ISTITUTO NAZIONALE DI FISICA NUCLEARE

Sezione di Trieste

INFN/AE-91/08

22 AGOSTO 1991

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MEAN-FIELD QUANTUM GRAVITY

Servizio Documentazione
dei Laboratori Nazionali di Frascati

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Abstract

We describe a new approach to quantum gravity, based on a kind of mean-field approximation. The action, which we choose to be quadratic in curvature and torsion, is made polynomial by replacing the inverse vierbein by its mean value. This action is used to compute the effective action for the vierbein and hence its vacuum expectation value. Self-consistency is then enforced by requiring that this vacuum expectation value be proportional to the mean field. We have explicitly carried out this self-consistent procedure at one-loop in the case of a mean field corresponding to Minkowski space, de Sitter space and in the long wavelength limit for a generic space. General Relativity is recovered as a low energy approximation.

Ref. S.I.S.S.A. 101/91/EP (July 1991)

1. INTRODUCTION

From the point of view of Elementary Particle Physics, Einstein's theory of gravity has many features in common with non-linear chiral models of QCD: both theories have derivative couplings, nonpolynomial interactions, a dimensionful coupling constant, are not renormalizable a.s.o. [1]. All this originates from more basic similarities at the kinematical level. In fact, it has been known for a long time that the metric carries a nonlinear realization of the group $GL(4)$, linear with respect to the Lorentz subgroup $O(1,3)$, and therefore is similar to the field variables of a nonlinear sigma model with values in the coset space $GL(4)/O(1,3)$ [2]. Another fruitful analogy is the one between gravity and Yang-Mills theory. This is particularly striking in the vierbein formalism and in first order formulations, where the Lorentz connection is an independent dynamical variable and the theory is invariant under local Lorentz transformations, in addition to coordinate transformations [3].

At first sight these two analogies have little in common and one may think that only one of them can be pursued at the time. However, this is not so. In fact, one can give a locally-$GL(4)$-invariant reformulation of General Relativity which makes both analogies apparent [4-6]. The best analogy is then between General Relativity and a chiral model in which the flavor group has been gauged. In this formulation, in addition to a $GL(4)$ connection, there are two nonlinear fields: the soldering form and an internal Lorentzian metric. The nonlinearity arises from the constraints that the soldering form be nondegenerate and the internal metric has Lorentzian signature. Either one of these fields (but not both at the same time) can be gauged away, leaving us with General Relativity either in metric or in vierbein formulation. Without the soldering form, the theory would describe a gauged $GL(4)/O(1,3)$-valued nonlinear sigma model. So this is more than just an analogy: one may say that gravity is very literally a soldered gauged sigma model.

If one takes this point of view seriously, one is led to believe that General Relativity should not be quantized, but rather be regarded as a low energy limit of some more fundamental theory. Now, the dynamical variables of QCD (quarks and gluons) are different from those of the chiral models (mesons): the former are described by spinor and vector fields, the latter by non-linear scalar fields. In the same way it may well be that the fundamental variables underlying gravity could have little or nothing to do with the metric, vierbein or connection. This possibility has been discussed for some time in the so-called "induced gravity" programme, where the Einstein action was seen as part of the effective action of some matter fields [7,8]. String and membrane theories also go in this direction. Our attempt here will be less radical: we shall assume that the familiar objects which appear in General Relativity (metric, vierbein, connection) are indeed fundamental variables. However, we shall try to go beyond the picture of an effective induced theory.

A nonlinear sigma model can always be regarded as a linear Higgs model with some constraint of the form $\Phi^a\Phi^a = \text{const}$, which forces the field to lie in an orbit of the gauge group. This constraint can be regarded as the effect of a Higgs potential in the strong coupling limit. A similar picture can be applied also to gravity, but with an important difference. In the usual models considered in particle physics, the constraints are holonomic and therefore the orbits of the gauge group are lower dimensional submanifolds of the space of the linear Higgs fields. On the contrary in gravity the constraints are of the anholonomic type and therefore the orbits corresponding to the nonlinear bosons are open subsets in certain tensor spaces. This geometrical fact means in more physical terms that while in particle physics the linear theory has more degrees of freedom than the nonlinear one, in the case of gravity the linear (unconstrained) theory has the same number of degrees of freedom of the nonlinear (constrained) theory.

Thus, if we were able to construct a theory of gravity in which the soldering form and the metric were not constrained a priori to be nondegenerate, then without having to introduce new degrees of freedom this theory would be more akin to a Higgs model than to a nonlinear sigma model [9]. Its quantum properties would presumably be improved and even if was not to be regarded as a fundamental theory, it would probably have the same status of the standard model of Particle physics. As an additional bonus, a theory of this type would provide a natural framework for the unification of gravity with the other interactions [10,6].

Our aim in this paper is to discuss the construction and quantization of such a "Higgs-like" or "unconstrained" theory of gravity, and its relation to the usual theory of General Relativity. In order to simplify the discussion we will assume that the internal metric is nondegenerate and choose the $GL(4)$ gauge so that it is equal to the Minkowski metric η_{ab}. Furthermore we will also assume that the $GL(4)$ connection is metric, i.e. reduces to an $O(1,3)$ connection. Thus we will effectively work within the framework of the vierbein formulation, but we will not assume a priori that the vierbein is nondegenerate, nor that torsion is zero.

In the construction of our model we try to follow as much as possible the example of the Higgs model as used in elementary particle and condensed matter physics. In doing so we encounter two different but related difficulties. First of all if the metric is allowed to become degenerate, then it is impossible to define its inverse, which is needed in the Lagrangian for contracting covariant indices (e.g. on derivatives). The other difficulty has to do with the construction of the potential, which is needed to guarantee that the vacuum expectation value of the Higgs fields (here the soldering form) is not zero. One can easily convince himself that it is impossible to write a nontrivial potential for the vierbein. For example, suppose we try to write down a term containing two vierbeins, no derivatives and invariant under coordinate and local Lorentz transformations. The only possibility is $g^{\mu\nu}\theta^a{}_\mu\theta^b{}_\nu\eta_{ab}$, where $g_{\mu\nu}$ is the space-time metric. If θ was unrelated to g, this would be a true mass-term for θ. However, in gravity one has the relation $g_{\mu\nu} = \theta^a{}_\mu\theta^b{}_\nu\eta_{ab}$ and so the term written above is not really quadratic: it is independent of θ and equal to 4. For the same reason one cannot write potential terms of higher order.

The origin of both these difficulties can be traced to the double role which is played in the gravitational Lagrangian by the metric (or vierbein). In any field theory one needs a metric to contract indices in the Lagrangian and to define the volume element; in this role the metric provides the geometrical standard according to which lengths and angles are measured. In addition in the theory of gravity the metric also plays the role of dynamical variable. This duality is the source of the beautiful geometrical interpretation of the classical theory, where the two roles coexist peacefully. However, it is also at the root of most difficulties, both conceptual and practical, which are encountered in the quantization of gravity. For example, the nonpolynomiality of gravitational Lagrangians is due to the fact that covariant indices must be contracted with the contravariant (inverse) metric. Also, the fact that the theory must contain a dimensionful coupling constant can be traced to the fact that geometrical and field theoretic arguments lead to different dimensions for the metric.

We overcome the two difficulties mentioned above by considering a mean-field quantum theory of gravity in which the two roles of the metric are kept separate: we assume that lengths and angles are not be measured with the dynamical, fluctuating metric, but rather with its vacuum expectation value (which we will refer to as "mean value"), assumed nondegenerate. On the other hand, the dynamical metric (or vierbein) can fluctuate without constraints and evolves on the background provided by its own mean value.

We modify the gravitational action by replacing the inverse metric $g^{\mu\nu}$ by the inverse mean metric $\bar{g}^{\mu\nu}$. Since the mean metric is assumed nondegenerate, the first difficulty does not arise anymore. The second difficulty is also avoided because one can now write terms like $\bar{g}^{\mu\nu}\theta^a{}_\mu\theta^b{}_\nu\eta_{ab}$, which are part of a genuine potential for the vierbein. The action will then look like a Higgs model action in a fixed background metric. One can use this action to perform quantum calculations in which the mean metric is kept fixed. Among other things one can compute the vacuum expectation value of the composite operator $g_{\mu\nu} = \theta^a{}_\mu\theta^b{}_\nu\eta_{ab}$. The whole scheme is self-consistent if one finds that this vacuum expectation value is equal (up to a dimensionful multiplicative constant ℓ^{-2}) to the mean metric that one had postulated in the beginning.

For technical reasons it is easier to compute the vacuum expectation value of θ rather than that of g. In this case self-consistency means that the vacuum expectation value of θ is equal (up to a dimensionful multiplicative constant ℓ^{-1}) to the mean vierbein $\bar{\theta}$, where $\bar{g}_{\mu\nu} = \bar{\theta}^a{}_\mu\bar{\theta}^b{}_\nu\eta_{ab}$. Note that since $\langle g_{\mu\nu}\rangle \neq \langle\theta^a{}_\mu\rangle\langle\theta^b{}_\nu\rangle\eta_{ab}$, the two procedures will lead to different values for the fundamental length ℓ, although qualitatively the results should be the same.

In Section 2 we will introduce a particularly simple action which incorporates the ideas illustrated above and discuss in more detail the general outline of the mean field approach.

In Section 3 we discuss the case in which the mean field corresponds to the flat Minkowski metric. In this particular case one can use Fourier analysis and the computations follow the familiar pattern from Elementary Particle models. We are led to an effective potential for the classical vierbein which is of the Coleman-Weinberg form. The minima of this potential occur at multiples of the unit matrix, thus ensuring selfconsistency.

In Section 4 we begin to take into account curvature effects by considering a de Sitter mean field. In this case one can still compute the one-loop effective action exactly. A closed form for the minimum can be given in the limit of large de Sitter radius, and in this approximation self-consistency can be explicitly checked.

In Section 5 we consider a generic mean field $\bar{g}$, and evaluate the first three terms in the long wavelength, low momentum expansion of the effective action. Evaluating this action at its minimum yields an effective action for the mean field which contains the Einstein term. Thus, at large distances General Relativity is recovered as an induced effect.

Finally, Section 6 contains further remarks and conclusions.

2. A SIMPLE MODEL

In order to make the previous discussion more concrete, we will illustrate the mean field approach to quantum gravity by discussing a particular model. We work within the context of the vierbein formulation of gravity, taking as independent dynamical variables the vierbein $\theta^a{}_\mu$ and an $O(1,3)$ gauge field $A_\mu{}^a{}_b$ (here $a, b = 0,1,2,3$ are internal indices and μ, $\nu = 0,1,2,3$ are spacetime indices). The spacetime metric is given by

$$g_{\mu\nu} = \theta^a{}_\mu \theta^b{}_\nu \eta_{ab} , \tag{2.1}$$

where $\eta_{ab} = \mathrm{diag}(-1,1,1,1)$. The analogy with the Higgs model of elementary particle physics suggests an action quadratic in the curvature of the $O(1,3)$ gauge field, $F_{\mu\nu}{}^a{}_b = \partial_\mu A_\nu{}^a{}_b - \partial_\nu A_\mu{}^a{}_b + g(A_\mu{}^a{}_c A_\nu{}^c{}_b - A_\nu{}^a{}_c A_\mu{}^c{}_b)$ and in the covariant derivative of the order parameter, $\nabla_\mu \theta^a{}_\nu = \partial_\mu \theta^a{}_\nu + g A_\mu{}^a{}_b \theta^b{}_\nu - \Gamma_\mu{}^\lambda{}_\nu \theta^a{}_\lambda$ (g is the gauge coupling constant and Γ are the Christoffel symbols of the composite metric (2.1)).

