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QUANTUM DEFORMATION OF THE LADDER REPRESENTATIONS OF U(1,1) FOR |q| = 1

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ABSTRACT

We construct $U_q(u(1,1))$ covariant creation and annihilation operators for q on the unit circle and consider the corresponding ladder and singleton representations.

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1. Introduction

We consider the quantum universal enveloping (QUE) algebra $U_q \equiv U_q(gl(2))$ with $U_q(sl(2))$ generators $q^{\pm H}$, E, F, satisfying the commutation relations (CR)

$$q^H E = E\, q^{H+2} \qquad q^H F = F\, q^{H-2} \tag{1.1a}$$

$$[\, E, F\,] = [H] \; := \frac{q^H - q^{-H}}{q - q^{-1}} \tag{1.1b}$$

and central subgroup $\{ q^{nN}, n \in \mathbb{Z} \}$. The coproduct is underformed for the Cartan generators $q^{\pm H}$, $q^{\pm N}$, while for E and F it has the form

$$\Delta(E) = E \otimes q^{-H} + 1 \otimes E \qquad \Delta(F) = F \otimes 1 + q^H \otimes F. \tag{1.2}$$

The counit (or trivial representation) ε is an algebra homomorphism $U_q(gl(2)) \to \mathbb{C}$ defined as 0 on the Lie algebra type generators E and F and as 1 on the group type elements q^{nH} and q^{nN}. The antipode S is an antihomomorphism of U_q satisfying

$$S(q^{\pm H}) = q^{\mp H}, \qquad S(q^{\pm N}) = q^{\mp N},$$
$$S(E) = -\, E\, q^H, \qquad S(F) = -\, q^{-H} F. \tag{1.3}$$

We now assume that $|q| = 1$ $(q \neq \pm 1)$ and introduce the antilinear antiinvolution * satisfying $(q^* = q^{-1}$ and)

$$(q^H)^* = q^{-H}, \quad (q^N)^* = q^{-N}, \quad E^* = -\, F \; (\Rightarrow F^* = -\, E\,) \tag{1.4}$$

and hence

$$S \circ * = * \circ S. \tag{1.5a}$$

Defining the star operation to satisfy [1]

$$(\, A \otimes B\,)^* = B^* \otimes A^*, \quad A, B \in U_q, \tag{1.5b}$$

we obtain that it is also a coalgebra antihomomorphism,

$$\Delta \circ * = (\, * \otimes *\,) \circ \sigma \circ \Delta \tag{1.5c}$$

(where σ is the permutation in $U_q^{\otimes 2}$, $\sigma(A \otimes B) := B \otimes A$).

Our complex QUE algebra equipped with such a conjugation will be identified with its "q-deformed noncompact real form" $U_q(u(1,1))$ for $|q| = 1$. We point out that the above star operation is usually not considered (cf. [2],[3]) because the unconventional property (1.5b) is being (implicitly) rejected.

The objective of the present paper will be to construct the singleton and the ladder representations of this QUE algebra that will be obtained as q-deformation of the corresponding representations of $sp(2,\mathbb{R}) \simeq su(1,1)$ (cf. [4],[5]). The latter being the simplest among the simple noncompact Lie algebras, this paper will be a first step to studying a relevant class of q-deformed representations of higher rank Lie algebras (like $sp(4,\mathbb{R})$ - cf. [6]) which may be viewed as deformations of relativistic symmetries.

We shall briefly recall the construction of the singleton and the ladder representations in the "classical" ($q = 1$) case. Let x belong to the Lie algebra $u(1,1)$, i.e. be a 2×2 complex matrix characterized by

$$x^* \tau_3 = - \tau_3 x \ , \ \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.6}$$

(where x^* is the hermitian conjugate of x), and let $\{a^*,a\}$ and $\{b^*,b\}$ be two (mutually commuting) pairs of Bose creation and annihilation operators:

$$[a, b] = 0 = [a, b^*] \quad , \quad [a, a^*] = 1 = [b, b^*] . \tag{1.7}$$

Then the ladder representation is an (infinite dimensional, reducible) unitary lowest weight representation of $u(1,1)$ defined by

$$x \to \mathfrak{X}(x) := (a^* \ \ -b) \ x \begin{pmatrix} a \\ b^* \end{pmatrix} \quad ; \tag{1.8a}$$

in particular the $su(1,1)$ generators are

$$E = \mathfrak{X}(\tau_+) = a^* b^* \quad , \quad F = \mathfrak{X}(\tau_-) = -b \ a$$
$$H = \mathfrak{X}(\tau_3) = a^* a + b \ b^* . \tag{1.8b}$$

