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SYMMETRIES OF THE CLASSICAL PATH INTEGRAL ON A GENERALIZED PHASE SPACE MANIFOLD

E.Gozzi[⋆], M.Reuter[†] and W.D.Thacker[*]

ABSTRACT

In this paper we generalize previous work done on the path-integral approach to classical mechanics and its symmetries. We study in particular the case that the components of the symplectic 2-form ω_{ab}, expressed in arbitrary coordinates, are allowed to depend on the phase space coordinates. This lifts the restriction that the path-integral and its symmetry generators be only expressed in terms of *canonical* coordinates. We show, in particular, that an extra term must be added to the anti-BRS charge in order to preserve the *ISp(2)* symmetry which reflects the geometry of phase space. The cohomology of this new anti-BRS operator is found to be isomorphic to the de Rham cohomology of phase space. The modification of the anti-BRS charge leads to a modification of one of the supersymmetry generators associated with the classical Hamiltonian. Despite this change in the form of the generators, the classical KMS conditions can still be derived from this supersymmetry. We also prove that the requirement of supersymmetric invariance of the states results in a *new* set of equations that, despite their new form, are still satisfied by the Gibbs states on a general phase space manifold.

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1. INTRODUCTION

In previous work[1] we developed a path integral formulation of *classical* Hamiltonian mechanics, assuming, for simplicity, that the components ω_{ab} of the symplectic 2-form, $\omega = \frac{1}{2}\omega_{ab} d\phi^a \wedge d\phi^b$, are constant in phase space $\mathcal{M}_{2n}$. However, in general, ω can be any closed and non-degenerate 2-form,*i.e.*, $d\omega = 0$ and $det(\omega_{ab}) \neq 0$. The measure for the classical path-integral is a delta-function which confines the system to paths satisfying the classical equations of motion. This measure can be expressed in terms of the exponential of an action $\widetilde{S}$ that depends not only on the phase space coordinates $\phi^a, a = 1, \cdots, 2n$, but also on dual auxiliary variables λ_a, anticommuting ghosts c^a, and antighosts $\bar{c}_a$. The ghosts c^a can be interpreted[1] as one-forms $d\phi^a$ making up a basis of the cotangent space $T^*_\phi \mathcal{M}_{2n}$, while the antighosts $\bar{c}_a$ form a basis of the tangent space $T_\phi \mathcal{M}_{2n}$. Written in terms of these variables, the classical path-integral (CPI) becomes more powerful than the delta-function measure from which it sprang. In fact, in this form it propagates not only scalar densities $\varrho(\phi)$, but also p-forms

$$\widetilde{\varrho}(\phi, c) = \frac{1}{p!}\varrho_{a_1 \cdots a_p}(\phi) c^{a_1} \cdots c^{a_p} \tag{1.1}$$

In the CPI the variables (ϕ, λ) and $(c, \bar{c})$ form conjugate pairs satisfying[1] the equal-time (graded) commutation relations:

$$\begin{aligned} [\phi^a, \lambda_b] &= i\delta^a_b \\ [c^a, \bar{c}_b] &= \delta^a_b \end{aligned} \tag{1.2}$$

while the rest of the commutators vanish. The commutation relations (1.2) can be realized by letting ϕ^a and c^a be multiplicative operators, while λ_a and $\bar{c}_a$ are realized as:

$$\begin{aligned} \lambda_a &= -i\frac{\partial}{\partial \phi^a} \equiv -i\partial_a \\ \bar{c}_a &= \frac{\partial}{\partial c^a} \end{aligned} \tag{1.3}$$

The super-Hamiltonian $\widetilde{\mathcal{H}}$ associated with the path integral action $\widetilde{S}$, once it is realized as an operator, acts on p-forms $\widetilde{\varrho}(\phi, c)$ by taking their Lie derivative along the Hamiltonian vector field with components $h^a = \omega^{ab}\frac{\partial H}{\partial \phi^b}$ where H is the classical Hamiltonian and $\omega^{ab}\omega_{bc} = \delta^a_c$.