The simplest such action has the form

$$S(\theta, A) = \int d^4x \, \sqrt{|\det g|} \left[-\frac{1}{4} g^{\mu\rho} g^{\nu\sigma} \eta_{ac} \eta^{bd} F_{\mu\nu}{}^a{}_b F_{\rho\sigma}{}^c{}_d - \frac{1}{2} g^{\mu\rho} g^{\nu\sigma} \eta_{ab} \nabla_\mu \theta^a{}_\nu \nabla_\rho \theta^b{}_\sigma \right] . \tag{2.2}$$

It is manifestly invariant under local Lorentz and general coordinate transformations. A more general action of this type would contain several other terms in which the indices are contracted in different ways, each term weighted with a different coefficient. For the purposes of this paper it will be sufficient to consider the particular case (2.2).

As discussed in the Introduction, the vierbein has two roles in the theory: it defines the geometry of spacetime and at the same time it is a dynamical variable. One can identify occurrences of the vierbein in the action where it plays the role of geometrical standard, and others where it plays the role of dynamical field. If one compares (2.2) with the action of the Higgs model, the only place where θ plays the role of dynamical variable is under the covariant derivative. The mean field approach consists in replacing θ by a "mean vierbein" $\bar\theta$ in all other places. Due to the particular form of the action, this is equivalent to replacing everywhere the composite metric $g_{\mu\nu}$ by $\bar g_{\mu\nu} = \bar\theta^a{}_\mu \bar\theta^b{}_\nu \eta_{ab}$.

As already mentioned, at this stage it becomes also possible to add to the action a potential term. In fact we will see later that to ensure renormalizability one has to consider terms containing arbitrary powers of the curvature tensor of $\bar g$ and up to four powers of the field θ. Since we will restrict ourselves to one-loop calculations, it will be sufficient to consider terms linear in curvature and quadratic in θ, and terms quadratic in curvature (which we do not write). Thus our starting action will be of the form

$$\begin{aligned} \bar S(\theta, A; \bar g) = \int d^4x \, \sqrt{|\det \bar g|} \Big[& -\frac{1}{4} \bar g^{\mu\rho} \bar g^{\nu\sigma} \eta_{ac} \eta^{bd} F_{\mu\nu}{}^a{}_b F_{\rho\sigma}{}^c{}_d - \frac{1}{2} \bar g^{\mu\rho} \bar g^{\nu\sigma} \eta_{ab} \bar\nabla_\mu \theta^a{}_\nu \bar\nabla_\rho \theta^b{}_\sigma \\ & - \frac{1}{2}(m^2 + \xi \bar R) \, \mathrm{tr} X - \frac{\lambda_1}{4} (\mathrm{tr} X)^2 - \frac{\lambda_2}{4} \mathrm{tr}(X^2) \Big] , \end{aligned} \tag{2.3}$$

where $X^a{}_b = \bar g^{\mu\nu} \theta^a{}_\mu \theta^c{}_\nu \eta_{cb}$, $\bar R$ is the scalar curvature of $\bar g$, $\bar\nabla$ denotes the covariant derivative constructed with A and the Christoffel symbols of $\bar g$ and m^2, ξ, λ_1, λ_2 are coupling constants. We shall refer to the second line in (2.3), with opposite sign, as to the tree-level potential $V^{(0)}$.

The step from the action (2.2) to the action (2.3) is not entirely unambiguous. The second term in (2.2) can be rewritten in terms of the torsion $\Theta_\mu{}^a{}_\nu = \nabla_\mu \theta^a{}_\nu - \nabla_\nu \theta^a{}_\mu$, and using the identity $\nabla_\mu \theta^a{}_\nu = \frac{1}{2}(\Theta_\mu{}^a{}_\nu - g^{\lambda\rho}\eta_{bc}\,\theta^a{}_\lambda\,\theta^b{}_\nu \Theta_\mu{}^c{}_\rho - g^{\lambda\rho}\eta_{bc}\,\theta^a{}_\lambda\,\theta^b{}_\mu \Theta_\nu{}^c{}_\rho)$ one can rewrite

$$-\frac{1}{2}\,g^{\mu\rho}g^{\nu\sigma}\eta_{ab}\nabla_\mu\theta^a{}_\nu\nabla_\rho\theta^b{}_\sigma = -\frac{3}{8}g^{\mu\rho}g^{\nu\sigma}\eta_{ab}\Theta_\mu{}^a{}_\nu\Theta_\rho{}^b{}_\sigma - \frac{1}{4}g^{\mu\rho}\theta^{-1}{}_a{}^\sigma\theta^{-1}{}_b{}^\nu\Theta_\mu{}^a{}_\nu\Theta_\rho{}^b{}_\sigma\,.$$

Had we started from the action S written in this alternative way, we would have arrived at an action $\bar{S}$ with a different kinetic term for θ. One could eliminate this ambiguity by restricting the possible forms of the kinetic term for θ. Anyway, the preceding discussion is meant as a motivation for, not as a derivation of the action $\bar{S}$. Our main reason for chosing the action (2.3) is that it leads to a simple form of the propagator.

Taking the coordinates to have dimensions of length and the "geometric" metric $\bar{g}$ to be dimensionless, the dynamical fields A and θ have canonical dimension of inverse length. Thus m^2 has the dimension of squared mass and the coupling constants λ_1, λ_2 and ξ are dimensionless, as usual. Note that if we assumed the composite metric (2.1) to be dimensionless, as required by geometric considerations, then for the field θ to have canonical dimension one would have to introduce in (2.1) a dimensionful constant ℓ. This is the approach that was adopted in [6,11]. Here we will let the composite metric (2.1) have dimension of mass squared, as required by the canonical dimension of θ, and $\bar{g}$ be dimensionless. The constant ℓ will then reappear in the self-consistency conditions which relate g to $\bar{g}$ (or θ to $\bar{\theta}$).

We emphasize that nothing goes wrong in the action $\bar{S}$ when the dynamical field $\theta^a{}_\mu$ becomes degenerate or even becomes identically zero. So in quantizing the theory with action (2.3) one need not worry at all about this aspect and one can functionally integrate over θ without constraints. Unlike the original action S, $\bar{S}$ is polynomial and contains interaction terms which are at most quartic in the dynamical fields; it has exactly the same type of interactions of the usual Higgs model. Thus the theory defined by (2.3) is power-counting renormalizable in flat space.

Since in the action $\bar{S}$ the metric $\bar{g}$ has to be treated as a fixed background, general coordinate invariance is lost. The choice of the mean vierbein $\bar{\theta}$ would also break local Lorentz invariance, but since $\bar{\theta}$ appears only through the combination $\bar{g}_{\mu\nu} = \bar{\theta}^a{}_\mu\bar{\theta}^b{}_\nu\eta_{ab}$, the action (2.3) is invariant under local Lorentz transformations. Furthermore, if we allow the transformations to act also on the background $\bar{g}$, the action (2.3) has the same invariances of the action (2.2).

We will evaluate the one-loop effective potential for $\theta^a{}_\mu$ using the saddle point approximation, treating $\bar{g}$ as a fixed background. As is well known this reduces to the calculation of a functional determinant. We first expand $\bar{S}$ up to second order around a classical solution $A_{(\mathrm{cl})}$, $\theta_{(\mathrm{cl})}$ of the field equations. In the expansion of $\bar{S}$ terms linear in the fluctuations are then absent. Defining

$$\theta^a{}_\mu = \theta_{(\mathrm{cl})}{}^a{}_\mu + \varphi^a{}_\mu\,, \tag{2.4a}$$

$$A_\mu{}^a{}_b = A_{(\mathrm{cl})\mu}{}^a{}_b + \omega_\mu{}^a{}_b\,, \tag{2.4b}$$

the quadratic action has the form

$$\bar{S}^{(2)}(\varphi,\omega;\theta_{(\mathrm{cl})},A_{(\mathrm{cl})};\bar{g}) = \frac{1}{2}\int d^4x\sqrt{|\det\bar{g}|}\ \mathrm{tr}\left[(\omega\ \varphi)\begin{pmatrix}\mathcal{O}_{[\omega\omega]} & \mathcal{O}_{[\omega\varphi]}\\ \mathcal{O}_{[\varphi\omega]} & \mathcal{O}_{[\varphi\varphi]}\end{pmatrix}\begin{pmatrix}\omega\\ \varphi\end{pmatrix}\right] \tag{2.5a}$$

$$= \frac{1}{2}\int d^4x\sqrt{|\det\bar{g}|}\ \Big[\omega_\mu{}^a{}_b\,\mathcal{O}_{[\omega\omega]}{}^\mu{}_a{}^{b\nu}{}_c{}^d\,\omega_\nu{}^c{}_d + 2\,\varphi^a{}_\mu\,\mathcal{O}_{[\varphi\omega]a}{}^{\mu\nu}{}_c{}^d\,\omega_\nu{}^c{}_d + \varphi^a{}_\mu\,\mathcal{O}_{[\varphi\varphi]a}{}^\mu{}_b{}^\nu\,\varphi^b{}_\nu\Big]\,. \tag{2.5b}$$

This linearized action is invariant under the linearized gauge transformations: the fields

$$A_\mu{}^a{}_b = \frac{1}{g}\bar{\nabla}_\mu\epsilon^a{}_b\,, \qquad \varphi^a{}_\mu = -\epsilon^a{}_b\theta_{(\mathrm{cl})}{}^b{}_\mu \tag{2.6}$$

are null vectors for the operator $\mathcal{O}$ which is the block matrix operator appearing in (2.5a). We therefore have to fix the gauge. We choose the t'Hooft gauge and add to the linearized action the gauge-fixing term

$$-\frac{1}{2\alpha}\int d^4x\ \sqrt{|\det\bar{g}|}\ \Big[\bar{g}^{\mu\nu}(\bar{\nabla}_\mu\omega_\nu{}^a{}_b + \alpha\, g\,\eta_{cb}\,\theta_{(\mathrm{cl})}{}^c{}_\mu\,\varphi^a{}_\nu)\Big]^2\,. \tag{2.7}$$

