \tag{2.29}$$

gives rise to a kind of topological field theory[11] in which the fermionic quantum fluctuations exactly compensate for the bosonic ones. In this manner the functional integral (2.22) is exactly localized on the solutions of the classical equations of motion resulting from (2.29). We observe that λ_a and $\bar{\eta}_\alpha$ enter $\widetilde{\mathcal{L}}$ as Lagrange multipliers for the equations of motion of ϕ^a and η^α:

$$\dot{\phi}^a(t) = h^a\big(\phi(t)\big) \tag{2.30}$$

$$\dot{\eta}^\alpha(t) = -\frac{i}{2} K_{ab}\big(\phi(t)\big) \big(\Sigma^{ab}\big)^\alpha{}_\beta \eta^\beta \tag{2.31}$$

For Σ^{ab} in the vector representation (2.12), the topological field theory aspects have been discussed in detail in ref.[12] to which the reader is referred for further information. Here we only remark that in the vector representation (where we used[7] the notation c^a, $\bar{c}_a$ for the Grassmannian variables) the equation of motion (2.31) reads

$$\dot{c}^a(t) = \partial_b h^a(\phi(t))c^b(t) \tag{2.32}$$

This equation is known as Jacobi's equation[13]. It describes small fluctuations around the solutions of Hamilton's equation (2.30). In fact, if we let $\phi(t) \to \phi(t) + \delta\phi(t)$ in (2.30) and expand to first order in $\delta\phi(t)$, we find exactly (2.32) with $c^a(t) = \delta\phi^a(t)$. In ref.[7] we argued that the time-dependent variable $c^a(t)$ should be interpreted as a basis in the tangent space to the space of classical trajectories $\phi^a_{cl}(t)$. A similar interpretation can be given for the time independent Grassmannian variables c^a in the Schrödinger-like picture we used before: at a fixed point $\phi \in \mathcal{M}_{2N}$ they provide a basis of the tangent space $T_\phi\mathcal{M}_{2N}$. Thus they can be identified with the differentials $d\phi^a$. Similarly the antighosts $\bar{c}_a$ span the cotangent space $T^*_\phi\mathcal{M}_{2N}$; they correspond to the basis elements ∂_a. In ref.[7] we have exploited this correspondence in order to reformulate all the operations of the exterior calculus on phase space (exterior derivative, contraction, Lie derivative, Lie brackets, etc.) in terms of the extended Poisson brackets (2.17). It is the purpose of the present paper to give a similar discussion for the case when the Grassmannian variables are in the metaplectic rather than in the vector representation. For metaplectic spinor fields most of the operations of the exterior calculus, such as exterior derivative, say, are not defined. Therefore we shall concentrate on the Lie-derivative (2.21) which, in any representation, can be represented by an extended Poisson bracket .

Before closing this section we mention a subtlety which is absent in the vector case. If the representation of Σ^{ab} is such that $\kappa_{ab}(\Sigma^{ab})^\alpha{}_\alpha \neq 0$, then the Hamiltonian (2.19) will suffer from an ordering ambiguity because, according to (2.23), pulling $\bar{\eta}_\alpha$ through η^β gives rise to a commutator term. For (2.21) and (2.26) to be exactly equivalent, the Grassmannian piece of the Hamiltonian must be "antinormal" ordered, *i.e.*, the $\bar{\eta}_\alpha = \frac{\partial}{\partial\eta^\alpha}$ operator should be to the right of η^β.

3. THE METAPLECTIC GROUP $Mp(2N)$

Let S denote a symplectic matrix in the vector representation. To any $S \in Sp(2N)$ we can associate a unitary operator $M(S), M(S)^\dagger = M(S)^{-1}$, acting on an infinite dimensional Hilbert space $\mathcal{V}$, such that the map $S \mapsto M(S)$ provides a representation of $Sp(2N)$. This map is a two-to-one homomorphism: both $+M(S)$ and $-M(S)$ represent the same group element. (Therefore the notation $M(S)$ is slightly misleading: we always have to carefully keep track of the correct sign!) Thus[3]

$$M(S_1)M(S_2) = \pm M(S_1 S_2) \tag{3.1}$$

The operators $M(S)$ form the metaplectic group $Mp(2N)$ which is the covering group of $Sp(2N)$. Objects transforming under $Sp(2N)$ as $\chi \mapsto M(S)\chi$ are called metaplectic spinors[1].

The representations of $Spin(1, n-1)$ follow from the representations of the Clifford algebra

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \tag{3.2}$$

in a well known manner. The representations of $Mp(2N)$ follow from the metaplectic Clifford algebra[10] [4]

$$\gamma^a \gamma^b - \gamma^b \gamma^a = 2i\omega^{ab} \tag{3.3}$$

in a similar way. Because of the crucial minus sign on the LHS of eq.(3.3) this algebra does not admit finite-dimensional matrix representations. The γ-matrices are rather operators on an infinite dimensional Hilbert space $\mathcal{V}$. Imitating the procedure known from spinors on space-time we specify a matrix $S \in Sp(2N)$ and try to find an operator $M(S)$ on $\mathcal{V}$ such that

$$M(S)^{-1}\gamma^a M(S) = S^a{}_b \gamma^b \tag{3.4}$$

We choose S infinitesimally close to the identity and parametrize $S^a{}_b$ as in eq. (2.12). For $M(S)$ we make the ansatz

$$M(S) = 1 - \frac{i}{2}\kappa_{ab}\Sigma^{ab}_{meta} \tag{3.5}$$

Eq. (3.4) leads to the condition

$$[\gamma^a, \Sigma^{bc}_{meta}] = i(\omega^{ab}\gamma^c + \omega^{ac}\gamma^b) \tag{3.6}$$

which is solved by

$$\Sigma^{ab}_{meta} = \frac{1}{4}(\gamma^a\gamma^b + \gamma^b\gamma^a) \tag{3.7}$$