The action $\tilde{S}$ is invariant under a set of transformations generated by the following conserved charges*

$$
\begin{aligned}
K &= \frac{1}{2}\omega_{ab}c^a c^b \\
\bar{K} &= \frac{1}{2}\omega^{ab}\bar{c}_a\bar{c}_b = \frac{1}{2}\omega^{ab}\frac{\partial}{\partial c^a}\frac{\partial}{\partial c^b} \\
Q_g &= c^a\bar{c}_a = c^a\frac{\partial}{\partial c^a} \\
Q &= ic^a\lambda_a = c^a\frac{\partial}{\partial \phi^a} \\
\bar{Q} &= i\omega^{ab}\bar{c}_a\lambda_b = \omega^{ab}\frac{\partial}{\partial c^a}\frac{\partial}{\partial \phi^b}
\end{aligned}
\tag{1.4}
$$

each of which is connected with the symplectic geometry[1] of phase space. The charges K and $\bar{K}$ can be identified with the symplectic two-form and symplectic bivector, respectively, and their conservation corresponds to the Liouville theorem of classical mechanics. Realized operatorially, the BRS charge Q acts as the exterior derivative on phase space, while the anti-BRS charge $\bar{Q}$ plays the role of exterior co-derivative, mapping p-vectors into (p+1)-vectors. Finally, the ghost charge Q_g counts form/vector number, attaching a weight of +1 to each one-form c^a and -1 to each tangent vector $\bar{c}_a$. The BRS and anti-BRS charges are nilpotent and (anti-)commute with each other

$$[Q, Q] = [\bar{Q}, \bar{Q}] = [Q, \bar{Q}] = 0 \tag{1.5}$$

⋆ The second equality gives the operatorial realisation of each charge containing derivative operators.

and, together with the rest of the charges, they make up the algebra of *ISp(2)*:

$$
\begin{aligned}
&[\, Q_g, Q] = Q \\
&[\, Q_g, \bar{Q}] = -\bar{Q} \\
&[\, K, Q] = [\bar{K}, \bar{Q}] = 0 \\
&[\, \bar{K}, Q] = \bar{Q} \\
&[\, K, \bar{Q}] = Q \\
&[\, Q_g, K] = 2K, \ [Q_g, \bar{K}] = -2\bar{K}, \ [K, \bar{K}] = Q_g
\end{aligned}
\tag{1.6}
$$

In addition to the *ISp(2)* charges, which reflect the symplectic geometry of phase space, independent of the classical Hamiltonian H, there is a pair of supercharges for every independent conserved quantity of the dynamical system[2] . In particular, if we take H as conserved quantity , the supercharges are

$$
\begin{aligned}
Q_H &= exp(\beta H)\ Q\ exp(-\beta H) \\
\bar{Q}_H &= exp(-\beta H)\ \bar{Q}\ exp(\beta H)
\end{aligned}
\tag{1.7}
$$

Q and $\bar{Q}$ are the BRS and anti-BRS charges introduced before and β plays the role of inverse temperature. The charges in (1.7) are genuine supersymmetry generators, satisfying

$$
\begin{aligned}
[Q_H, Q_H] &= [\bar{Q}_H, \bar{Q}_H] = 0 \\
[Q_H, \bar{Q}_H] &= 2i\beta\widetilde{\mathcal{H}}
\end{aligned}
\tag{1.8}
$$

Physical states of ghost number 2n which are invariant under this supersymmetry turn out[2] to be just the Gibbs states[3] [4] $\varrho(\phi) = k\ exp\ [-\beta\ H(\phi)]$ characterizing thermodynamic equilibrium. This supersymmetry can also be used to derive[2] the classical KMS condition[5] characterizing the Gibbs distribution.

The classical path-integral, the generators of its *ISp(2)* symmetry, and the Hamiltonian supercharges were all constructed in Refs.[1,2] assuming that the coefficients ω_{ab} of the symplectic form were constant. We know that it is always possible[6] to cover a symplectic manifold with *local* charts for which

ω_{ab} is constant (Darboux theorem). However, if the path-integral and symmetry generators are to retain their *global* geometric significance, independent of the particular choice of coordinates, they should be formulated for the general case that ω_{ab} depends on the phase space position.

In this paper (Sect.II) we show that the action $\tilde{S}$ of the CPI, formulated with a position dependent symplectic form, is invariant under nonlinear as well as linear canonical transformations. Considering the generators of the *ISp(2)* symmetry, we find in Sect. III that the anti-BRS charge must be modified by a term proportional to the derivative of ω^{ab} so that it remains conserved, and retains its algebraic properties (1.5), (1.6) and its geometric interpretation as the exterior co-derivative on phase space. In Sect.IV. we study the cohomology problem associated to the modified anti-BRS operator and prove it to be isomorphic to the standard deRham cohomology of phase space.