Collecting all terms, the operators governing the dynamics of small fluctuations are

$$\mathcal{O}_{[\omega\omega]\mu ab}{}^{\nu cd} = \delta^{[c}_{[a}\,\delta^{d]}_{b]}\ \Big[\delta^\nu_\mu\,\bar{\nabla}_\lambda\bar{\nabla}^\lambda - \Big(1-\frac{1}{\alpha}\Big)\bar{\nabla}_\mu\bar{\nabla}^\nu - \bar{R}_\mu{}^\nu\Big] + 4g\,\delta^{[c}_{[a}\,F_{(\mathrm{cl})\mu}{}^\nu{}_{b]}{}^{d]} - \delta^\nu_\mu\ M_{(\mathrm{cl})ab}{}^{cd}\,, \tag{2.8a}$$

$$\mathcal{O}_{[\varphi\omega]a\mu}{}^{\nu cd} = 2g\,\delta^{[c}_a\,\bar{\nabla}^\nu\theta_{(\mathrm{cl})}{}^{d]}{}_\mu\,, \tag{2.8b}$$

$$\mathcal{O}_{[\varphi\varphi]a\mu}{}^{b\nu} = \delta^b_a\delta^\nu_\mu\ \big[\bar{\nabla}_\lambda\bar{\nabla}^\lambda - (m^2+\xi\,\bar{R})\big] - N_{(\mathrm{cl})a\mu}{}^{b\nu}\,, \tag{2.8c}$$

where

$$M_{(\mathrm{cl})ab}{}^{cd} = g^2\,\delta^{[c}_{[a}\,X_{(\mathrm{cl})b]}{}^{d]} = g^2\,\delta^{[c}_{[a}\,\theta_{(\mathrm{cl})b]}{}^\mu\theta_{(\mathrm{cl})}{}^{d]}{}_\mu\,, \tag{2.9a}$$

$$N_{(\mathrm{cl})a\mu}{}^{b\nu} = \alpha\,g^2\delta^b_a\,X_{(\mathrm{cl})\mu}{}^\nu + \lambda_1\,\big[X_{(\mathrm{cl})c}{}^c\ \delta^b_a\delta^\nu_\mu + 2\,\theta_{(\mathrm{cl})a\mu}\,\theta_{(\mathrm{cl})}{}^{b\nu}\big] + \lambda_2\,\big[X_{(\mathrm{cl})a}{}^b\ \delta^\nu_\mu + \delta^b_a\ X_{(\mathrm{cl})\mu}{}^\nu + \theta_{(\mathrm{cl})a}{}^\nu\theta_{(\mathrm{cl})}{}^b{}_\mu\big]\,. \tag{2.9b}$$

Here and in the following indices are raised and lowered with η_{ab} and $\bar{g}_{\mu\nu}$ and transformed from latin to greek by means of $\bar{\theta}$. The ghost operator is

$$\mathcal{O}_{[\mathrm{gh}]ab}{}^{cd} = \delta^{[c}_{[a}\,\delta^{d]}_{b]}\ \bar{\nabla}_\lambda\bar{\nabla}^\lambda - \alpha\,M_{(\mathrm{cl})ab}{}^{cd}\,. \tag{2.10}$$

In order to compute the functional determinants, one has to analytically continue the differential operators to the Euclidean sector. This amounts simply to changing the overall sign of the operators (2.8) and (2.10). The one-loop effective action is then formally given by

$$\Gamma^{(1)}(\theta_{(\mathrm{cl})}, A_{(\mathrm{cl})}; \bar{g}) = \frac{1}{2} \ln \det \mathcal{O} - \ln \det \mathcal{O}_{[\mathrm{gh}]} \ . \tag{2.11}$$

The explicit evaluation of these determinants requires a regularization due to the presence of ultraviolet divergences. In the subsequent three sections we will use a simple cutoff, zeta function and heat kernel regularization for the cases when the mean field is flat space, de Sitter space and a generic space, respectively. These methods will be found to give entirely consistent results.

If the regularization procedure respects general coordinate invariance, as is the case with the method we adopt in Section 5, the effective action (2.11) will have the same invariance properties of the classical action $\bar{S}$. In particular, it will be invariant under coordinate transformations when all fields, including $\bar{g}$, are transformed.

The vacuum expectation values $\langle\theta\rangle$ and $\langle A\rangle$ are obtained by minimizing the total effective action $\Gamma = \bar{S} + \Gamma^{(1)}$ with respect to $\theta_{(\mathrm{cl})}$ and $A_{(\mathrm{cl})}$. These minima depend upon $\bar{g}$. Since we want to interpret the background vierbein $\bar{\theta}$ as the vacuum expectation value of θ, the theory will be self-consistent if $\langle\theta\rangle = \ell^{-1}\bar{\theta}$, where ℓ is a constant with the dimension of length. In the next Sections we shall see that self-consistency can be achieved in the case of a flat or de Sitter background, and, in the long wavelength limit, for a generic background.

3. FLAT SPACE

By the equivalence principle, any macroscopic metric can be approximated in sufficiently small regions by a flat metric. On the other hand, quantum gravity is supposed to supersede the classical theory precisely at short distances. Thus it is perfectly appropriate that quantum gravity should begin by explaining flat space. Note that in view of the analogy with the Higgs model, the Minkowski metric is already a nontrivial background: it is the same as having a constant nonzero Higgs field.

We take then $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ and $\bar{\theta}^a{}_\mu = \delta^a_\mu$. For the classical fields we choose $A_{(\mathrm{cl})} = 0$ and $\theta_{(\mathrm{cl})} = \mathrm{const}$. The operators governing the dynamics of small fluctuations become, in momentum space,

$$\mathcal{O}_{[\omega\omega]\mu ab}{}^{\nu cd} = \delta^{[c}_{[a}\delta^{d]}_{b]} \left[-\delta^\nu_\mu \, k_\lambda k^\lambda + \left(1 - \frac{1}{\alpha}\right) k_\mu k^\nu \right] - \delta^\nu_\mu \, M_{(\mathrm{cl})ab}{}^{cd} \ , \tag{3.1a}$$

$$\mathcal{O}_{[\varphi\omega]a\mu}{}^{\nu cd} = 0 \ , \tag{3.1b}$$

$$\mathcal{O}_{[\varphi\varphi]a\mu}{}^{b\nu} = \delta^b_a \delta^\nu_\mu \, \left(-k_\lambda k^\lambda - m^2\right) - N_{(\mathrm{cl})a\mu}{}^{b\nu} \ , \tag{3.1c}$$

$$\mathcal{O}_{[\mathrm{gh}]ab}{}^{cd} = -\delta^{[c}_{[a}\delta^{d]}_{b]} \, k_\lambda k^\lambda - \alpha \, M_{(\mathrm{cl})ab}{}^{cd} \ . \tag{3.1d}$$

Due to (3.1b), the one-loop effective action reduces to

$$\Gamma^{(1)} = \frac{1}{2} \ln \det \mathcal{O}_{[\omega\omega]} + \frac{1}{2} \ln \det \mathcal{O}_{[\varphi\varphi]} - \ln \det \mathcal{O}_{[\mathrm{gh}]} \ . \tag{3.2}$$

Since the classical fields are constant, the effective action is the spacetime integral of the effective potential V. For every k, the matrices in (3.1) can be explicitly diagonalized.

This is achieved by first bringing $\theta_{(\mathrm{cl})}$ to diagonal form by means of independent global Lorentz transformations on the internal and spacetime indices. Up to an irrelevant multiplicative factor, the eigenvalues of every operator $\mathcal{O}$ are of the form $-(k^2+\lambda_i)$. Then, in the Euclidean regime, $\ln\det\mathcal{O}=\frac{1}{(2\pi)^4}\sum_i\int^\Lambda d^4k\ln(k^2+\lambda_i)$, where Λ is an ultraviolet cutoff. The integral over k can be performed explicitly and one finds, for Λ large,

$$
\begin{aligned}
V^{(1)} = & \frac{3}{64\pi^2}\left[2\Lambda^2\mathrm{tr}M_{(\mathrm{cl})}+\mathrm{tr}\left({M_{(\mathrm{cl})}}^2\left(\ln\frac{M_{(\mathrm{cl})}}{\Lambda^2}-\frac{1}{2}\right)\right)\right] \\
& -\frac{1}{64\pi^2}\left[2\alpha\,\Lambda^2\mathrm{tr}M_{(\mathrm{cl})}+\alpha^2\,\mathrm{tr}\left({M_{(\mathrm{cl})}}^2\left(\ln\frac{\alpha M_{(\mathrm{cl})}}{\Lambda^2}-\frac{1}{2}\right)\right)\right] \\
& +\frac{1}{64\pi^2}\left[2\Lambda^2\,\mathrm{tr}\tilde{N}_{(\mathrm{cl})}+\mathrm{tr}\left({\tilde{N}_{(\mathrm{cl})}}^2\left(\ln\frac{\tilde{N}_{(\mathrm{cl})}}{\Lambda^2}-\frac{1}{2}\right)\right)\right],
\end{aligned}
\tag{3.3}
$$

where $\tilde{N}_{(\mathrm{cl})}=m^2+N_{(\mathrm{cl})}$, the traces are over double indices and an infinite, field-independent constant has been dropped. In (3.3) the first line is the contribution of the transverse components of ω, the second line comes from the longitudinal components of ω and from the ghosts, the last line is the φ contribution. This effective potential contains divergences proportional to $\mathrm{tr}M_{(\mathrm{cl})}$, $\mathrm{tr}({M_{(\mathrm{cl})}}^2)$ and $\mathrm{tr}\tilde{N}_{(\mathrm{cl})}$, $\mathrm{tr}({\tilde{N}_{(\mathrm{cl})}}^2)$. These terms are proportional to $\mathrm{tr}X_{(\mathrm{cl})}$ and to suitable combinations of $(\mathrm{tr}X_{(\mathrm{cl})})^2$ and $\mathrm{tr}({X_{(\mathrm{cl})}}^2)$, and thus of the same form of the potential terms in the starting action. The infinities in (3.3) can then be absorbed in a renormalization of the coupling constants of the tree-level potential $V^{(0)}$. In fact, we see that the addition of the tree-level potential was necessary to ensure renormalizability of the theory. This is very similar to what happens in scalar electrodynamics, where renormalizability of the meson-photon interaction demands the presence of a quartic self-interaction of the scalars.

With a suitable choice of the renormalization scale μ the total renormalized effective potential takes the form

$$
\begin{aligned}
V=V^{(0)}+V^{(1)}= & \frac{1}{2}m^2\mathrm{tr}X_{(\mathrm{cl})}+\frac{\lambda_1}{4}(\mathrm{tr}X_{(\mathrm{cl})})^2+\frac{\lambda_2}{4}\mathrm{tr}({X_{(\mathrm{cl})}}^2) \\
& +\frac{1}{64\pi^2}\left[(3-\alpha^2)\mathrm{tr}\left({M_{(\mathrm{cl})}}^2\left(\ln\frac{M_{(\mathrm{cl})}}{\mu^2}-\frac{3}{2}\right)\right)-\alpha^2\ln\alpha\;\mathrm{tr}{M_{(\mathrm{cl})}}^2\right] \\
& +\frac{1}{64\pi^2}\,\mathrm{tr}\left({\tilde{N}_{(\mathrm{cl})}}^2\left(\ln\frac{\tilde{N}_{(\mathrm{cl})}}{\mu^2}-\frac{3}{2}\right)\right),
\end{aligned}
\tag{3.4}
$$

where the coupling constants are the renormalized ones. Notice that for $\alpha=0$, the t'Hooft gauge reduces to the Landau gauge; in that case the potential (3.4) reduces, modulo a finite redefinition of the renormalization scale μ, to that already computed in [11].