We see that any representation of the γ's gives rise to a representation of the generators Σ^{ab}_{meta}, in terms of which the group elements of $Mp(2N)$ are given then by eq. (3.5). We consider only representations in which the γ^a are hermitian operators with respect to the inner product on $\mathcal{V}$. Then the generators are hermitian too and $M(S)$ is unitary:

$$\begin{aligned}(\gamma^a)^\dagger &= \gamma^a \\ (\Sigma^{ab})^\dagger &= \Sigma^{ab} \\ M(S)^\dagger &= M(S)^{-1}\end{aligned} \tag{3.8}$$

For later use we note that eq. (3.7) implies

$$[\Sigma^{ab}, \Sigma^{cd}] = i(\omega^{ac}\Sigma^{bd} + \omega^{bc}\Sigma^{ad} + \omega^{ad}\Sigma^{bc} + \omega^{bd}\Sigma^{ac}) \tag{3.9}$$

Next we give two explicit examples of the above construction.

THE OSCILLATOR REPRESENTATION: We choose the Hilbert space $\mathcal{V}$ to be the N-fold tensor power of the Fock space of the harmonic oscillator. Let a_i and $a_i^\dagger, i = 1 \cdots N$, denote the corresponding annihilation and creation operators with the usual commutation relation

$$[a_i, a_j^\dagger] = \delta_{ij}, \; i,j = 1 \cdots N \tag{3.10}$$

$\mathcal{V}$ is spanned by the basis vectors

$$|\{n\}\rangle \equiv |n_1\rangle\,|n_2\rangle \cdots |n_N\rangle$$

with $|n_i\rangle \equiv (n_i!)^{-\frac{1}{2}}(a_i^\dagger)^{n_i}\,|0\rangle$ where $n_i = 0, 1, 2, 3, \cdots$. Hence the generic "representation index" α is now a set of N non-negative integers: $\alpha \to \{n\} \equiv (n_1, n_2, \cdots, n_N), n_i = 0, 1, 2 \cdots$ Defining

$$\begin{aligned}\gamma^k &= a_k + a_k^\dagger \\ \gamma^{N+k} &= i(a_k - a_k^\dagger), \; k = 1 \cdots N\end{aligned} \tag{3.11}$$

it is easy to see that these operators satisfy the commutation relations (3.3).

THE $\widehat{\varphi}$-REPRESENTATION: Here we exploit that the metaplectic Clifford algebra is essentially the same as the Heisenberg algebra. We choose $\mathcal{V}$ to be the Hilbert space of a quantum mechanical system with N degrees of freedom whose phase-space has the topology of R^{2N}. The position operators $\widehat{x}^k$ and the momentum operators $\widehat{\pi}^k$ satisfy the canonical commutation relations

$$[\widehat{x}^k, \widehat{\pi}^j] = i\hbar\delta^{kj} \quad , \quad [\widehat{x}^k, \widehat{x}^j] = 0 = [\widehat{\pi}^k, \widehat{\pi}^j], \quad k, j = 1 \cdots N \tag{3.12}$$

Combining $\widehat{x}^k$ and $\widehat{\pi}^k$ into

$$\widehat{\varphi}^a \equiv (\widehat{\pi}^k, \widehat{x}^k) \ , a = 1 \cdots 2N \tag{3.13}$$

eqs. (3.12) read

$$\widehat{\varphi}^a\widehat{\varphi}^b - \widehat{\varphi}^b\widehat{\varphi}^a = i\hbar\omega^{ab} \tag{3.14}$$

Comparing (3.3) with (3.14) we see that the metaplectic γ-matrices can be realized as position and momentum operators:

$$\gamma^a = \Big(\frac{2}{\hbar}\Big)^{\frac{1}{2}} \widehat{\varphi}^a \tag{3.15}$$

In the representation in which $\widehat{x}^k$ is diagonal, say, we have ($k = 1 \cdots N$):

$$\begin{aligned} (\,\gamma^k)^x{}_y &= \Big(\frac{2}{\hbar}\Big)^{\frac{1}{2}} \langle x|\,\widehat{x}^k\,|y\rangle = \Big(\frac{2}{\hbar}\Big)^{\frac{1}{2}} x^k\delta^N(x-y) \\ (\,\gamma^{N+k})^x{}_y &= \Big(\frac{2}{\hbar}\Big)^{\frac{1}{2}} \langle x|\,\widehat{\pi}^k\,|y\rangle = -i(2\hbar)^{\frac{1}{2}}\partial_k\delta^N(x-y) \end{aligned} \tag{3.16}$$

so that the generic representation index α stands now for the eigenvalue of $\widehat{x}^k$: $\alpha \to x \equiv (x^k) \in R^N$. In the bra-ket notation upper and lower indices corresponds to ket and bra vectors, respectively. In this representation the generators of $Mp(2N)$ are second-order differential operators:

$$\Big(\frac{1}{2}\kappa_{ab}\Sigma^{ab}_{meta}\Big)^x{}_y = \Big[-\frac{1}{2}\kappa_{kj}\partial^k\partial^j - \frac{i}{2}\kappa_{N+k,j}(x^k\partial^j + \partial^j x^k) + \kappa_{N+k,N+j}x^k x^j\Big]\delta^N(x-y) \tag{3.17}$$