The additional term in $\bar{Q}$ leads to a modification of the supersymmetric charge $\bar{Q}_H$ derived from it. The requirement of supersymmetric invariance of the 2n-ghost states[2] now leads to a *new* equation, but its solutions are still the Gibbs states on a generalized phase space manifold. Also the KMS condition continues to be derivable from this supersymmetry using the modified $\bar{Q}_H$: all this is reported in Sect. V. We confine some computational details to three appendices A,B,C.

2. CANONICAL COVARIANCE.

Let us briefly review the derivation of Ref.[1] of the CPI, now allowing the coefficients ω_{ab} of the symplectic form to depend on the phase space coordinates ϕ. In classical mechanics the propagator $P\Big(\phi,t|\phi_0,0\Big)$, which gives the classical probability for a particle to be at the point with coordinates ϕ at time t, given that it was at the point ϕ_0 at time 0, is just a delta function

$$P\Big(\phi,t|\phi_0,0\Big)=\delta^{2n}\Big(\phi-\phi_{cl}(t,\phi_0)\Big) \tag{2.1}$$

where $\phi_{cl}(t,\phi_0)$ is a solution of Hamilton's equation

$$\dot{\phi}^a(t)=\omega^{ab}(\phi)\partial_b H(\phi(t)),\ with\ \omega^{ab}(\phi)\omega_{bc}(\phi)=\delta^a_c \tag{2.2}$$

subject to the initial conditions $\phi^a(0)=\phi^a_0$.

The delta function in (2.1) can be rewritten as

$$\delta^{2n}\Big(\phi-\phi_{cl}(t,\phi_0)\Big)=\Big\{\prod_{i=1}^{N-1}\int d\phi_{(i)}\delta^{2n}\Big(\phi_{(i)}-\phi_{cl}(t_i,\phi_0)\Big)\Big\}\delta^{2n}\Big(\phi-\phi_{cl}(t,\phi_0)\Big) \tag{2.3}$$

where we have sliced the interval [0,t] in N intervals and labelled the various instants as t_i and the fields at t_i as $\phi_{(i)}$. Each delta function contained in the product on the RHS of (2.3) can be written as:

$$\delta^{2n}\Big(\phi_{(i)}-\phi_{cl}(t_i,\phi_0)\Big)=\prod_{a=1}^{2n}\delta\Big(\dot{\phi}^a-\omega^{ab}\partial_b H\Big)_{|t_i}det\Big[\delta^a_b\partial_t-\partial_b\Big(\omega_{ac}(\phi)\partial_c H(\phi)\Big)\Big]_{|t_i} \tag{2.4}$$

where the argument of the determinant is obtained from the functional derivative of the equation of motion (2.2) with respect to $\phi_{(i)}$. Introducing anticommuting variables c^a and $\bar{c}_a$ to exponentiate the determinant, and the commuting auxiliary variables λ_a to exponentiate the delta functions, we can re-write the propagator as a path-integral using the slicing (2.3)* :

$$P\Big(\phi,t|\phi_0,0\Big)=\int_{\phi_0}^{\phi}\mathcal{D}\phi\ \mathcal{D}\lambda\ \mathcal{D}c\ \mathcal{D}\bar{c}\ exp\ i\widetilde{S} \tag{2.5}$$

where $\widetilde{S}=\int_0^t dt'\ \widetilde{\mathcal{L}}$ with

$$\widetilde{\mathcal{L}}\equiv\lambda_a\Big[\dot{\phi}^a-\omega^{ab}(\phi)\partial_b H(\phi)\Big]+i\bar{c}_a\Big(\delta^a_b\partial_t-\partial_b[\omega^{ac}(\phi)\partial_c H(\phi)]\Big)c^b \tag{2.6}$$

Holding ϕ and c both fixed at the endpoints of the path-integral, we obtain the kernel[1] , $K(\phi,c,t|\phi_0,c_0,0)$, which propagates the phase space p-forms of

---

⋆ The limit of $N\to\infty$ has to be taken with some care and some normalization factors might appear in eq.[2.5], but they are of no importance for our discussion.

eq. (1.1):

$$\tilde{\varrho}(\phi, c, t) = \int\int d^{2n}\phi_0 \; d^{2n}c_0 \; K\big(\phi, c, t|\phi_0, c_0, 0\big) \; \tilde{\varrho}(\phi_0, c_0) \tag{2.7}$$