The minimum of the effective potential (3.4) cannot easily be written in analytic form. To simplify calculations we follow [12] in setting $m^2=0$, and assume that the coupling constants λ_1 and λ_2 are of order g^4. Then, in the last term of (3.4) the dependence on λ_1 and λ_2 can be neglected. For a physically acceptable range of the parameters λ_1, λ_2 and g, the absolute minimum of the effective potential (3.4) occurs for

$$
{\theta_{(\mathrm{cl})}}^a{}_\mu=\frac{\mu}{g}\exp\left[\frac{1}{2}-\frac{16\pi^2}{9+5\alpha^2}\left(\frac{4\lambda_1+\lambda_2}{g^4}+\frac{5}{32\pi^2}\alpha^2\ln\alpha\right)\right]\,\delta^a_\mu\,,
\tag{3.5}
$$

and global Lorentz transformations thereof. Thus, the quantum dynamics of the model drives the vacuum expectation value of the vierbein to be nondegenerate. This is a gravitational analog of the so called Coleman-Weinberg mechanism [12]. In fact, the vacuum expectation value of θ is proportional to the mean vierbein $\bar{\theta}$, with proportionality constant ℓ^{-1} of the order of the renormalization mass μ. We will in Section 5 that ℓ^2 can be identified with Newton's constant.

The effective potential (3.4) and its minimum (3.5) depend explicitly on the gauge parameter α. This is a common feature of effective potential calculations [13]. In order to obtain a gauge-independent result one could compute the Vilkovisky-de Witt effective action for our problem [14]. We do not expect this to yield qualitatively different results. In any event, notice that the existence of a non-degenerate absolute minimum is guaranteed for any finite value of α (an infinite value for α is in any case excluded because it would give rise to an effective potential unbounded from below). Flat space is then a self-consistent solution of the theory in any gauge.

4. DE SITTER SPACE

In this Section we start taking into account effects due to the curvature of the mean metric. We choose $\bar{g}$ to be the de Sitter metric, which enables us to compute the one-loop effective action exactly. The results we obtain will also be used as an independent check of more general calculations in the next section. One loop effective actions in de Sitter space have been computed before in a variety of contexts [15-20]. Since calculations are performed in the Euclidean sector, de Sitter space is just a four-dimensional sphere. We will not need to choose a coordinate system to write down the metric explicitly. The only properties that we will need are

$$\bar{R}_{\mu\nu\rho\sigma} = \frac{1}{r^2}\left(\bar{g}_{\mu\rho}\bar{g}_{\nu\sigma} - \bar{g}_{\mu\sigma}\bar{g}_{\nu\rho}\right) , \qquad \bar{R}_{\mu\nu} = \frac{3}{r^2}\,\bar{g}_{\mu\nu} , \qquad \bar{R} = \frac{12}{r^2} , \tag{4.1}$$

where r is the radius of the de Sitter space (related to the cosmological constant λ_0 by $r^2 = 3/\lambda_0$). For the classical fields we take

$$\theta_{(\mathrm{cl})}{}^a{}_\mu = \rho\,\bar{\theta}^a{}_\mu , \qquad gA_{(\mathrm{cl})\lambda}{}^a{}_b = \bar{\theta}^a{}_\mu\,\bar{\Gamma}_\lambda{}^\mu{}_\nu\,\bar{\theta}^{-1}{}_b{}^\nu + \bar{\theta}^a{}_\mu\partial_\lambda\,\bar{\theta}^{-1}{}_b{}^\mu , \tag{4.2}$$

with ρ a constant with the dimensions of mass. We thus reduce the freedom in the classical vierbein to a single constant ρ and our task in this Section will be to show that the quantum dynamics of the theory requires ρ to be nonzero. The curvature of $A_{(\mathrm{cl})}$ is related now to the Riemann tensor of the mean metric by $gF_{(\mathrm{cl})\mu\nu}{}^a{}_b = \bar{\theta}^a{}_\rho\,\bar{R}_{\mu\nu}{}^\rho{}_\sigma\,\bar{\theta}^{-1}{}_b{}^\sigma$. Using this, one can verify that (4.2) solve the classical equations of motion. We will now compute the one-loop effective action, which will depend on the proportionality constant ρ and, parametrically, on the radius r.

With the assumptions above, the differential operators (2.8) and (2.10) simplify; in particular

$$M_{(\mathrm{cl})ab}{}^{cd} = g^2\rho^2\,\delta^{[c}_{[a}\delta^{d]}_{b]} , \tag{4.3a}$$

$$N_{(\mathrm{cl})a\mu}{}^{b\nu} = \left(\alpha g^2 + 2(2\lambda_1 + \lambda_2)\right)\rho^2\,\delta^b_a\delta^\nu_\mu + 2\lambda_1\,\rho^2\,\bar{\theta}_{a\mu}\bar{\theta}^{b\nu} + \lambda_2\rho^2\bar{\theta}_a{}^\nu\bar{\theta}^b{}_\mu . \tag{4.3b}$$

Also, $\mathcal{O}_{[\omega\varphi]} = 0$, so that $\Gamma^{(1)}$ is again given by (3.2).

In order to deal with the nonminimal term $\bar{\nabla}_\mu \bar{\nabla}^\nu$ in $\mathcal{O}_{[\omega\omega]}$, it is convenient to decompose ω in its transverse and longitudinal parts, satisfying respectively $\bar{\nabla}_\mu {\omega^T}^\mu{}_{ab} = 0$ and ${\omega^L}_{\mu ab} = \bar{\nabla}_\mu \epsilon_{ab}$. The operator $\mathcal{O}_{[\omega\omega]}$ maps transverse fields to transverse fields and longitudinal fields to longitudinal fields. Therefore, $\ln\det \mathcal{O}_{[\omega\omega]} = \ln\det^T \mathcal{O}_{[\omega\omega]} + \ln\det^L \mathcal{O}_{[\omega\omega]}$. Using the formula

$$\mathcal{O}_{[\omega\omega]\,\mu ab}{}^{\nu cd}\, \bar{\nabla}_\nu \epsilon_{cd} = \bar{\nabla}_\mu \left(\mathcal{O}'_{ab}{}^{cd}\, \epsilon_{cd} \right) , \tag{4.4}$$

where $\mathcal{O}'_{ab}{}^{cd} = \frac{1}{\alpha}\, \mathcal{O}_{[gh]ab}{}^{cd}$, we can rewrite: $\ln\det^L \mathcal{O}_{[\omega\omega]} = \ln\det \mathcal{O}'$. On the other hand when the operator $\mathcal{O}_{[\omega\omega]}$ acts on transverse fields the nonminimal term drops out.

The spectrum of the operator $\mathcal{O}_{[\omega\omega]}$ on transverse fields, and the spectra of $\mathcal{O}'$, $\mathcal{O}_{[\varphi\varphi]}$ and $\mathcal{O}_{[gh]}$ can be determined explicitly using group-theoretic arguments. We begin by decomposing the fields ω and φ in their irreducible components with respect to the group $O(4)$. Transforming all indices to latin by means of $\bar{\theta}$, we have

$$\omega_{abc} = \frac{2}{3}(t_{abc} - t_{acb}) + \frac{1}{3}(\eta_{ab}\, v_c - \eta_{ac}\, v_b) + \varepsilon_{abcd}\, w^d , \tag{4.5a}$$

$$\varphi_{ab} = \psi_{ab} + \chi_{ab} + \eta_{ab}\, \tau , \tag{4.5b}$$

where $t_{abc} = \frac{1}{2}(\omega_{abc} + \omega_{bac}) + \frac{1}{6}(\eta_{ac}\, \omega^d{}_{db} + \eta_{bc}\, \omega^d{}_{da}) - \frac{1}{3}\eta_{ab}\omega^d{}_{dc}$ carries the 16-dimensional representation $\left(\frac{3}{2},\frac{1}{2}\right) \oplus \left(\frac{1}{2},\frac{3}{2}\right)$, $v_a = \omega^b{}_{ba}$ and $w_a = \frac{1}{6}\varepsilon_{abcd}\, \omega^{bcd}$ carry the 4-dimensional representation $\left(\frac{1}{2},\frac{1}{2}\right)$, $\psi_{ab} = \frac{1}{2}(\varphi_{ab}+\varphi_{ba}) - \frac{1}{4}\eta_{ab}\, \varphi^c{}_c$ transforms according to the 9-dimensional representation $(1,1)$, $\chi_{ab} = \frac{1}{2}(\varphi_{ab} - \varphi_{ba})$ transforms according to the 6-dimensional representation $(1,0) \oplus (0,1)$ and $\tau = \frac{1}{4}\varphi^a{}_a$ transforms according to the 1-dimensional representation $(0,0)$ [21].

It can be shown that the transverse part of ω is given by the tensor t and the longitudinal parts v^L and w^L of the vectors v and w, while the longitudinal part of ω is given by the transverse parts of v and w. This is made plausible by observing that the 18 degrees of freedom of ω^T match the 16 of t plus one each for v^L and w^L; similarly ω^L has 6 degrees of freedom, matching the ones of v^T and w^T. A rigorous proof of this result is given in the Appendix. When acting on transverse fields the operator $\mathcal{O}_{[\omega\omega]}$ respects the $O(4)$ decomposition. Symbolically,

$$\omega^T \mathcal{O}_{[\omega\omega]} \omega^T = \frac{4}{3}\, t\, \mathcal{O}_{[tt]}\, t + \frac{2}{3}\, v^L\, \mathcal{O}_{[v^L v^L]}\, v^L + 6\, w^L\, \mathcal{O}_{[w^L w^L]}\, w^L , \tag{4.6}$$

where

$$\mathcal{O}_{[tt]} = -\bar{\nabla}_\lambda \bar{\nabla}^\lambda + \frac{5}{r^2} + z , \tag{4.7a}$$

$$\mathcal{O}_{[v^L v^L]} = \mathcal{O}_{[w^L w^L]} = -\bar{\nabla}_\lambda \bar{\nabla}^\lambda - \frac{1}{r^2} + z , \tag{4.7b}$$

with $z = g^2 \rho^2$ (tensor indices have been suppressed since these operators are multiples of the identity in the appropriate tensor space). Similarly one finds

$$\varphi\, \mathcal{O}_{[\varphi\varphi]}\, \varphi = \psi\, \mathcal{O}_{[\psi\psi]}\, \psi + \chi\, \mathcal{O}_{[\chi\chi]}\, \chi + 4\, \tau\, \mathcal{O}_{[\tau\tau]}\, \tau , \tag{4.8}$$

where

$$\mathcal{O}_{[\psi\psi]} = -\bar{\nabla}_\lambda \bar{\nabla}^\lambda + z_\psi\,, \qquad z_\psi = m^2 + \frac{12}{r^2}\xi + \rho^2\left(4\lambda_1 + 3\lambda_2 + g^2\alpha\right)\,, \tag{4.9a}$$

$$\mathcal{O}_{[\chi\chi]} = -\bar{\nabla}_\lambda \bar{\nabla}^\lambda + z_\chi\,, \qquad z_\chi = m^2 + \frac{12}{r^2}\xi + \rho^2\left(4\lambda_1 + \lambda_2 + g^2\alpha\right)\,, \tag{4.9b}$$

$$\mathcal{O}_{[\tau\tau]} = -\bar{\nabla}_\lambda \bar{\nabla}^\lambda + z_\tau\,, \qquad z_\tau = m^2 + \frac{12}{r^2}\xi + \rho^2\left(12\lambda_1 + 3\lambda_2 + g^2\alpha\right)\,. \tag{4.9c}$$

Finally, $\mathcal{O}_{[\mathrm{gh}]}$ and $\mathcal{O}'$ act on the antisymmetric tensor representation $(1,0)\oplus(0,1)$.