(We set $\hbar = 1$ from now on.) We shall use the implicit summation (integration) convention also for the variable x which runs over an infinite set of values. Let $|\psi\rangle$ be a vector in the representation space $\mathcal{V}$ with complex components

$$\psi^x \equiv < x|\psi > \tag{3.18}$$

and let $\langle\psi| \in \mathcal{V}^*$ be the vector dual to $|\psi\rangle$ with

$$< \psi|x > \equiv \psi_x = (\psi^x)^* \tag{3.19}$$

Then the dual pairing

$$< \chi|\psi > \equiv \chi_x \psi^x \equiv (\chi^x)^* \psi^x \equiv \int d^N x\, (\chi^x)^* \psi^x \tag{3.20}$$

is the usual inner product on $L^2(R^N, d^N x)$.

4. SPINOR FIELDS ON $\mathcal{M}_{2N}$

The tangent bundle over phase-space, $T\mathcal{M}_{2N}$, has $\mathcal{M}_{2N}$ as its base manifold, and the fibers over each point $\phi \in \mathcal{M}_{2N}$ (the local tangent spaces $T_\phi \mathcal{M}_{2N}$) are copies of R^{2N}. The structure group $Sp(2N)$ acts on the fibers in the vector representation. Let us now consider the associated "spin bundle" where the structure group acts on the fibers in the metaplectic representation. The fibers are now copies of the Hilbert space $\mathcal{V}$ introduced in the previous section. At each point $\phi \in \mathcal{M}_{2n}$ in the base manifold we erect a local Hilbert space $\mathcal{V}_\phi$ which plays a role analogous to $T_\phi \mathcal{M}_{2N}$. Depending on the concrete realization of Σ^{ab}_{meta} we may think of $\mathcal{V}_\phi$, for ϕ fixed, as a Fock space or as the space of square integrable functions over R^N, say. A section through the spin bundle is locally given by a function

$$\psi : \mathcal{M}_{2N} \to \mathcal{V}\ ,\ \phi \mapsto |\psi; \phi\rangle \in \mathcal{V}_\phi \tag{4.1}$$

Here the notation $|\cdots; \phi\rangle$ indicates that this vector "lives" in the local Hilbert space at the point ϕ. At the level of matrix elements the function (4.1) is defined by the components

$$\psi^x(\phi) = < x|\psi; \phi > \tag{4.2}$$

i.e., for each ϕ we specify an (infinite-component) vector in the local Hilbert space $\mathcal{V}_\phi$. We shall use the "$\widehat{\varphi}$-representation" from now on. The operators $\widehat{\varphi} = (\widehat{\pi}^k, \widehat{x}^k)$ should be carefully distinguished from the operators $\widehat{\phi}^a = (\widehat{p}^k, \widehat{q}^k)$ appearing in the conventional canonical quantization of the system $(\mathcal{M}_{2N}, H)$. The operators $\widehat{\phi}^a$ act on the quantum mechanical Hilbert space $\mathcal{V}_{qm}$, which is a *global* object, unrelated to any specific point of phase-space. Here, instead, we are dealing with an infinite family of Hilbert spaces $\mathcal{V}_\phi$ labelled by the points of phase space. The operators $\widehat{\varphi}$ act on those local Hilbert spaces.

By replacing $\mathcal{V}$ by the dual Hilbert space $\mathcal{V}^*$ we arrive at the dual spin bundle. Its sections are locally described by functions

$$\chi_x(\phi) = <\chi;\phi|x> \quad , <\chi;\phi| \in \mathcal{V}^*_\phi \tag{4.3}$$

In our formalism it is natural to consider also "multispinor" fields

$$\phi \mapsto \chi^{x_1 \cdots x_q}_{y_1 \cdots y_p}(\phi) \tag{4.4}$$

which assume values in the tensor product

$$\underbrace{\mathcal{V}^*_\phi \otimes \mathcal{V}^*_\phi \otimes \cdots \otimes \mathcal{V}^*_\phi}_{p\ factors} \otimes \underbrace{\mathcal{V}_\phi \otimes \mathcal{V}_\phi \otimes \cdots \otimes \mathcal{V}_\phi}_{q\ factors} \tag{4.5}$$

which is spanned by vectors of the form

$$|\psi_1;\phi> \otimes \cdots \otimes |\psi_q;\phi> \otimes <\chi_1;\phi| \otimes \cdots \otimes <\chi_p;\phi| \tag{4.6}$$

We assume that χ of eq. (4.4) is completely antisymmetric in its upper and lower indices. Therefore it can be represented by a monomial in η and $\bar\eta$:

$$\widehat{\chi}(\phi,\eta,\bar\eta) = \frac{1}{p!q!}\, \chi^{x_1 \cdots x_q}_{y_1 \cdots y_p}\, \bar\eta_{x_1} \cdots \bar\eta_{x_q} \eta^{y_1} \cdots \eta^{y_p} \tag{4.7}$$