From the Lagrangian (2.6), we can immediately read off the associated super-Hamiltonian $\widetilde{\mathcal{H}}$:

$$\widetilde{\mathcal{H}} = \lambda_a \omega^{ab} \partial_b H + i\bar{c}_a \partial_b (\omega^{ac} \partial_c H) c^b \tag{2.8}$$

which, with the help of eqn. (1.3), can be written in operatorial form

$$\widetilde{\mathcal{H}} = i\omega^{ab}(\partial_b H)\partial_a - i\partial_b(\omega^{ac}\partial_c H)c^b \frac{\partial}{\partial c^a} \tag{2.9}$$

where we have taken the convention of ordering derivative operators to the right. Phase space p-forms, propagating according to (2.7), then obey the "Schroedinger-like"[1] equation

$$i\partial_t \tilde{\varrho} = \widetilde{\mathcal{H}} \tilde{\varrho} \tag{2.10}$$

Recognizing in $\widetilde{\mathcal{H}}$ the combination

$$h^a = \omega^{ac} \partial_c H \tag{2.11}$$

as the components of the Hamiltonian vector field† $h = (dH)^\sharp$, we see that $\widetilde{\mathcal{H}}$ acts as a Lie derivative l_h (along a vector field h) on p-forms

$$\begin{aligned}(l_h \tilde{\varrho})_{a_1 \cdots a_p} &= h^b \partial_b \tilde{\varrho}_{a_1 \cdots a_p} + (\partial_{a_1} h^b) \tilde{\varrho}_{b a_2 \cdots a_p} \\ &\quad + (\partial_{a_2} h^b) \tilde{\varrho}_{a_1 b a_3 \cdots a_p} + \cdots \end{aligned} \tag{2.12}$$

so

$$\begin{aligned}\widetilde{\mathcal{H}} &= -il_h \\ &= -i\Big[h^a \partial_a + (\partial_b h^a) c^b \frac{\partial}{\partial c^a}\Big]\end{aligned} \tag{2.13}$$

† We use the notation of Abraham and Marsden of ref.[6].

The Schroedinger-like equation (2.10) is therefore just the Liouville equation

$$\partial_t \tilde{\varrho} = -l_h \tilde{\varrho} \tag{2.14}$$

generalized to p-forms.

We now wish to show that the path-integral Lagrangian $\tilde{\mathcal{L}}$ is a scalar under infinitesimal canonical transformations

$$\phi^a \to \phi'^a = \phi^a - \varepsilon^a(\phi) \tag{2.15}$$

where $\varepsilon^a(\phi)$ are components of a vector field $\varepsilon(\phi)$ along which the Lie derivative of the symplectic form $\omega = \frac{1}{2}\omega_{ab} d\phi^a \wedge d\phi^b \equiv \frac{1}{2}\omega_{ab} c^a c^b$ vanishes:

$$l_\varepsilon \omega = 0 \tag{2.16}$$

Locally this vector field is given in terms of a generating function $G(\phi, t)$ by

$$\varepsilon^a(\phi, t) = \omega^{ab}(\phi)\partial_b G(\phi, t) \tag{2.17}$$

Under the transformation (2.15), we have*

$$\begin{aligned}
H &\to H'(\phi') = H(\phi) - \frac{\partial G}{\partial t} \\
h^a &\to h'^a(\phi') = h^a(\phi) - h^b \partial_b \varepsilon^a - \partial_t \varepsilon^a \\
\lambda_a &\to \lambda'_a = \lambda_a + \lambda_b \partial_a \varepsilon^b \\
c^a &\to c'^a = c^a - c^b \partial_b \varepsilon^a \\
\bar{c}_a &\to \bar{c}'_a = \bar{c}_a + \bar{c}_b \partial_a \varepsilon^b \\
\dot{\phi}^a &\to \dot{\phi}'^a = \dot{\phi}^a - \dot{\phi}^b \partial_b \varepsilon^a - \partial_t \varepsilon^a \\
\dot{c}^a &\to \dot{c}'^a = \dot{c}^a - \dot{c}^b \partial_b \varepsilon^a - \dot{\phi}^e c^b \partial_e \partial_b \varepsilon^a
\end{aligned} \tag{2.18}$$

Inserting these expressions into the Lagrangian $\tilde{\mathcal{L}}'(\phi') = \lambda'_a \dot{\phi}'^a + i\bar{c}'_a \dot{c}'^a - \widetilde{\mathcal{H}}'$, we‡

* See appendix A for details.

obtain:

$$\tilde{\mathcal{L}}'(\phi') = \tilde{\mathcal{L}}(\phi) + i\bar{c}_a c^b \partial_b[(h^e - \dot{\phi}^e)\partial_e \varepsilon^a] \tag{2.19}$$

and, when the classical equation of motion $\dot{\phi}^e = h^e$ are satisfied, we get

$$\tilde{\mathcal{L}}'(\phi') = \tilde{\mathcal{L}}(\phi)$$

Therefore, the path-integral action $\tilde{S}$ is invariant under canonical transformations.