The spectra of all these operators can be determined using the method explained in [22]. The eigenvalues λ_n and the corresponding multiplicities d_n are given in the following table.

Eigenvalues and multiplicities			
operator	λ_n	d_n	
$\mathcal{O}_{[tt]}$	$r^{-2}(n^2+3n+2)+z$ $r^{-2}(n^2+3n-2)+z$	$\frac{5}{4}(n+4)(n-1)(2n+3)$ $n(n+3)(2n+3)$	$n\geq 2$ $n\geq 2$
$\mathcal{O}_{[v^L v^L]}$	$r^{-2}(n^2+3n-4)+z$	$\frac{1}{6}(n+1)(n+2)(2n+3)$	$n\geq 1$
$\mathcal{O}_{[\psi\psi]}$	$r^{-2}(n^2+3n-2)+z_\psi$ $r^{-2}(n^2+3n-6)+z_\psi$ $r^{-2}(n^2+3n-8)+z_\psi$	$\frac{5}{6}(n-1)(n+4)(2n+3)$ $\frac{1}{2}n(n+3)(2n+3)$ $\frac{1}{6}(n+1)(n+2)(2n+3)$	$n\geq 2$ $n\geq 2$ $n\geq 2$
$\mathcal{O}_{[\chi\chi]}$	$r^{-2}(n^2+3n-2)+z_\chi$	$n(n+3)(2n+3)$	$n\geq 1$
$\mathcal{O}_{[\tau\tau]}$	$r^{-2}(n^2+3n)+z_\tau$	$\frac{1}{6}(n+1)(n+2)(2n+3)$	$n\geq 0$
$\mathcal{O}_{[\mathrm{gh}]}$	$r^{-2}(n^2+3n-2)+\alpha\, z$	$n(n+3)(2n+3)$	$n\geq 1$

We note that for $\rho=0$ the operators $\mathcal{O}_{[v^L v^L]}$ and $\mathcal{O}_{[w^L w^L]}$ have five zero eigenvalues each. This is because for $\rho=0$ the linearized Lagrangian (2.5) is the one of pure Yang-Mills theory for the group $SO(4)=SU(2)\times SU(2)$, and the classical field $A_{(\mathrm{cl})}$ is the direct sum of an instanton and an anti-instanton. Thus, these are the familiar zero-modes due to the $O(5)$-invariance of the instanton background [15]. When $m^2=0$, $\xi=0$, the operator $\mathcal{O}_{[\tau\tau]}$ also has a zero eigenvalue for $\rho=0$.

The complete one-loop effective action is given by

$$\begin{aligned}\Gamma^{(1)} = {} & \frac{1}{2}\ln\det\mathcal{O}_{[tt]} + \frac{1}{2}\ln\det\mathcal{O}_{[v^L v^L]} + \frac{1}{2}\ln\det\mathcal{O}_{[w^L w^L]} + \frac{1}{2}\ln\det\mathcal{O}' \\ & + \frac{1}{2}\ln\det\mathcal{O}_{[\psi\psi]} + \frac{1}{2}\ln\det\mathcal{O}_{[\chi\chi]} + \frac{1}{2}\ln\det\mathcal{O}_{[\tau\tau]} - \ln\det\mathcal{O}_{[\mathrm{gh}]}\,.\end{aligned} \tag{4.10}$$

For each operator $\mathcal{O}$, we define the dimensionless zeta-function $\zeta(\mathcal{O},s) = \sum_n d_n (r^2\lambda_n)^{-s}$ [23]. Then, $\ln\left(\det \mathcal{O}/\mu^2\right) = \sum_n d_n \ln\left(\lambda_n/\mu^2\right) = -\zeta'(\mathcal{O},0) - \ln(\mu^2 r^2)\zeta(\mathcal{O},0)$, where the prime signifies derivative with respect to s and μ is the renormalization scale. These zeta functions can be evaluated exactly in terms of digamma functions [18,20]. The complete expression is complicated and not very revealing. Instead we will present the approximated form of $\Gamma^{(1)}$ for $r\rho$ large and $\alpha \neq 0$:

$$\begin{aligned}\Gamma^{(1)}(\rho;r) = {} & \frac{3}{4}r^4 z^2\left(\ln\frac{z}{\mu^2} - \frac{3}{2}\right) + 3r^2 z\left(\ln\frac{z}{\mu^2} - 1\right) + \frac{59}{10}\ln\frac{z}{\mu^2} \\ & + \frac{3}{8}r^4 z_\psi^2\left(\ln\frac{z_\psi}{\mu^2} - \frac{3}{2}\right) - \frac{3}{2}r^2 z_\psi\left(\ln\frac{z_\psi}{\mu^2} - 1\right) + \frac{9}{20}\ln\frac{z_\psi}{\mu^2} \\ & + \frac{1}{4}r^4 z_\chi^2\left(\ln\frac{z_\chi}{\mu^2} - \frac{3}{2}\right) - r^2 z_\chi\left(\ln\frac{z_\chi}{\mu^2} - 1\right) + \frac{19}{30}\ln\frac{z_\chi}{\mu^2} \\ & + \frac{1}{24}r^4 z_\tau^2\left(\ln\frac{z_\tau}{\mu^2} - \frac{3}{2}\right) - \frac{1}{6}r^2 z_\tau\left(\ln\frac{z_\tau}{\mu^2} - 1\right) + \frac{29}{180}\ln\frac{z_\tau}{\mu^2} \\ & - \frac{1}{4}r^4\alpha^2 z^2\left(\ln\frac{\alpha z}{\mu^2} - \frac{3}{2}\right) + r^2\alpha z\left(\ln\frac{\alpha z}{\mu^2} - 1\right) - \frac{19}{30}\ln\frac{\alpha z}{\mu^2}\,.\end{aligned} \tag{4.11}$$

We assume again $m^2 = 0$, $\xi = 0$ and $\lambda_1 \approx \lambda_2 \approx g^4$. The one-loop effective potential $V^{(1)}$ is defined by $\Gamma^{(1)}(\rho;r) = \Omega(r)V^{(1)}(\rho;r)$, where $\Omega(r) = \frac{8}{3}\pi^2 r^4$ is the volume of de Sitter space and ρ is constant. The effective potential is given to order g^4 by

$$\begin{aligned}V^{(1)} = \frac{1}{32\pi^2}\Bigg\{ & \left(9 + 5\alpha^2\right)z^2\left(\ln\frac{z}{\mu^2} - \frac{3}{2}\right) + \frac{5}{3}z^2\alpha^2\ln\alpha \\ & + \frac{4}{r^2}\left[\left(9 - 5\alpha - 18\frac{2\lambda_1 + \lambda_2}{g^2}\right)z\ln\frac{z}{\mu^2} - \left(9 - 5\alpha + \ln\alpha\left(5\alpha + 18\frac{2\lambda_1+\lambda_2}{g^2}\right)\right)z\right] \\ & + \frac{560}{3}\frac{1}{r^4}\ln\frac{z}{\mu^2}\Bigg\}\,;\end{aligned} \tag{4.12}$$

an additive ρ-independent costant has been dropped. Notice that in the flat space limit $r \to \infty$ only the first line remains. It coincides with the one-loop effective potential in flat space discussed in Section 3, when we set $\theta_{(\mathrm{cl})} = \rho\bar{\theta}$. Therefore the method of the zeta function and the method of the cutoff give the same result for the effective action in flat space.

The minimum of the total effective potential $V = V^{(0)} + V^{(1)}$ occurs, for any finite nonzero value of the gauge parameter α, at a non-zero value of z, whose explicit expression can not be given analytically. However, expanding around the flat-space minimum ${\theta_{(\mathrm{cl})}}^a{}_\mu = \rho_0\delta^a_\mu$ given in (3.5) and keeping only terms of lowest order in r^{-2}, one gets:

$$\begin{aligned}\rho^2 = \rho_0^2 - \frac{1}{r^2 g^2}\frac{2}{9 + 5\alpha^2}\Bigg[& \left(9 - 5\alpha - 18\frac{2\lambda_1+\lambda_2}{g^2}\right)\ln\left(\frac{g^2\rho_0^2}{\mu^2}\right) \\ & - 5\alpha\ln\alpha - 18(1 + \ln\alpha)\frac{2\lambda_1+\lambda_2}{g^2}\Bigg] + O\left(\frac{1}{r^4}\right)\,.\end{aligned} \tag{4.13}$$

For large μr the minimum of V is in the region for which the expression (4.12) can be trusted.

For small values of ρ the exact expression of the one-loop effective potential is found to be logarithmically divergent. This is not surprising since, as observed, the operators $\mathcal{O}_{[v^L v^L]}$ and $\mathcal{O}_{[w^L w^L]}$ have five zero-modes each at $\rho = 0$. These clearly give the dominant contribution to the effective potential, which then diverges like $\ln(\rho/\mu)$ for ρ close to zero. Of course this behaviour can not be trusted within the one-loop approximation. Indeed as briefly discussed in the following, by resumming an infinite number of loop contributions due to the zero-modes one gets a sensible (finite) result. Similar behaviour had been observed before [24]. Notice that (4.11) and (4.12) are also logarithimically divergent for z close to zero; however, these infinities are not significant since they occur for values of ρ for which those expressions are no longer valid.

In order to compute higher-loop contributions to the effective action, it is necessary to go beyond the quadratic approximation in the expansion of the action (2.3), taking into account also interaction terms in the fluctuating fields ω and φ. Since it is v^L and w^L that have zero modes, we shall concentrate on the quartic self-interactions $\frac{g^2}{27}(v^L{}_\mu v^{L\mu})^2$ and $3g^2(w^L{}_\mu w^{L\mu})^2$ and add them to the quadratic part of the action, symbolically given by the last two terms in (4.6). Other quartic interaction terms can be added, but the ones written above will be sufficient for our purposes.