This is a special case of eq. (2.18) : the variables η^x transform in the metaplectic representation now, and $\bar\eta_x$ in the representation dual to it. Their extended Poisson bracket is

$$\left\{\bar\eta_x, \eta^y\right\}_{epb} = -i\delta_x^{\ y} \equiv -i\delta^{(N)}(x-y) \tag{4.8}$$

Now we are in a position to apply the formalism of section 2 to the metaplectic multispinors. The "super-Hamiltonian" is defined as

$$\widetilde{\mathcal{H}} = h^a(\phi)\lambda_a + \frac{1}{2} K_{ab}(\phi) : \bar\eta_x \left(\Sigma^{ab}_{meta}\right)^x_{\ y} \eta^y : \tag{4.9}$$

where : $\cdots$: denotes the anti-normal ordering. (Cf. the remarks at the end of section 2.) Hence, as we have discussed already, the Lie derivative along the hamiltonian vector field is given by

$$\{\widetilde{\mathcal{H}},\widehat{\chi}\}_{epb} = -(l_h\chi)^\wedge \tag{4.10}$$

Inserting (4.7) one obtains

$$\begin{aligned} l_h\chi^{x_1\cdots x_q}_{y_1\cdots y_p}(\phi) &= h^a(\phi)\partial_a\chi^{x_1\cdots x_q}_{y_1\cdots y_p}(\phi) + \frac{i}{2}\sum_{j=1}^{q} K_{ab}(\phi)\big(\Sigma^{ab}_{meta}\big)^{x_j}{}_z\,\chi^{y_1\cdots y_{j-1}zy_{j+1}\cdots y_q}_{x_1\cdots x_p}(\phi) \\ &\quad - \frac{i}{2}\sum_{j=1}^{p} K_{ab}(\phi)\,\chi^{x_1\cdots x_q}_{y_1\cdots y_{j-1}zy_{j+1}\cdots y_p}(\phi)\big(\Sigma^{ab}_{meta}\big)^z{}_{y_j} \end{aligned} \tag{4.11}$$

We introduce time dependent fields $\chi(\phi,t)$ in such a way that the only effect of the time-evolution is to drag the field along the hamiltonian flow:

$$\partial_t\chi^{x_1\cdots x_q}_{y_1\cdots y_p}(\phi,t) = -l_h\chi^{x_1\cdots x_q}_{y_1\cdots y_p}(\phi,t) \tag{4.12}$$

It is interesting to study bilinears formed from a spinor field $\psi^x(\phi)$ and a dual spinor field $\chi_x(\phi)$. (In the following, χ is not necessarily the dual of ψ.) We define:

$$\begin{aligned} E(\phi) &\equiv \chi(\phi)\psi(\phi) \equiv \chi_x(\phi)\psi^x(\phi) \\ T^a(\phi) &\equiv \chi(\phi)\gamma^a\psi(\phi) \equiv \chi_x(\phi)(\gamma^a)^x{}_y\psi^y(\phi) \\ R^{ab}(\phi) &\equiv \chi(\phi)\Sigma^{ab}_{meta}\psi(\phi) \equiv \chi_x(\phi)(\Sigma^{ab}_{meta})^x{}_y\psi^y(\phi) \end{aligned} \tag{4.13}$$

where, as usual, integration over x and y is understood. Using the Leibniz rule (which, in our approach, is equivalent to the Jacobi identity for the extended Poisson brackets) we find for the Lie-derivative of the bilinears (4.13):

$$l_hE = h^c\partial_cE \tag{4.14}$$

$$l_hT^a = h^c\partial_cT^a - \partial_ch^aT^c \tag{4.15}$$

$$l_hR^{ab} = h^c\partial_cR^{ab} - \partial_ch^aR^{cb} - \partial_ch^bR^{ac} \tag{4.16}$$

Here eqs. (3.6) and (3.7) have been used. We observe that $E(\phi)$ transforms under symplectic diffeomorphisms as a scalar, $T^a(\phi)$ as a vector, and $R^{ab}(\phi)$ as a symmetric tensor. We shall see shortly that T^a and R^{ab} are related to translation and rotation generators, hence their names. Eq. (4.14) shows that $E=\chi\psi$ transforms under any canonical transformations and, in particular, under the time evolution, like a scalar density $\varrho(\phi,t)$, see eq. (2.4). Similarly eqs. (4.15) and (4.16) are special cases of eq. (2.5) for vectors and second rank tensors provided the dynamics of χ_x and ψ^x is governed by the equation (4.12).