3. NEW anti-BRS CHARGE.

Let us now turn our attention to the generators of the *ISp(2)* algebra in eq.(1.4). The symplectic two-form $K = \frac{1}{2}\omega_{ab}c^a c^b$ and symplectic bivector $\bar{K} = \frac{1}{2}\omega^{ab}\bar{c}_a\bar{c}_b$ are invariant geometric objects which should retain their expressions when ω_{ab} and ω^{ab} depend on ϕ. The ghost charge $Q_g = c^a\frac{\partial}{\partial c^a}$ and the BRS charge $Q = c^a\frac{\partial}{\partial \phi^a}$ are independent of ω, so they shall not change when the components of ω depend on ϕ. The anti-BRS charge $\bar{Q} = \omega^{ab}\frac{\partial}{\partial c^a}\frac{\partial}{\partial \phi^b}$, on the other hand, will have to be modified when ω^{ab} depends on ϕ, since it is no longer nilpotent and does not commute with the super-Hamiltonian $\widetilde{\mathcal{H}}$. In order to find a suitable $\bar{Q}$, we take advantage of the commutators of the *ISp(2)*-algebra (1.6), assuming that this symmetry is intrinsic to the CPI on a general phase space manifold. In particular, we have $\bar{Q} = [\bar{K}, Q]$, and this relation is the one which gives to $\bar{Q}$ its meaning as an *exterior co-derivative*[1] . So if we want to maintain this geometrical meaning of $\bar{Q}$, we have to keep the above relation. It is easily checked that Q and $\bar{K}$ both commute with $\widetilde{\mathcal{H}}$, and therefore, by the Jacobi identity, their commutator $\bar{Q}$ should also

‡ $\widetilde{\mathcal{H}}'$ is given by (2.8) with the un-primed variables replaced by the primed ones

be conserved. Evaluating this commutator[♮], we obtain

$$\bar{Q} = i\omega^{ab}\bar{c}_a\lambda_b - \frac{1}{2}\frac{\partial\omega^{ab}}{\partial\phi^e}c^e\bar{c}_a\bar{c}_b \tag{3.1}$$

which becomes in the representation (3):

$$\bar{Q} = \omega^{ab}\frac{\partial}{\partial c^a}\partial_b - \frac{1}{2}\frac{\partial\omega^{ab}}{\partial\phi^e}c^e\frac{\partial}{\partial c^a}\frac{\partial}{\partial c^b} \tag{3.2}$$

With the help of the identity $\omega^{ae}\frac{\partial\omega^{bc}}{\partial\phi^e} + cyc.\ perm.\ of\ (a,b,c) = 0$, which follows from $d\omega = 0$, one can show that the anti-BRS charge (3.1) is nilpotent and (anti-)commutes with the BRS charge[◇]

$$[\bar{Q},\bar{Q}] = [\bar{Q},Q] = 0$$

In Ref.[1] we showed that, when ω^{ab} is constant, the anti-BRS charge acts as an exterior co-derivative on p-vectors

$$V_{(p)}(\phi,\bar{c}) = \frac{1}{p!}V^{a_1\cdots a_p}(\phi)\bar{c}_{a_1}\cdots\bar{c}_{a_p} \tag{3.3}$$

We now wish to show that the new anti-BRS charge (3.1) retains this role for $\omega^{ab}(\phi)$.

The exterior co-derivative is obtained[6] as follows. First we use the symplectic structure to associate with the p-vector $V_{(p)}$ the p-form

$$V^{\flat}_{(p)} = \frac{1}{p!}\omega_{f_1g_1}\cdots\omega_{f_pg_p}V^{g_1\cdots g_p}\, c^{f_1}\ldots c^{f_p} \tag{3.4}$$

then we take the exterior derivative $d = c^e\partial_e$ of the resulting p-form

$$dV^{\flat}_{(p)} = \frac{1}{p!}\partial_e[\omega_{f_1g_1}\cdots\omega_{f_pg_p}\, V^{g_1\cdots g_p}]c^e c^{f_1}\ldots c^{f_p} \tag{3.5}$$

Finally, we use the canonical correspondence[1][6] between forms and vectors to

♮ See appendix B for details.

