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TWO-PARAMETER QUANTUM DEFORMATION OF *GL* (1/1)

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Two-parameter quantum deformation of $GL(1|1)$

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ABSTRACT

Two-parameter quantum deformation of the algebra of functions on the supergroup $GL(1|1)$ and of the universal enveloping algebra $Ugl(1|1)$ is presented.

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where

$$
\begin{aligned}
(R_{12})^{i_1 i_2 i_3}_{j_1 j_2 j_3} &= R^{i_1 i_2}_{j_1 j_2} \delta^{i_3}_{j_3} \ , \\
(R_{13})^{i_1 i_2 i_3}_{j_1 j_2 j_3} &= (-1)^{i_2(i_3+j_3)} R^{i_1 i_3}_{j_1 j_3} \delta^{i_2}_{j_2} \ , \\
(R_{23})^{i_1 i_2 i_3}_{j_1 j_2 j_3} &= (-1)^{i_1(i_2+i_3+j_2+j_3)} R^{i_2 i_3}_{j_2 j_3} \delta^{i_1}_{j_1} \ .
\end{aligned} \tag{3}
$$

Here, and later on, in exponents like $(-1)^i$ for simplicity we write i instead of the Z_2-grade of i, which is 0 for the first row or column of a 2×2 matrix, and 1 for the second.

Let T be a 2×2 super-matrix

$$
T = \begin{pmatrix} a & \beta \\ \gamma & d \end{pmatrix} \ , \tag{4}
$$

where a, d are even and β, γ are odd variables. We use the tensoring convention

$$
\begin{aligned}
(T_1)^{ab}_{cd} &= (T \otimes I)^{ab}_{cd} = (-1)^{c(b+d)} T^a_c \delta^b_d \ , \\
(T_2)^{ab}_{cd} &= (I \otimes T)^{ab}_{cd} = (-1)^{a(b+d)} T^a_d \delta^a_c \ .
\end{aligned} \tag{5}
$$

The explicit form of T_1 and T_2 is

$$
T_1 = \begin{pmatrix} a & 0 & \beta & 0 \\ 0 & a & 0 & \beta \\ \gamma & 0 & d & 0 \\ 0 & \gamma & 0 & d \end{pmatrix} \quad , \quad T_2 = \begin{pmatrix} a & \beta & 0 & 0 \\ \gamma & d & 0 & 0 \\ 0 & 0 & a & -\beta \\ 0 & 0 & -\gamma & d \end{pmatrix} \ . \tag{6}
$$

We define a two-parameter deformation of the algebra of functions on the Lie supergroup $GL(1|1)$ as an associative algebra with unit, generated by the variables a, d, β and γ, with the commutation relations which can be written in the matrix form as

$$
R \, T_1 \, T_2 = T_2 \, T_1 \, R \ . \tag{7}
$$

$TD = DT$. Therefore by imposing the relation $D = 1$, we may define a two-parameter deformation of $SL(1|1)$. (This is impossible for $SL(2)$ as the determinant is not central for two parameters [15,16]).

Now, we describe the corresponding deformation of the universal enveloping algebra $Ugl(1|1)$. Let

$$L^+ = \begin{pmatrix} U_+ & w\chi_+ \\ 0 & W_+ \end{pmatrix} , \quad L^- = \begin{pmatrix} U_- & 0 \\ -w\chi_- & W_- \end{pmatrix} . \tag{14}$$

These matrices obey the commutation relations

$$R^+ L_1^\pm L_2^\pm = L_2^\pm L_1^\pm R^+ \quad , \tag{15}$$

$$R^+ L_1^+ L_2^- = L_2^- L_1^+ R^+ \quad , \tag{16}$$

where, $R^+ = PRP$, and

$$P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad , \tag{17}$$

is the super-permutation matrix. In terms of generators, these relations read

$$\begin{aligned}
\chi_+ U_+ &= pU_+\chi_+ \, , \quad \chi_+^2 = 0 \quad , \\
\chi_+ W_+ &= pW_+\chi_- \, , \quad U_+W_+ = W_+U_+ \quad , \\
\chi_- U_- &= qU_-\chi_- \, , \quad \chi_-^2 = 0 \quad , \\
\chi_- W_- &= qW_-\chi_- \, , \quad U_-W_- = W_-U_- \quad , \\
\chi_+ U_- &= \frac{1}{q}U_-\chi_+ \, , \quad \chi_- U_+ = \frac{1}{p}U_+\chi_- \quad , \\
\chi_+ W_- &= \frac{1}{q}W_-\chi_+ \, , \quad \chi_- W_+ = \frac{1}{p}W_+\chi_- \quad , \\
\{\chi_+, \chi_-\}_{\frac{q}{p}} &= \frac{1}{w'}(U_+W_- - W_+U_-) \, ,
\end{aligned} \tag{18}$$

where

$$\{A, B\}_{\frac{q}{p}} = (\frac{q}{p})^{\frac{1}{2}} AB + (\frac{q}{p})^{-\frac{1}{2}} BA \, , \tag{19}$$

$$w' = (qp)^{\frac{1}{2}} - (qp)^{-\frac{1}{2}} \, . \tag{20}$$

One easily sees that when $p = q$, these relations go back to these of $gl_q(1|1)$ in [6].

We close with some comments.

Any solution R of the graded Yang-Baxter equation (2) provides a solution of the braid equation by $\check{R} = PR$. Clearly, the matrix $\mathcal{P}PR$, where $\mathcal{P}$ is the ordinary permutation matrix, solves the (nongraded) Yang-Baxter equation and a quantum group can be associated by the usual Leningrad method (cf. [18]). However, a family of such R-matrices does not contain the unit matrix but rather the matrix $diag(\pm 1)$ and it can be easily verified that some generators are nilpotent in the deformed algebra of functions. Therefore, in such situations it seems more appropriate to use the graded version we have adopted.

Other issues related to the two-parameter quantisation of $GL(1|1)$, like the Clebsch-Gordan coefficients and the differential calculus [22,6] on the associated superplane, will be reported elsewhere.

After this work has been completed, we obtained the paper [23], containing (up to a redefinition of parameters) the matrix (1); and we were informed that the central property of D was known to the authors.

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