5. THE "WORLD-LINE" SPINORS η AND $\bar{\eta}$

Next we consider the path-integral (2.22) specialized for $\Sigma^{ab} = \Sigma^{ab}_{meta}$ in the $\widehat{\varphi}$-representation. First we look at the classical equations of motion resulting from the corresponding Lagrangian

$$\widetilde{\mathcal{L}} = \lambda_a \big(\dot{\phi}^a - h^a(\phi)\big) + i\bar{\eta}_x \big[\delta^x{}_y \partial_t + \frac{i}{2}\partial_a\partial_b H(\phi)(\Sigma^{ab}_{meta})^x{}_y\big]\eta^y \tag{5.1}$$

For ϕ^a and λ_a they read

$$\partial_t \phi^a = h^a(\phi) \tag{5.2}$$

$$\partial_t \lambda_a = -\partial_a h^b \lambda_b - \frac{1}{2}\partial_a\partial_b\partial_c H \; \bar{\eta}_x (\Sigma^{bc}_{meta})^x{}_y \eta^y \tag{5.3}$$

and for the spinors they are

$$i\partial_t \eta^x(t) = \frac{1}{2}\partial_a\partial_b H(\phi(t)) \; (\Sigma^{ab}_{meta})^x{}_y \eta^y(t) \tag{5.4}$$

$$i\partial_t \bar{\eta}_x(t) = \frac{1}{2}\partial_a\partial_b H(\phi(t)) \; \bar{\eta}_y (\Sigma^{ab}_{meta})^y{}_x \tag{5.5}$$

Recalling eq. (3.17) we see that eq. (5.4) has the form of the Schrödinger equation

$$i\partial_t \eta = \widehat{H}^{(2)}\eta$$

of a particle whose configuration space is R^N and whose dynamics is governed by a Hamiltonian $\widehat{H}^{(2)} \equiv \frac{1}{2}K_{ab}\Sigma^{ab}_{meta}$ with a quadratic potential:

$$\widehat{H}^{(2)} = -\frac{1}{2}\partial_k\partial_j H \partial^k \partial^j - \frac{i}{2}\partial_{N+k}\partial_j H \; (x^k \partial^j + \partial^j x^k) + \partial_{N+k}\partial_{N+j} H \; x^k x^j \tag{5.6}$$

This Hamiltonian is explicitly time-dependent: the second derivatives of H have to be evaluated at the point $\phi^a(t)$. Eqs. (5.4) and (5.5) are the metaplectic analogs of the equations

$$\partial_t c^a = \partial_b h^a c^b \tag{5.7}$$

$$\partial_t \bar{c}_a = -\partial_a h^b \bar{c}_b \tag{5.8}$$

in the vector representation where the Grassmannian variables are denoted by c^a and $\bar{c}_a$. We mentioned already that eq. (5.7) is the Jacobi equation describing small fluctuations

$\delta\phi^a(t) = c^a(t)$ around classical solutions $\phi(t)$ of Hamilton's equation (5.2). The solution of (5.7) reads

$$c^a(t) = S^a{}_b(t)c^b(0) \tag{5.9}$$

with the Jacobi matrix

$$S(t) = \widehat{T}\ exp \int_0^t dt'\ N(t') \tag{5.10}$$

where $\widehat{T}$ is the time-ordering operator and where N is the matrix whose elements are

$$N^a{}_b(t) \equiv \partial_b h^a(\phi(t)) \equiv \omega^{ac}\partial_b\partial_b H(\phi(t)) \tag{5.11}$$

It is easy to see[3][9] that $S(t)$ is symplectic: $S(t) \in Sp(2N)$. The matrix $N^a{}_b$ is the counterpart of the quadratic Schrödinger operator $\widehat{H}^{(2)}$ in the vector representation. The solution of eq.(5.4) is given by

$$\eta^x(t) = M(S(t))^x{}_y\eta^y(0) \tag{5.12}$$

with the unitary time evolution operator[3]

$$\begin{aligned} M(S(t)) &= \widehat{T}\ exp\ [-\frac{i}{2}\int_0^t dt'\ \partial_a\partial_b H(\phi(t'))\ \Sigma^{ab}_{meta}] \\ &\equiv \widehat{T}\ exp[-i\int_0^t dt' \widehat{H}^{(2)}(\phi(t'))] \end{aligned} \tag{5.13}$$

Note that $S(t)$ and $M(S(t))$ are functionals of the path $\phi^a(t)$. The operator $M(S(t))$ is an element of the metaplectic group, $M(S(t)) \in Mp(2N)$. It represents the Jacobi matrix $S^a{}_b$ in the spinor representation. In general the notation $M = M(S)$ would be ambiguous because of the two-to-one homomorphism between $S \in Sp(2N)$ and $M \in Mp(2N)$. However, in the present case, the continuity of the time-evolution and the fixed initial condition $S(t = 0) = 1$, $M(t = 0) = 1$ makes the relation between S and M unique[3].

Eqs. (5.9) and (5.12) should be interpreted as folows. We start from a trajectory $\phi^a(t)$ and calculate the corresponding Jacobi matrix $S^a{}_b(t)$ for $t > 0$. At t=0, the variable $c^a(0)$ is a vector in the tangent space $T_{\phi(0)}\mathcal{M}_{2N}$. For $t \geq 0$, the $c^a(t)$ given by (5.9) "lives" in a different tangent space: $c^a(t) \in T_{\phi(t)}\mathcal{M}_{2N}$. Eq. (5.9) describes how $c^a(t)$ is "transported" along the one-parameter family of tangent spaces $T_{\phi(t)}\mathcal{M}_{2N}$ which is associated to the classical path $\phi^a(t)$. An analogous argument applies to the metaplectic spinors η. For different times, $\eta(t)$ is a vector in different local Hilbert spaces: $\eta(t) \in \mathcal{V}_{\phi(t)}$. Eq.(5.12) describes the change of η as it is dragged along $\phi^a(t)$ from $\mathcal{V}_{\phi(0)}$ to $\mathcal{V}_{\phi(t)}$.