$\mathfrak{X}(x)$ acts in the Fock space $\mathfrak{F}$ of $a^{(*)}$ and $b^{(*)}$. By definition, $\mathfrak{F}$ contains a distinguished vacuum vector $|0\rangle$ such that

$$a\,|0\rangle = 0 = b\,|0\rangle \;\; (\Rightarrow F\,|0\rangle = 0, \;\; H\,|0\rangle = |0\rangle)\,, \qquad \langle 0|\, a^* = 0 = \langle 0|\, b^*, \;\; \langle 0|0\rangle = 1; \tag{1.9}$$

the vector $|0\rangle$ is determined from these properties up to a phase factor. The irreducible components of the ladder representation are singled out by the (integer) eigenvalues of the central element

$$N = a^* a - b^* b\,. \tag{1.10}$$

The singleton representation is obtained for $a = b$ by replacing (1.8) and (1.9) by

$$x \to \frac{1}{2}\,(a^* - a)\; x \begin{pmatrix} a \\ a^* \end{pmatrix}, \quad a\,|0\rangle = 0 = (H - \frac{1}{2})\;|0\rangle\,. \tag{1.11}$$

2. $U_q(u(1,1))$ covariant creation and annihilation operators

We demand that the pair $(a^*, -b)$ (of the type appearing in the construction (1.8) of the ladder representation of $u(1,1)$) transforms covariantly under $U_q(u(1,1))$. In other words, it satisfies essentially the same CR with the $U_q(gl(2))$ generators as the $U_q(su(2))$ - covariant creation operators of Pusz and Woronowicz [7] (see also [8] where both cases q real and $|q| = 1$ have been considered):

$$[E, a^*] = 0\,, \qquad -[E, b] = a^* q^{-H} \tag{2.1a}$$

$$F a^* - q a^* F = -\,b\,, \qquad F\,b = q^{-1} b\, F \tag{2.1b}$$

$$q^H a^* = a^* q^{H+1}\,, \qquad q^H b = b\, q^{H-1} \tag{2.1c}$$

$$q^N a^* = a^* q^{N+1}\,, \qquad q^N b = b\, q^{N+1}\,. \tag{2.1d}$$

The corresponding transformation properties of the conjugate pair (a, b^*) are deduced from (2.1) by hermitian conjugation taking into account (1.4).

The q-deformed Bose CR that are covariant under (2.1) (when $|q| = 1$) are

$$a^* b = q\; b\; a^* \qquad (\Leftrightarrow a\; b^* = q\; b^* a)\,. \tag{2.2}$$

We are again looking for a Fock space representation of the deformed CR (to be determined) with a vacuum vector satisfying (1.9). According to the general idea that Cartan generators remain stable against deformation, we define the hermitian number operators N_a and N_b that annihilate the vacuum and satisfy

$$N = N_a - N_b \ , \qquad H = N_a + N_b + 1 \tag{2.3}$$

(cf. (1.8b)). It would be plausible to suppose that the deformed CR between the creation operators a^* and b^* are still homogeneous, i.e.

$$b^* a^* = f \ a^* b^* \qquad (\Leftrightarrow a \ b = \bar{f} \ b \ a) \tag{2.4}$$

where $f = f(q)$ is an yet undetermined function of q. The E-invariance of (2.4) is automatic, since (2.1b) implies $q \ E \ b^* = b^* E$. Eq. (2.1a) gives $[b^*, F] = q^H a$, so that

$$F \ b^* a^* = q \ b^* a^* F - (a \ a^* q^H + b^* b) \tag{2.5a}$$

$$F \ a^* b^* = q \ a^* b^* F - (a^* a \ q^H + b \ b^*) . \tag{2.5b}$$

Thus, invariance with respect to F requires

$$a \ a^* q^H + b^* b = f \ (a^* a \ q^H + b \ b^*) . \tag{2.5c}$$

Normalizing the creation and annihilation operators by the condition

$$\langle 0 | a \ a^* | 0 \rangle = 1 = \langle 0 | b \ b^* | 0 \rangle \ , \tag{2.6}$$