We shall now compute, in the limit of small ρ, the contributions to the effective action given by graphs with an arbitrary number of loops generated by these new interaction terms. We use the method of the auxiliary field, as explained for example in [25]. We introduce two auxiliary fields ϕ_1 and ϕ_2 and rewrite the action for v^L and w^L as

$$\int d^4x \, \sqrt{|\det \bar{g}|}\Big[\frac{2}{3} v^L \, \tilde{\mathcal{O}}_{[v^L v^L]}\, v^L + 6\, w^L \, \tilde{\mathcal{O}}_{[w^L w^L]}\, w^L \\ - \frac{3}{g^2}(\phi_1 - g^2\rho^2)^2 - \frac{3}{g^2}(\phi_2 - g^2\rho^2)^2\Big] \, , \tag{4.14}$$

where $\tilde{\mathcal{O}}_{[v^L v^L]}$ and $\tilde{\mathcal{O}}_{[w^L w^L]}$ are given by (4.7b) in which z is replaced by ϕ_1 and ϕ_2, respectively. Then the one-loop effective action, parametrically depending on ϕ_1 and ϕ_2, is formally given by

$$\Gamma(\rho,\phi_1,\phi_2;r) = \frac{1}{2}\ln\det\tilde{\mathcal{O}}_{[v^L v^L]} + \frac{1}{2}\ln\det\tilde{\mathcal{O}}_{[w^L w^L]} \\ - \frac{3}{g^2}\int d^4x\, \sqrt{|\det\bar{g}|}\, \big[(\phi_1 - g^2\rho^2)^2 + (\phi_2 - g^2\rho^2)^2\big] \, . \tag{4.15}$$

As already observed, only zero-mode contributions need to be included in the computation of the determinants, so that one finds

$$\Gamma(\rho,\phi_1,\phi_2;r) = \frac{5}{2}\ln\frac{\phi_1}{\mu^2} - \frac{3\Omega}{g^2}(\phi_1 - g^2\rho^2)^2 + \frac{5}{2}\ln\frac{\phi_2}{\mu^2} - \frac{3\Omega}{g^2}(\phi_2 - g^2\rho^2)^2 \, , \tag{4.16}$$

where ϕ_1 and ϕ_2 are now constant. The dependence of Γ on the auxiliary variables ϕ_1 and ϕ_2 can be eliminated by using their corresponding equations of motion,

$$\phi_i = g^2\rho^2 + \frac{5g^2}{12\Omega}\frac{1}{\phi_i} \, , \qquad i = 1,2 \, . \tag{4.17}$$

Inserting the solutions of (4.17) back in (4.16) one obtains the expression of the effective action which includes the contributions of the zero-modes to all loop-orders. The first terms in the expansion for small ρ of the total effective potential can be easily determined

$$V = -\frac{15}{16\pi^2 r^4}\left[1 + \ln\left(\frac{32\pi^2\mu^4 r^4}{5g^2}\right)\right] + \sqrt{\frac{5}{2}\frac{3g}{\pi r^2}}\,\rho^2 - (3g^2 - 4\lambda_1 - \lambda_2)\,\rho^4 + \ldots \quad (4.18)$$

where again we have set $m^2 = 0$ and $\xi = 0$. The resummation of a certain class of graphs (the so called "daisy graphs") has thus eliminated from the effective potential the logarithmic divergence of the one-loop contribution, producing a regular power-law behaviour.

In conclusion we have found that for large radius of the background de Sitter metric $\bar{g}$, the effective potential has its absolute minimum for some nonzero value of ρ and hence de Sitter space gives a self-consistent solution of the theory. The constant ℓ appearing in the self-consistency condition $\theta_{(\mathrm{cl})} = \ell^{-1}\bar{\theta}$ is given just by $\ell = \rho^{-1}_{(\mathrm{min})}$. For reasonable values of the coupling constants the second term on the r.h.s. of (4.13) is negative, so ℓ^{-1} decreases for decreasing values of r (increasing curvature). We expect that, as in similar models [17,18], ρ will go to zero for some sufficiently small value of r. This is usually taken as a signal of a phase transition. In the present model it also signals a breakdown of selfconsistency. We then come the remarkable conclusion that for given values of the coupling constants, only a certain range of values of the radius may be permitted.

5. GRAVITY-INDUCED GRAVITY

The fact that the dynamical variables of our theory have the right tensorial structure is not enough to qualify it as a theory of gravity. This comes from the identification of $\langle g \rangle$, via $\bar{g}$, with the classical, macroscopic metric. Further indications come from studying the effective dynamics for $\bar{g}$. Such a dynamics can be obtained by evaluating the effective action $\Gamma(\theta_{(\mathrm{cl})}, A_{(\mathrm{cl})}; \bar{g})$ at its minimum values for $\theta_{(\mathrm{cl})}$ and $A_{(\mathrm{cl})}$. It is of course impossible to compute exactly the one-loop effective action for a generic $\bar{g}$, but one can study its behaviour at large distances. Note that this is really all that is needed, since $\bar{g}$ is supposed to represent the macroscopic metric.

In computing the Euclidean one-loop effective action (2.11) for an arbitrary $\bar{g}$ we will use the method of the heat kernel. Given an operator $\mathcal{O}$, acting on a space of tensors possibly carrying also internal indices, we define its determinant through the formula:

$$\ln\det\mathcal{O} = -\int_{\frac{1}{\Lambda^2}}^{\infty} ds\, s^{-1}\int d^4x\sqrt{|\det\bar{g}|}\;\mathrm{tr}K(x,x;s)\ , \quad (5.1)$$

where Λ is an ultraviolet cutoff and K is the heat kernel of the operator $\mathcal{O}$, satisfying $\frac{d}{ds}K + \mathcal{O}K = 0$ and tr means trace over both tensorial and internal indices. For small s, the trace of the heat kernel has the well-known asymptotic expansion

$$\int d^4x\sqrt{|\det\bar{g}|}\;\mathrm{tr}K(x,x;s) \approx B_0 s^{-2} + B_2 s^{-1} + B_4 + O(s)\ , \quad (5.2)$$

where $B_n = \int d^4x\sqrt{|\det \bar{g}|}\ \mathrm{tr}\, b_n(x)$. For an operator of the form $\mathcal{O} = -\bar{\nabla}_\lambda\bar{\nabla}^\lambda + Z$, one has [26]

$$b_0 = \frac{1}{(4\pi)^2}\mathbf{1} \tag{5.3a}$$

$$b_2 = \frac{1}{(4\pi)^2}\left(\frac{\bar{R}}{6}\mathbf{1} - Z\right) \tag{5.3b}$$

$$\begin{aligned} b_4 = \frac{1}{(4\pi)^2}\Bigg[&\left(\frac{1}{180}\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} - \frac{1}{180}\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} + \frac{1}{72}\bar{R}^2 + \frac{1}{30}\bar{\nabla}_\mu\bar{\nabla}^\mu\bar{R}\right)\mathbf{1} \\ &- \frac{1}{6}\bar{R}Z - \frac{1}{6}\bar{\nabla}_\mu\bar{\nabla}^\mu Z + \frac{1}{2}Z^2 + \frac{1}{12}\mathcal{F}_{\mu\nu}\mathcal{F}^{\mu\nu}\Bigg], \end{aligned} \tag{5.3c}$$

where $\mathbf{1}$ is the unity in the appropriate internal space and $\mathcal{F}$ acts both on spacetime and internal indices and is defined by $[\bar{\nabla}_\mu, \bar{\nabla}_\nu] = \mathcal{F}_{\mu\nu}$. In order to be able to apply these formulae to the operators $\mathcal{O}$ and $\mathcal{O}_{[\mathrm{gh}]}$ given in (2.8) and (2.10), we choose henceforth the Feynman-t'Hooft gauge for which $\alpha = 1$. To extract the dependence of $\Gamma^{(1)}$ on $\theta_{(\mathrm{cl})}$, we split $\mathcal{O} = \bar{\mathcal{O}} + Q$, where Q contains all the terms of Z quadratic in $\theta_{(\mathrm{cl})}$ and $\bar{\mathcal{O}} = -\bar{\nabla}_\lambda\bar{\nabla}^\lambda + \bar{Z}$. Then,

$$\begin{aligned} &\int d^4x\sqrt{|\det \bar{g}|}\ \mathrm{tr}\left(b_0(\mathcal{O})s^{-3} + b_2(\mathcal{O})s^{-2} + b_4(\mathcal{O})s^{-1} + \ldots\right) \\ &\qquad = \int d^4x\sqrt{|\det \bar{g}|}\mathrm{tr}\left(b_0(\bar{\mathcal{O}})e^{-sQ}s^{-3} + b_2(\bar{\mathcal{O}})e^{-sQ}s^{-2} + b_4(\bar{\mathcal{O}})e^{-sQ}s^{-1} + \ldots\right). \end{aligned} \tag{5.4}$$

Inserting in (5.1), the integration over s can be performed explicitly. The result is

$$\begin{aligned} \ln\det\mathcal{O} = \int d^4x\sqrt{|\det \bar{g}|}\ \mathrm{tr}\Bigg[&-\frac{\Lambda^4}{2}b_0(\mathcal{O}) - \Lambda^2 b_2(\mathcal{O}) - \left(\ln\frac{\Lambda^2}{\mu^2} - \gamma\right)b_4(\mathcal{O}) \\ &+ \frac{1}{2}b_0(\bar{\mathcal{O}})\,Q^2\left(\ln\frac{Q}{\mu^2} - \frac{3}{2}\right) - b_2(\bar{\mathcal{O}})\,Q\left(\ln\frac{Q}{\mu^2} - 1\right) + b_4(\bar{\mathcal{O}})\ln\frac{Q}{\mu^2}\Bigg], \end{aligned} \tag{5.5}$$

where γ is Euler's constant. The first line contains all the divergent parts, while the second contains only finite terms. We will use a renormalization procedure which amounts to taking the second line as the definition of the finite part of $\ln\det\mathcal{O}$.

In the case of the operator $\mathcal{O}$ given in (2.8), in the gauge $\alpha = 1$, $Z = \bar{Z} + Q$ where

$$\bar{Z} = \begin{bmatrix} \bar{Z}_{[\omega\omega]\,\mu ab}{}^{\nu cd} & \bar{Z}_{[\omega\varphi]\,\mu ab}{}^{c\nu} \\ \bar{Z}_{[\varphi\omega]\,a\mu}{}^{\nu cd} & \bar{Z}_{[\varphi\varphi]\,a\mu}{}^{c\nu} \end{bmatrix} = \begin{bmatrix} \delta^{[c}_{[a}\,\delta^{d]}_{b]}\bar{R}_\mu{}^\nu - 4g\,F_{(\mathrm{cl})\mu}{}^\nu{}_{[a}{}^{[c}\delta^{d]}_{b]} & -2g\,\delta^c_{[a}\bar{\nabla}_\mu\theta_{(\mathrm{cl})b]}{}^\nu \\ -2g\,\delta^{[c}_a\bar{\nabla}^\nu\theta_{(\mathrm{cl})}{}^{d]}{}_\mu & (m^2 + \xi\,\bar{R})\,\delta^c_a\delta^\nu_\mu \end{bmatrix} \tag{5.6}$$

$$Q = \begin{bmatrix} Q_{[\omega\omega]\,\mu ab}{}^{\nu cd} & Q_{[\omega\varphi]\,\mu ab}{}^{c\nu} \\ Q_{[\varphi\omega]\,a\mu}{}^{\nu cd} & Q_{[\varphi\varphi]\,a\mu}{}^{c\nu} \end{bmatrix} = \begin{bmatrix} \delta^\nu_\mu\,M_{(\mathrm{cl})ab}{}^{cd} & 0 \\ 0 & N_{(\mathrm{cl})a\mu}{}^{c\nu} \end{bmatrix} \tag{5.7}$$