The importance of the classical solution (5.12) stems from the fact that the path-integral

$$Z_{meta}(\phi_2, \eta_2, t_2; \phi_1, \eta_1, t_1) \propto \int \mathcal{D}\phi\ \mathcal{D}\lambda\ \mathcal{D}\eta\ \mathcal{D}\bar{\eta}\ exp\big[i \int_{t_1}^{t_2} dt \widetilde{\mathcal{L}}\big] \tag{5.14}$$

with the boundary conditions $\phi^a(t_{1,2}) = \phi^a_{1,2}$ and $\eta^x(t_{1,2}) = \eta^x_{1,2}$ can be solved exactly in terms of the classical solution ϕ_{cl} of eq. (5.2) and the operator $M(S(t))$:

$$Z_{meta}(\phi_2, \eta_2, t_2; \phi_1, \eta_1, t_1) = \delta(\phi_2 - \phi_{cl}(t; \phi_1))\ \delta(\eta_2 - M(S(t_2; \phi_1))\eta_1) \tag{5.15}$$

Here the solution $\phi_{cl}(t; \phi_1)$ is subject to the initial condition $\phi_{cl}(0; \phi_1) = \phi_1$, and $S(t; \phi_1)$ is the Jacobi matrix (5.10) for exactly this classical path. Because of the ordering ambiguity mentioned in section 2, eq.(5.15) obtains only for a specific discretization of the path-integral (5.14), namely the one corresponding to "antinormal" ordering. Any other discretization would lead to an additional path-dependent phase factor on the RHS of eq. (5.15).

Next we investigate the time evolution of the bilinears formed from the variables η^x and $\bar{\eta}_x$. We consider

$$\begin{aligned}
\mathcal{E}(t) &\equiv \bar{\eta}(t)\eta(t) \equiv \bar{\eta}_x(t)\eta^x(t) \\
\mathcal{T}^a &\equiv \bar{\eta}(t)\gamma^a\eta(t) \equiv \bar{\eta}_x(t)(\gamma^a)^x{}_y\eta^y(t) \\
\mathcal{R}^{ab}(t) &\equiv \bar{\eta}(t)\Sigma^{ab}_{meta}\eta(t) \equiv \bar{\eta}_x(t)(\Sigma^{ab}_{meta})^x{}_y\eta^y(t)
\end{aligned} \tag{5.16}$$

These bilinears should not be confused with the similar ones in eq. (4.13): in eq. (4.13) the fields are defined on all of phase-space, whereas the objects in (5.16) are defined along some fixed trajectories $\phi(t)$ only. In an obvious analogy, $\psi^x(\phi)$ and $\eta^x(t)$ can be visualized as "space-time" and "world-line" fermions, respectively. Using eqs.(5.4) and (5.5) one finds that

$$\partial_t \mathcal{E}(t) = 0 \tag{5.17}$$

$$\partial_t \mathcal{T}^a(t) = \partial_b h^a \mathcal{T}^b(t) \tag{5.18}$$

$$\partial_t \mathcal{R}^{ab}(t) = \partial_c h^a \mathcal{R}^{cb}(t) + \partial_c h^b \mathcal{R}^{ac}(t) \tag{5.19}$$

We can look at these equations in two rather different ways. From a geometrical point of view they express the fact that the time-evolution drags $\mathcal{E}, \mathcal{T}^a$ and $\mathcal{R}^{ab}$ along the path $\phi(t)$ whereby they behave as a scalar, a vector and a symmetric tensor, respectively. With this interpretation eqs.(5.17), (5.18), (5.19) may be thought of as a restriction of eqs. (4.14), (4.15) and (4.16) (with l_h replaced by $-\partial_t$) to the points of the trajectory $\phi^a(t)$. We may look at eqs. (5.17), (5.18), (5.19) also as a manifestation of Ehrenfest's theorem of elementary quantum mechanics which says that the expectation values of observables, which are at most quadratic in the coordinates and momenta, evolve exactly according to the classical equations of motion. In the present case the theorem applies to the operators $\widehat{x}^k$ and $\widehat{\pi}^k$ acting on the local Hilbert spaces $\mathcal{V}_\phi$. In fact, $\mathcal{T}^a$ is the "expectation value" $\bar{\eta}\gamma^a\eta$ of the "observable" γ^a, *i.e.*, of $\widehat{x}^k$ and $\widehat{\pi}^k$, with respect to the "wave-function" $\eta^x(t) \equiv \eta(x,t)$. Similarly, $\mathcal{R}^{ab}$ is the "expectation value" of products such as $\widehat{x}^k\widehat{\pi}^j$. With this interpretation, eq. (5.17) expresses the conservation of the "norm" $\bar{\eta}\eta$ in the local Hilbert space $\mathcal{V}_\phi$.

We observe that eq. (5.18) coincides with eq. (5.7) for the vector ghosts. This suggests the correspondence

$$c^a \sim \bar{\eta}\gamma^a\eta \tag{5.20}$$

indicating that the metaplectic world-line spinors may be thought of as "square roots" of the vectors c^a, *i.e.*, of the Jacobi fields.