we obtain from (2.5) and (2.3)

$$f = q \ . \tag{2.7}$$

For this value of f, (2.5) and its hermitian conjugate become equivalent to

$$b^* b = [H-1] \ a \ a^* - [H] \ a^* a \tag{2.8a}$$

$$b \ b^* = [H] \ a \ a^* - [H+1] \ a^* a \tag{2.8b}$$

or, taking once more into account (2.3), to

$$(b\ b^* - q^{\alpha} b^* b)\, q^{\alpha N_b} = (a\ a^* - q^{-\alpha}\, a^* a)\, q^{-\alpha N_a} := A_\alpha\ , \tag{2.9}$$

$\alpha = \pm 1$ (obviously, $A_+ = A_-^*$). Now, it turns out that the commutation relation of, say, A_+ with E (which can be obtained from (2.1), (2.3) and (1.1)), depend on whether A_+ is expressed in terms of $\{a^*, a\}$ or $\{b^*, b\}$; we obtain in the first case

$$E\ A_+ = q^2 A_+ E + (1-q^2)\ C \tag{2.10a}$$

with

$$C = a^*\ b^*\ q^{-N_a}\ , \tag{2.10b}$$

and in the second case

$$E\ A_+ = q^{-2} A_+ E + (1-q^{-2})\ C\ . \tag{2.10c}$$

Consistency of (2.10a) and (2.10c) then requires

$$C = A_+ E \tag{2.10d}$$

which , in turn, implies

$$[\, E,\ A_+ \,] = 0\ . \tag{2.11a}$$

Combining this with the representation independent equality

$$[\, E,\ A_- \,] = 0\ , \tag{2.11b}$$

and taking into account the relations conjugate to (2.11a,b), one concludes that A_+ and A_- are $U_q(u(1,1))$ invariant operators; therefore, one may choose

$$A_\alpha = c_\alpha\ 1\ ,\ \alpha = \pm\ . \tag{2.12a}$$

The normalization condition (2.6) then dictates that

$$A_+ = 1 = A_-\ . \tag{2.12b}$$

As a byproduct we obtain from (2.10d) and (2.12b)

$$E\ (= C) = a^*\ b^*\ q^{-N_a}\ , \tag{2.13a}$$

so that

$$F\ (= -\ E^*) = -\ q^{N_a}\ b\ a\ . \tag{2.13b}$$

Moreover, from

$$(a\, a^* - q^{\alpha} a^* a) = q^{-\alpha N_a} , \quad (b\, b^* - q^{\alpha} b^* b) = q^{-\alpha N_b} \tag{2.14}$$

(see (2.9) and (2.12b)), one immediately obtains

$$a^* a = [N_a] , \quad a\, a^* = [N_a + 1] \tag{2.15a}$$

$$b^* b = [N_b] , \quad b\, b^* = [N_b + 1] . \tag{2.15b}$$

Note that the CR (1.1b) follows from (2.13) because of the q-number relation

$$[x + 1] [y + 1] - [x] [y] = [x + y + 1] . \tag{2.16a}$$

Since also

$$q^{-y} [x] - q^{-x} [y] = [x - y] , \tag{2.16b}$$

we can write down the $U_q(u(1,1))$ invariant "particle minus antiparticle q-number operator" $[N]$ (cf. (2.3), (2.15) and compare with (1.10)) as

$$[N] = q^{-N_b} a^* a - q^{-N_a} b^* b . \tag{2.17}$$

3. Ladder representations of $U_q(u(1,1))$

For generic q (i.e., q not a root of 1), the eigenvalues N of N label the irreducible components $\mathfrak{F}_N$ of the Fock space representation. Unitarity (Hilbert space positivity) restricts in this case N to zero. Indeed, the norm square of an eigenvector of H

$$\Psi_{Nm} = \frac{E^m}{[m]!} \Psi_{No} , \quad m \in \mathbb{Z}_+ , \tag{3.1}$$