For the ghost operator, $\tilde{Z}_{[\mathrm{gh}]} = 0$ and $Q_{[\mathrm{gh}]} = \alpha\, M_{(\mathrm{cl})}$. It is now a straightforward task to insert these formulae in (5.3), (5.5) and compute the traces. The part of the one-loop effective action (2.11) that contains the divergent contributions is given by

$$\begin{aligned}\Gamma^{(1)}_{\mathrm{div}} = \frac{1}{(4\pi)^2}\int d^4x\ \sqrt{|\det \bar{g}|}\Bigg\{ & -7\,\Lambda^4 + \Lambda^2\Big[8\,m^2 + 8(\xi-\frac{1}{6})\,\bar{R} + \mathrm{tr}M_{(\mathrm{cl})} + \frac{1}{2}\mathrm{tr}N_{(\mathrm{cl})}\Big] \\ & - \left(\ln\frac{\Lambda^2}{\mu^2} - \gamma\right)\Big[\frac{1}{2}\mathrm{tr}M_{(\mathrm{cl})}{}^2 + \frac{1}{4}\mathrm{tr}N_{(\mathrm{cl})}{}^2 + \frac{\bar{R}}{3}\,\mathrm{tr}M_{(\mathrm{cl})} + \frac{1}{2}\Big(m^2 + (\xi-\frac{1}{6})\bar{R}\Big)\mathrm{tr}N_{(\mathrm{cl})} \\ & + 4m^2\Big(1 + 2m^2(\xi-\frac{1}{6})\bar{R}\Big) - \frac{61}{180}\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} + \frac{64}{45}\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} - \Big(\frac{11}{24} - 4(\xi-\frac{1}{6})^2\Big)\,\bar{R}^2 \\ & + \frac{5\,g^2}{3}\,F_{(\mathrm{cl})\mu\nu ab}\,F_{(\mathrm{cl})}{}^{\mu\nu ab} + 3\,g^2\,\bar{\nabla}_\lambda\theta_{(\mathrm{cl})}{}^a{}_\mu\bar{\nabla}^\lambda\theta_{(\mathrm{cl})a}{}^\mu\Big]\Bigg\}\,. \end{aligned} \tag{5.8}$$

It has terms that are either of the same form of the starting action (2.3), or quadratic in the curvature of $\bar{\theta}$. The former can be eliminated by a suitable renormalization of the coupling constants of (2.3), while the latter are cancelled by adding suitable counterterms (we did not write these terms in the action (2.3) because they are independent of the dynamical variables θ and A). The remaining finite part, written in terms of the renormalized coupling constants, reads

$$\begin{aligned}\Gamma^{(1)} = \frac{1}{2}\frac{1}{(4\pi)^2}\int d^4x\ \sqrt{|\det\bar{g}|}\Bigg\{ & \mathrm{tr}\left(M_{(\mathrm{cl})}{}^2\left(\ln\frac{M_{(\mathrm{cl})}}{\mu^2} - \frac{3}{2}\right)\right) + \frac{1}{2}\mathrm{tr}\left(N_{(\mathrm{cl})}{}^2\left(\ln\frac{N_{(\mathrm{cl})}}{\mu^2} - \frac{3}{2}\right)\right) \\ & + \frac{2}{3}\bar{R}\,\mathrm{tr}\left(M_{(\mathrm{cl})}\left(\ln\frac{M_{(\mathrm{cl})}}{\mu^2} - 1\right)\right) + \Big(m^2 + (\xi-\frac{1}{6})\bar{R}\Big)\mathrm{tr}\left(N_{(\mathrm{cl})}\left(\ln\frac{N_{(\mathrm{cl})}}{\mu^2} - 1\right)\right) \\ & + \left(-\frac{13}{180}\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} + \frac{22}{45}\bar{R}_{\mu\nu}\bar{R}^{\mu\nu} - \frac{5}{36}\bar{R}^2 - \frac{1}{10}\bar{\nabla}_\lambda\bar{\nabla}^\lambda\bar{R}\right)\mathrm{tr}\left(\ln\frac{M_{(\mathrm{cl})}}{\mu^2}\right) \\ & + \left(2g^2\delta^c_a\bar{\nabla}_\mu\theta_{(\mathrm{cl})b}{}^\nu\bar{\nabla}^\mu\theta_{(\mathrm{cl})}{}^d{}_\nu - \frac{22}{3}g^2\,\delta^{[c}_a\,F_{(\mathrm{cl})\mu\nu b}{}^{e]}F_{(\mathrm{cl})}{}^{\mu\nu}{}_e{}^d\right)\left(\ln\frac{M_{(\mathrm{cl})}}{\mu^2}\right)_{cd}{}^{ab} \\ & + \bigg(\frac{1}{180}\Big(\bar{R}_{\mu\nu\rho\sigma}\bar{R}^{\mu\nu\rho\sigma} - \bar{R}_{\mu\nu}\bar{R}^{\mu\nu}\Big) + \frac{1}{2}(\xi-\frac{1}{6})^2\bar{R}^2 - \frac{1}{30}(5\xi-1)\bar{\nabla}_\lambda\bar{\nabla}^\lambda\bar{R} \\ & + \frac{m^4}{2} + (\xi-\frac{1}{6})\,m^2\bar{R}\bigg)\,\mathrm{tr}\left(\ln\frac{N_{(\mathrm{cl})}}{\mu^2}\right) + \bigg(2g^2\,\delta^{[b}_a\,\bar{\nabla}_\lambda\theta_{(\mathrm{cl})}{}^{c]}{}_\mu\bar{\nabla}^\lambda\theta_{(\mathrm{cl})c}{}^\nu \\ & - \frac{1}{12}\Big(\delta^b_a\,\bar{R}_{\lambda\rho\sigma\mu}\bar{R}^{\lambda\rho\sigma\nu} - 2g\,\bar{R}_{\rho\sigma\mu}{}^\nu\,F_{(\mathrm{cl})}{}^{\rho\sigma}{}_a{}^b - g^2\,\delta^\nu_\mu\,F_{(\mathrm{cl})\rho\sigma a}{}^c\,F_{(\mathrm{cl})}{}^{\rho\sigma}{}_c{}^b\Big)\bigg)\left(\ln\frac{N_{(\mathrm{cl})}}{\mu^2}\right)_{b\nu}{}^{a\mu}\Bigg\} \end{aligned} \tag{5.9}$$

This formula reduces exactly to (4.11) in the case of de Sitter space and for $\theta_{(\mathrm{cl})} = \rho\bar{\theta}$. This shows two things. Firstly, the expansion in inverse powers of the de Sitter radius used in (4.11) coincides with the expansion in powers of curvature used in (5.9). Moreover, the renormalization scheme we have adopted here to extract the finite part of the effective action is equivalent to the renormalization procedure which is implicit in the zeta function approach.

To proceed further one needs to compute the vacuum expectation value of the fields θ and A. These are the solutions of the equations of motion obtained by varying the total effective action $\Gamma = \bar{S} + \Gamma^{(1)}$ with respect to $\theta_{(\mathrm{cl})}$ and $A_{(\mathrm{cl})}$. For simplicity, we shall keep in Γ only the dominant parts in the long wavelength expansion, those containing at most terms linear in the curvature of $\bar{\theta}$ and without derivatives of the classical fields. As in the previous Sections, we set $m^2 = 0$, assume $\lambda_1 \approx \lambda_2 \approx g^4$ and choose $\xi = \frac{1}{6}$ to simplify a bit the expressions. The total effective action Γ is then given by the potential terms in (2.3) plus the first three terms in (5.9).

We notice that to this order Γ is independent on $A_{(\mathrm{cl})}$ so that the vacuum expectation value $\langle A\rangle$ is left arbitrary. This slightly unpleasant fact depends on our choice of action (2.3) and is not a general consequence of the mean-field approach. If we added to the action (2.3) a term linear in the curvature, such as the second term in (6.1) below, this term would appear in Γ to the order we are discussing and by itself would give vanishing torsion as an equation of motion.

The variation of Γ with respect to $\theta_{(\mathrm{cl})}$ gives the equation for $\langle\theta\rangle$. Since we have to impose self-consistency on the theory anyway, we shall look for solutions for which $\langle\theta\rangle$ is proportional to $\bar{\theta}$, and check that the minimum of Γ occurs for a non-zero value of the proportionality constant ρ. Indeed, one finds that in our approximation the absolute minimum of Γ occurs for

$$\rho^2 = \frac{\mu^2}{g^2} e^{1-\frac{16\pi^2}{7}\left(\frac{4\lambda_1+\lambda_2}{g^4}\right)} - \frac{\bar{R}}{7g^2}\left[1 + \frac{8\pi^2}{3g^2} - \frac{16\pi^2}{7}\frac{4\lambda_1+\lambda_2}{g^4}\right] . \qquad (5.10)$$

Note that since ρ has to be a constant, self-consistency requires $\bar{R} =$ constant. In particular this condition is satisfied by all solutions of Einstein's vacuum equations.

One can now obtain an effective action for $\bar{g}$ by evaluating Γ at its minimum (5.10). The explicit computation gives

$$\Gamma(\bar{g}) = \int d^4x \sqrt{|\det \bar{g}|}\left\{\frac{\mu^2}{3g^2} e^{1-\frac{16\pi^2}{7}\left(\frac{4\lambda_1+\lambda_2}{g^4}\right)}\left[1 - \frac{6}{7}\frac{4\lambda_1+\lambda_2}{g^4}\right]\bar{R} + \ldots\right\} , \qquad (5.11)$$

where Γ has been normalized such that it vanishes in flat space. The Einstein-Hilbert action is thus recovered as the action that governs gravity at large distances. The mechanism by which this happens is very similar to the one discussed in [27]. There are also close ties with the "induced gravity" programme [7,8] and with the ideas in [28], where the method of the effective potential was applied to the gravitational field. Newton's constant G_N, which appears in the Einstein-Hilbert action in the form $-\frac{1}{16\pi G_N}\int d^4x\bar{R}$, is seen to be of the order of the renormalization point μ, which in turn appears in the theory as an arbitrary dimensionful constant. Newton's constant therefore appears in this theory to arise through a sort of dimensional transmutation.

6. CONCLUSIONS

The original motivation for this work was the recognition that the metric and/or vierbein play in the theory of gravity the role of order parameter. In particular one can see a kind of Higgs phenomenon occurring already in the standard formulation of General Relativity [6]. Since the Higgs phenomenon plays such an important role in the description of Elementary Particle physics, it is tempting to try and construct a theory of gravity following the same lines. The questions that the Higgs model is designed to answer are: Why is the order parameter nonzero? What is the origin of the mass of the gauge fields? In the context of gravity, analougous questions are: Why are the metric and/or vierbein nondegenerate? Why is the connection metric and torsionfree? These are the questions that we have tried to answer with our mean-field model of quantum gravity.

In the traditional approach to quantum gravity it is implicitly assumed that the geometry of spacetime is determined by the quantum metric or, in the path integral framework, by the fluctuating metric. One could try to give some conceptual foundation to the mean-field approach by postulating that lengths and angles should not be measured with the quantum (fluctuating) metric but rather with its vacuum expectation value. Having thus two metrics at our disposal, we can write the action $\bar{S}$ given in (2.3) in which the mean metric and the fluctuating metric play different roles. In several respects this action lends itself better to treatment by traditional field-theoretic methods than an action of the form (2.2). All quantum calculations are to be performed by keeping $\bar{g}$ fixed, so the technical aspects of our approach are identical to those of quantum field theory in a fixed curved background metric, a well-studied subject [29]. The difference with more traditional approaches to quantum gravity lies in the appearance of the vacuum state in the action through $\bar{g}$, so that the mean-field theory is not a quantum field theory in the traditional sense. In practice, this is reflected in the necessity of verifying, at the end of the day, the self-consistency conditions $\langle\theta\rangle = \ell^{-1}\bar{\theta}$.