In the operatorial formalism equivalent to the path-integral (5.14) the anticommuting variables satisfy

$$[\bar{\eta}_x, \eta^y] = \delta^y_x \equiv \delta^{(N)}(x-y) \tag{5.21}$$

This implies the following commutators for the bilinears (5.16) :

$$\begin{aligned}
[\mathcal{E},\mathcal{T}^a] &= 0 \;\; , \;\; [\mathcal{E},\mathcal{R}^{ab}] = 0 \\
[\mathcal{T}^a,\mathcal{T}^b] &= 2i\omega^{ab}\mathcal{E} \\
[\mathcal{T}^a,\mathcal{R}^{bc}] &= i\left(\omega^{ab}\mathcal{T}^c + \omega^{ac}\mathcal{T}^b\right) \\
[\mathcal{R}^{ab},\mathcal{R}^{cd}] &= i\left(\omega^{ac}\mathcal{R}^{bd} + \omega^{ad}\mathcal{R}^{bc} + \omega^{bc}\mathcal{R}^{ad} + \omega^{bd}\mathcal{R}^{ac}\right)
\end{aligned} \tag{5.22}$$

These are the commutation relations for the Lie algebra of the inhomogeneous metaplectic group[3] $IMp(2N)$. It is the semi-direct product of $Mp(2N)$ generated by $\mathcal{R}^{ab}$, and the Weyl group generated by $\mathcal{T}^a$ and $\mathcal{E}$. Comparing eqs. (5.22) to the Poincaré algebra we see that $\mathcal{R}^{ab}$ and $\mathcal{T}^a$ play the role of Lorentz rotations and of translations on phase-space. This analogy is incomplete, however. On phase-space the translations $\mathcal{T}^a$ do not commute: The Weyl group contains the central extension $\mathcal{E}$. Usually this fact is expressed by saying that the Weyl operators $T(\phi) \equiv exp\ [i\phi^a \omega_{ab} \widehat{\phi}^b]$ provide a projective representation of the translations: $T(\phi_1)T(\phi_2) = exp[\frac{i}{2}\phi_1^a \omega_{ab} \phi_2^b] T(\phi_1 + \phi_2)$. This implies that we pick up a phase when we go around a loop in phase-space (see ref.[3] for a detailed discussion).

6. THE METAPLECTIC SPINORS AS SEMICLASSICAL WAVE FUNCTIONS

In this section we reinterprete the variables $\eta^x(t)$ and $\bar{\eta}_x(t)$ from a slightly different point of view. Let us consider the quantum mechanical system defined by the Hamiltonian $H(\phi^a) \equiv H(p^i, q^i)$. We know that transition amplitudes can be expressed as Feynman path-integrals

$$< q_2, t_2 | q_1, t_1 > = \int \mathcal{D}\phi^a(t)\ exp\{i \int_{t_1}^{t_2} dt \big[\frac{1}{2} \phi^a \omega_{ab} \dot{\phi}^b - H(\phi)\big]\} \tag{6.1}$$

The integration is over paths $\phi^a(t) \equiv \big(p^i(t), q^i(t)\big)$ subject to the boundary conditions $q^i(t_{1,2}) = q^i_{1,2}$. Let us evaluate (6.1) semiclassically. To this end we split the integration variable $\phi^a(t)$ in a classical part $\phi_{cl}(t)$, which solves Hamilton's equation, and a quantum fluctuation $\varphi^a(t) \equiv \big(\pi^i(t), x^i(t)\big)$: $\phi^a(t) = \phi^a_{cl}(t) + \varphi^a(t)$. In this way we obtain

$$< q_2, t_2 | q_1, t_1 > = \sum_{\phi_{cl}} e^{iS[\phi_{cl}]} \Delta(x_2, t_2; x_1, t_1; [\phi_{cl}]) \tag{6.2}$$

where the sum is over all relevant classical solutions, and

$$\Delta(x_2, t_2; x_1, t_1; [\phi_{cl}]) \equiv \int \mathcal{D}\varphi^a(t)\ exp\{i \int_{t_1}^{t_2} dt \big[\frac{1}{2} \varphi^a \omega_{ab} \dot{\varphi}^b - H^{(2)}(\varphi^a; \phi_{cl})\big]\} \tag{6.3}$$

with the quadratic Hamitonian

$$H^{(2)}(\varphi^a; \phi^a_{cl}) = \frac{1}{2} \partial_a \partial_b H(\phi_{cl}(t))\ \varphi^a \varphi^b \tag{6.4}$$

The boundary conditions for the integral (6.3) are $x^i(t_{1,2}) = x^i_{1,2}$ where $q^i_{1,2} \equiv q^i_{cl}(t_{1,2}) + x^i_{1,2}$. If $q^i_{cl}(t)$ passes *exactly* through the points $q^i_{1,2}$ at time $t = t_{1,2}$ then $x^i_{1,2} = 0$. Let us also consider the possibility that there is no classical path which connects q^i_1 and q^i_2 exactly but that the amplitude is nevertheless dominated by a classical trajectory (or several) which passes near q_1^i and q_2^i at $t = t_{1,2}$. In this case $x_{1,2}^i \neq 0$. From eq. (6.3) with (6.4) we learn that the quantum correction Δ is a solution of the Schrödinger equation

$$i\partial_t \Delta(x,t;x_0,t_0;[\phi_{cl}]) = \widehat{H}^{(2)} \Delta(x,t;x_0,t_0;[\phi_{cl}]) \tag{6.5}$$