where Ψ_{No} is a lowest weight vector of the form

$$\Psi_{No} = \begin{cases} (a^*)^N |0\rangle & . \quad N \geq 0 \\ (b^*)^{-N} |0\rangle & . \quad N \leq 0 \end{cases} \tag{3.2}$$

is

$$\| \Psi_{Nm} \|^2 = \frac{[m+|N|]!}{[m]!} . \tag{3.3}$$

If $[k] > 0$ for $1 \leq k \leq [\,|N|\,]$, then positivity of (3.3) would imply positivity of $[n+|N|]$ for all positive n. This is impossible, however, since for $q = \exp it$ (t real, $t \neq 0$) the function

$$[\nu] = \frac{\sin \nu t}{\sin t} \tag{3.4}$$

cannot be positive for all positive ν. For $N = 0$ the vectors Ψ_{0m} form an orthonormal basis and the representation of $U_q(u(1,1))$ is unitary.

Let now q be a root of 1. We consider the cases of an even and an odd root separately.

If p is the (minimal natural) number such that

$$q^p = -1 \Rightarrow [p] = 0 \tag{3.5}$$

then the lowest weight vector Ψ_{No} has zero norm for $|N| \geq p$ (its norm being $[\,|N|\,]!$). For $|N| < p$ the space $\mathfrak{F}_N$ has a (p-|N|)-dimensional invariant subspace spanned by $E^m \Psi_{No}$, $m = 0,1,\ldots, p-|N|-1$. If, in addition,

$$[2] = q + q^{-1} = 2 \cos \pi/p \Leftrightarrow q = \exp \frac{\pm i\pi}{p} , \tag{3.6}$$

then the finite dimensional representation of $U_q(u(1,1))$ in this subspace is unitary. The unitarity property fails for q an odd root of 1, since e.g.

$$[m+1] < 0 \quad \text{for} \quad q = \exp i \frac{2\pi}{2m+1} . \tag{3.7}$$

4. A q-singleton representation

In order to avoid fractional powers of q in considering a q-deformed singleton we shall first rescale the generators (viewed now as forming a (Chevalley) basis of $U_q(sp(2,\mathbb{R}))$), setting

$$[2]\,[E,F] = [H] , \quad [H,E] = 4E , \quad [H,F] = -4F \tag{4.1}$$

$$[H, a^*] = 2a^* , \quad [H, \tilde{a}] = -2\tilde{a} . \tag{4.2}$$

We shall assume further the following $U_q(sp(2,\mathbb{R}))$ covariance properties for the components of the doublet $(a^*, -\tilde{a})$ (cf. (2.1), (4.1) and (4.2)):

$$[E, a^*] = 0 , \quad -[E, \tilde{a}] = a^* q^{-H} \tag{4.3a}$$

$$F a^* - q^2 a^* F = -\tilde{a} , \quad F \tilde{a} = q^{-2} \tilde{a} F . \tag{4.3b}$$

All these CR together with the $U_q(sp(2,\mathbb{R}))$-invariant deformation of the canonical CR $[a, a^*] = 1$ (valid for $q = 1$),

$$(a^* \quad -\tilde{a}) \; \varepsilon \begin{pmatrix} a^* \\ -\tilde{a} \end{pmatrix} = \tilde{a} a^* - q^{-2} a^* \tilde{a} = 1 , \tag{4.4a}$$

$$\varepsilon = \begin{pmatrix} 0 & q^{-2} \\ -1 & 0 \end{pmatrix} \tag{4.4b}$$

are satified for

$$\tilde{a} = q^{-N} a \tag{4.5}$$

where N is the "particle number":

$$[N, a^*] = a^* , \quad [N, a] = -a , \quad N|0\rangle = 0 . \tag{4.6}$$

The $U_q(sp(2,\mathbb{R}))$ generators are expressed in terms of a^*, a and N as follows (see also [9]):

$$E = \frac{a^{*2}}{[2]} , \quad F\,(= -E^*) = -\frac{a^2}{[2]} , \quad H = 2N+1. \tag{4.7}$$

The resulting Fock space representation of $U_q(sp(2,\mathbb{R}))$ splits into two "q-singleton" representations with lowest weight vectors $|0\rangle$ and $a^*|0\rangle$, respectively .

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