We have found that Minkowski space is a self-consistent solution of the one-loop quantum dynamics of the theory. Furthermore, insofar as quantum field theory in flat space preserves global Lorentz invariance, it seems very likely that Minkowski space will be a self-consistent solution at all orders of perturbation theory. In the case of de Sitter space we have found that for given values of the coupling constants self-consistency may be achieved at least for a certain range of values of the de Sitter radius. When one considers a more general mean field, it becomes much more difficult to establish conditions of self-consistency. Our approach has been to deal with this problem order by order in an expansion in powers of momentum. It appears that to lowest order in such an expansion any solution of Einstein's equations in vacuum will give a self-consistent solution of the theory. At the next order, terms quadratic in the curvature appear, suggesting that some kind of Yang-Mills-type equation will become relevant at short distances.

Any theory of quantum gravity has to reproduce General Relativity in the classical limit. In our approach, General Relativity appears as an effective low-energy theory, much as in certain "induced gravity" schemes: the effective action depends on $\theta_{(\mathrm{cl})}$ and $\bar{\theta}$, so when it is evaluated at its minimum with respect to $\theta_{(\mathrm{cl})}$ one remains with an effective action for $\bar{\theta}$, which contains the Einstein-Hilbert term. In particular, Newton's constant appears as the vacuum expectation value of the vierbein, thus providing a concrete realization of an old idea [30].

The small fluctuations of the fields θ and A have masses of the order of the vacuum expectation value of the vierbein, *i.e.* of Planck's mass. This is in accordance with the empirical observation that there are no massless spin-one bosons related to gravity, and is also related to the absence of torsion and nonmetricity at low energy. The graviton only appears in the low-energy effective theory and its masslessness is a consequence of the invariances of the theory.

We have tried to present the essential ideas of the mean field approach keeping technical complications at a minimum. There are several directions in which our work can be improved and generalized, some of which are rather straightforward while others require substantial work.

A simple generalization would consist in replacing the gauge group $O(1,3)$ by $O(1,N-1)$. In this case the internal (latin) indices run from 0 to $N-1$, while the spacetime (greek) indices remain as before. In this generalized theory, gravity is unified with an $O(N-4)$ Yang-Mills theory, the form $\theta^a{}_\mu$ playing the role of order parameter [10,6]. At least in the case of Minkowski space, the calculation of the effective potential proceeds as in Section 3 and the results are the same [11].

Another possibility is to add to the action $\bar{S}$ a "cosmological" term and an "Einstein" term, written in polynomial form:

$$\frac{\lambda_0}{24}\int d^4x\,\varepsilon_{abcd}\theta^a{}_\mu\theta^b{}_\nu\theta^c{}_\rho\theta^d{}_\sigma\varepsilon^{\mu\nu\rho\sigma}+\kappa\int d^4x\;\varepsilon_{abcd}\theta^a{}_\mu\theta^b{}_\nu F_{\rho\sigma}{}^{cd}\varepsilon^{\mu\nu\rho\sigma}\;. \tag{6.1}$$

Note that the "cosmological" term should be regarded as part of the quartic potential, while the "Einstein" term describes cubic and quartic interactions between θ and A. We observe that, perhaps surprisingly, the addition of the "cosmological" term to $\bar{S}$ does not ruin the results of section 3: Minkowski space is still a minimum of the effective action, except for a λ_0-dependent modification of (3.5).

The action (2.3), while giving a simple form for the propagators, suffers from a serious drawback: the massive modes $A_\mu{}^0{}_i$ and $A_\mu{}^i{}_0$ have propagators with negative residues at the poles. This is because the first term in the action (2.3) is a Yang-Mills action for a noncompact gauge group. Had we chosen a more general action S containing also terms with different contractions of the indices, then in the presence of a potential for θ it would probably have been possible to choose the parameters in the action so that the theory was free from ghosts and tachyons. One could also argue that since the ghosts occur at the Planck mass, where the theory probably ceases to be meaningful anyway, their presence may not be fatal. Both these possibilities have to be investigated in greater detail.

We have chosen to study the vacuum expectation value of the gauge-variant order parameter θ. This is analogous to studying the vacuum expectation value of the Higgs field Φ in the Standard Model. There, a more rigorous procedure would be to compute the vacuum expectation value of the gauge-invariant operator $\mathrm{tr}(\Phi^2)$. Similarly, in our case, one could try to compute $\langle g\rangle$, which is invariant under local Lorentz transformations, although not under general coordinate transformations. Our result can be regarded as a calculation of $\langle g\rangle$ in the approximation in which $\langle\theta^2\rangle=\langle\theta\rangle^2$. As we have already mentioned in Section 2, a direct determination of $\langle g\rangle$ is technically more complicated, since g has to be treated as a composite operator, but we expect it to yield essentially equivalent results.

Also, we are aware that from a rigorous point of view the effective potential has to be convex and hence what we have said can only be true in a metaphoric sense. We believe that a correct treatment of this point can be given, for instance using the concept of constrained effective potential [31]. The use of this idea seems to be quite natural to our problem, where self-consistency demands $\langle \theta \rangle$ to take a fixed value.

It may seem that since the fluctuating metric is no longer used as the geometrical standard of lengths and angles, the theory has lost some of its geometrical flavor. In fact, this is only partly so. The geometric nature of gravity lies therein, that the geometry of spacetime is dynamically determined. This is still true in our approach, since the vacuum expectation value of the metric, which we take as the standard of lengths and angles, is determined selfconsistently by the quantum dynamics of the theory. What is gone is the idea of quantum fluctuations of the geometry (and consequently also of the topology). While this idea may be fascinating, it is also at the origin of most difficulties of quantum gravity and a more conservative approach like the one we are proposing may have better chances of success. Whether this will be the case requires much more work to establish.

Acknowledgements

We thank S. Randjbar-Daemi for explaining the spectra on the sphere to us and E. Spallucci for much help and advice through all stages of this work.

APPENDIX

In Section 3 we have introduced two decompositions in the space of all tensors $\omega_{\mu ab}$, antisymmetric in the "internal" indices a, b. The first was the Hodge decomposition of ω, regarded as a Lie-algebra valued one-form, into its exact (longitudinal) and coexact (transverse) part (recall that the first cohomology group $H^1(S^4) = 0$ and there are no harmonic one-forms on S^4). The second was the decomposition (4.5a) into $O(4)$-invariant parts. We will refer to the three terms on the r.h.s. of (4.5a) as the tensor, vector and axial vector parts of ω. In this appendix we examine the relationship between these two decompositions. We prove the following theorem:

1) *the tensor part of ω is coexact;*
2) *the vector part of ω is coexact iff v is exact and is exact iff v is coexact;*
3) *the axial vector part of ω is coexact iff w is exact and is exact iff w is coexact.*

In the proofs we will use $\bar{\theta}$ to transform all indices from latin to greek and vice-versa when convenient. These operations can be performed freely under the covariant derivatives because, due to the second condition in (4.2), $\bar{\nabla}\bar{\theta} = 0$.

We begin by proving 3). Assume that $\omega_{\lambda\mu\nu} = \varepsilon_{\lambda\mu\nu\rho} w^\rho$. Clearly if $w = df$ for some function f, $\bar{\nabla}^\lambda \omega_{\lambda\mu\nu} = 0$. Conversely if $\bar{\nabla}^\lambda \omega_{\lambda\mu\nu} = 0$, then $\bar{\nabla}_\lambda w_\rho - \bar{\nabla}_\rho w_\lambda = 0$, and therefore $w = df$ for some function f. Thus the axial vector part of ω is coexact if and only if w is exact. Next suppose that ω is exact, *i.e.* that $\omega_{\lambda ab} = \bar{\nabla}_\lambda \epsilon_{ab}$ for some antisymmetric tensor ϵ_{ab}. Then we have $\bar{\nabla}_\lambda w^\lambda = \bar{\nabla}^\mu \frac{1}{\sqrt{\bar{g}}} \varepsilon^{\mu\nu\rho\sigma} \bar{\nabla}_\nu \epsilon_{\rho\sigma} = \frac{1}{\sqrt{\bar{g}}} \varepsilon^{\mu\nu\rho\sigma} [\bar{\nabla}_\mu, \bar{\nabla}_\nu] \epsilon_{\rho\sigma} = 0$, *i.e.* w is coexact. Conversely, if w is coexact, one sees immediately that as a three-form ω is closed, and therefore there exists a two-form $\epsilon_{\rho\sigma}$ such that $\omega = d\epsilon$. Since the axial vector part of ω is totally antisymmetric, this is equivalent to saying that $\omega_{\lambda ab} = \bar{\nabla}_\lambda \epsilon_{ab}$.

Next to prove 2), assume that $\omega_{\lambda\mu\nu} = \frac{1}{3}(\bar{g}_{\lambda\mu} v_\nu - \bar{g}_{\lambda\nu} v_\mu)$. Clearly if $v_\mu = \partial_\mu f$ for some function f, $\bar{\nabla}^\mu \omega_{\mu\nu\rho} = 0$. Conversely if $\bar{\nabla}^\mu \omega_{\mu\nu\rho} = 0$, then $(\bar{\nabla}_\nu v_\rho - \bar{\nabla}_\rho v_\nu) = 0$, i.e. v is closed. But every closed one-form on the sphere is exact, so $v_\mu = \partial_\mu f$. Thus the vector part of ω is coexact if and only if v is exact. Next suppose that ω is exact, *i.e.* that $\omega_{\lambda ab} = \bar{\nabla}_\lambda \epsilon_{ab}$ for some antisymmetric tensor ϵ_{ab}. We have $v_\lambda = \omega^\mu{}_{\mu\lambda} = \bar{\nabla}^\mu \epsilon_{\mu\lambda}$, so v is coexact. Conversely if v is coexact, *i.e.* there exists an antisymmetric tensor ϵ such that $v_\mu = \bar{\nabla}^\nu \epsilon_{\nu\mu}$, then $\omega_{\lambda\mu\nu} = \frac{1}{3}(\bar{g}_{\lambda\mu} \bar{\nabla}^\tau \epsilon_{\tau\nu} - \bar{g}_{\lambda\nu} \bar{\nabla}^\tau \epsilon_{\tau\mu})$. This is precisely the vector part of $\bar{\nabla}_\lambda \epsilon_{\mu\nu}$, and since by assumption ω was purely vectorial, it must be itself of this form.

Finally to prove 1), assume $\omega_{\lambda\mu\nu} = \frac{2}{3}(t_{\lambda\mu\nu} - t_{\lambda\nu\mu})$. We show that ω cannot be exact as a one-form. In fact, if $\omega_{\mu ab} = \bar{\nabla}_\mu \epsilon_{ab}$, since t is symmetric in the first pair of indices, $0 = \varepsilon^{\lambda\mu\nu\rho} \omega_{\lambda\mu\nu} = \varepsilon^{\lambda\mu\nu\rho} \bar{\nabla}_\lambda \epsilon_{\mu\nu}$. Regarding ϵ as a two-form, this implies $d\epsilon = 0$ and since the second cohomology group $H^2(S^4) = 0$, $\epsilon = du$ for some one-form u. On the other hand, since t is traceless, also the tensor part of ω is traceless, so $\bar{\nabla}^\mu \epsilon_{\mu\nu} = 0$. This implies $\delta du = 0$. The one-form u can also be decomposed into exact and coexact parts. The exact part does not contribute to ϵ, so we can assume without loss of generality that $\delta u = 0$. But then $(d\delta + \delta d)u = 0$. Since there are no harmonic one-forms on the sphere, $u = 0$. So ω must be coexact as a one-form.

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