In this equation

$$\widehat{H}^{(2)} = \frac{1}{2}\partial_a\partial_b H(\phi_{cl}(t))\, \widehat{\varphi}^a \widehat{\varphi}^b = \frac{1}{2} K_{ab}(\phi_{cl}(t))\, \Sigma^{ab}_{meta} \tag{6.6}$$

is exactly the Schrödinger Hamiltonian which appears in the equation of motion of the world-line spinors $\eta(t)$, see eq.(5.4) and eq. (3.7) with (3.15). As a consequence, eq. (6.5) coincides with the equation of motion (5.4) for $\eta^x(t) \equiv \Delta(x,t)$. This allows for a geometric interpretation of the "index" $x \in R^N$ which labels the matrix elements of Σ^{ab}_{meta} in the $\widehat{\varphi}$-representation. Originally the operators $\widehat{\varphi}^a = (\widehat{\pi}^i, \widehat{x}^i)$ were introduced in (3.15) in order to represent the metaplectic γ-matrices. Here we see that they are the operatorial counterpart of the fluctuation variable $\varphi^a(t) \equiv \phi^a(t) - \phi^a_{cl}(t)$ in the path-integral. In this way the eigenvalues of $\widehat{x}$, *i.e.*, the "representation indices" x , acquire a direct physical interpretation related to the *base manifold* $\mathcal{M}_{2n}$ rather than the *fiber* $\mathcal{V}_\phi$: given some point $\phi = (p,q) \in \mathcal{M}_{2n}$ they parametrize nearby points with the same coordinate p as $(p, q+x)$. This is obvious from the initial condition of the path-integral: $q_{1,2} = q_{cl}(t_{1,2}) + x_{1,2}$. (Since the initial and final values of the momenta are integrated over, no corresponding shift of p is obtained.)

The analogous situation in the vector representation, *i.e.*, in classical mechanics, is well known[12]. Let us fix a point $\phi \in \mathcal{M}_{2N}$ and let us consider the vectors $c \in T_\phi \mathcal{M}_{2N}$ living in the tangent space at ϕ. Then, heuristically speaking, these vectors parametrize the points in the neighbourhood of ϕ as $\phi' = \phi + \varepsilon c$ (with ε an anticommuting parameter[7,12]). This means that (locally) points in the fiber are related to points in the base space. Switching on the hamiltonian flow and repeating this construction at each point along some classical trajectory $\phi(t)$, the "world-line vector fields" $c(t)$, living in the tangent space to the space of classical paths, parametrize nearby trajectories via $\phi'(t) = \phi(t) + \varepsilon c(t)$. Hence $c(t)$ obeys the Jacobi equations (5.7). Its solution tells us the influence of a change of initial conditions, $\delta\phi(t_1) \equiv \varepsilon c(t_1)$, on the trajectory for any later time $t_2 > t_1$: $\delta\phi(t_2) = \varepsilon c(t_2)$.

In the metaplectic case the situation is similar: the world-line spinors $\eta(t)$ are the "square-root" of the Jacobi fields $c(t)$ in the sense that $c^a \sim \bar{\eta}\gamma^a\eta$. Bilinears formed from

η living in the fiber $\mathcal{V}_\phi$ can be used to parametrize nearby points in the base as

$$\phi'^{\,a} = \phi^a + \bar{\eta}\gamma_a\eta \tag{6.7}$$

In this sense $\mathcal{V}_\phi^* \otimes \mathcal{V}_\phi$ is equivalent to $T_\phi\mathcal{M}_{2n}$. More importantly, the spinors $\eta^x(t) = \Delta(x,t;x_1,t_1;[\phi_{cl}])$ themselves are semiclassical wave functions. They give us the probability amplitude of finding the particle at time t at a distance x relative to $q_{cl}(t)$ given that, at time t_1, it was at distance x_1 from $q_{cl}(t_1)$.

7. CONCLUSIONS

In the previous sections we introduced a path-integral for the time evolution of metaplectic spinor fields which can be defined on the phase-space of any hamiltonian system. The path-integral includes an integration over world-line spinors $\eta(t)$ which assume values in local Hilbert spaces $\mathcal{V}_\phi$. They are the quantum mechanical analogue of the world-line vectors $c^a(t)$, living in $T_\phi\mathcal{M}_{2N}$, which appear in classical mechanics. The classical quantities can be interpreted as Jacobi fields, the metaplectic spinors are semiclassical wave-functions.

The kinematical framework outlined in this paper lends itself for further investigation in various direction. A particularly interesting aspect concerns the fact that not every symplectic manifold can support metaplectic structures[2]. This is analogous to the well-known fact that there are space-times which cannot carry spin-structure. Witten has shown[14] that in this case the supersymmetric quantum mechanics of a spinning particle on this space-time is suffering from a global anomaly. Using similar techniques, it can be shown that the path-integral discussed in this paper has a global anomaly if one attempts to define it on a symplectic manifold which cannot carry metaplectic structures. We shall come back to this point elsewhere[15].

Further open problems to be explored include the implementation of the background split-symmetry[16] in the metaplectic path-integral. In the classical case this symmetry is implemented by a BRS invariance[7] which guarantees that a shift in the classical trajectory, $\phi_{cl}(t)$, can be compensated for by a corresponding contribution from $c^a(t)$. Since $c^a(t)$ is replaced by the composite $\bar{\eta}\gamma^a\eta$ in the metaplectic case, a variant of the classical BRS symmetry might be present also here, but more work is needed.

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