◇ See Appendix B.

obtain the exterior co-derivative of $V_{(p)}$

$$
\begin{aligned}
(dV^{\flat}_{(p)})^{\sharp} &= \frac{1}{p!}\omega^{ae}\omega^{a_1f_1}\cdots\omega^{a_pf_p}\partial_e[\omega_{f_1g_1}\cdots\omega_{f_pg_p}V^{g_1\cdots g_p}]\bar{c}_a\bar{c}_{a_1}\cdots\bar{c}_{a_p} \\
&= \frac{1}{p!}[\omega^{ae}\partial_e V^{a_1\cdots a_p} + p\omega^{ae}\omega^{a_1f_1}\frac{\partial\omega_{f_1g_1}}{\partial\phi^e}V^{g_1a_2\cdots a_p}]\bar{c}_a\bar{c}_{a_1}\cdots\bar{c}_{a_p}
\end{aligned}
\tag{3.6}
$$

Calculating the graded commutator of the anti-BRS charge (3.1) with the p-vector (3.3), we find

$$
[\bar{Q}, V_{(p)}] = \frac{1}{p!}[\omega^{ae}\partial_e V^{a_1\cdots a_p} - \frac{p}{2}\frac{\partial\omega^{aa_1}}{\partial\phi^{g_1}}V^{g_1a_2\cdots a_p}]\bar{c}_a\bar{c}_{a_1}\cdots\bar{c}_{a_p} \tag{3.7}
$$

The right hand sides of (3.6) and (3.7) are equivalent because of the identity

$$
\omega^{ae}\omega^{a_1f_1}\frac{\partial\omega_{f_1g_1}}{\partial\phi^e}\bar{c}_a\bar{c}_{a_1} = -\frac{1}{2}\frac{\partial\omega^{aa_1}}{\partial\phi^{g_1}}\bar{c}_a\bar{c}_{a_1} \tag{3.8}
$$

and therefore we have proven that

$$
[\bar{Q}, V_{(p)}] = (dV^{\flat}_{(p)})^{\sharp} \tag{3.9}
$$

This means that the new $\bar{Q}$ has the same geometrical meaning as the old one[1] .

4. Anti-BRS COHOMOLOGY

From our previous work[7] , we know already that the cohomology of the BRS operator Q acting on p-form fields $\tilde{\varrho}^{(p)}$ is isomorphic to the deRham cohomology. In the following we compute the cohomology of the new anti-BRS operator $\bar{Q}$. This means that we solve the equations

$$
\bar{Q}\tilde{\varrho}^{(p)}(\phi, c) = 0, \quad \tilde{\varrho}^{(p)} \neq \bar{Q}\tilde{\varrho}^{(p+1)}(\phi, c) \tag{4.1}
$$

where $\bar{Q}$ is given by (30) and where the "Schroedinger state"[1]

$$
\tilde{\varrho}^{(p)}(\phi, c) = \frac{1}{p!}\varrho^{(p)}_{a_1\cdots a_p}(\phi)c^{a_1}\cdots c^{a_p} \tag{4.2}
$$

is considered a function of the (anticommuting) c-numbers c^a rather than an operator. Note that $\bar{Q}$ maps p-forms onto (p-1)-forms. In order to solve eqs.(4.1) , we change the representation[1] of the operator algebra (2). Instead of the "position space representation" (3), we shall use the "momentum space representation" for the ghosts. Then $\bar{c}_a$ is a multiplicative operator and $c^a = \frac{\partial}{\partial \bar{c}_a}$ acts as a derivative. Previously[1] "states" were represented by p-form fields $\tilde{\varrho}^{(p)}(\phi, c)$, now this role is played by the p-vector fields $V_{(p)}(\phi, \bar{c})$ of eq.(3.3)*. The Grassmannian Fourier transform

$$V_{(2n-p)}(\phi, \bar{c}) = \int d^{2n}c \; exp(c^a \bar{c}_a) \tilde{\varrho}^{(p)}(\phi, c) \tag{4.3}$$

establishes a one-to-one correspondence between vector fields and form-valued fields. Therefore the solution of the cohomology problem (4.1) follows from the solution of

$$\bar{Q} V_{(p)} = 0, \; V_{(p)} \neq \bar{Q} V_{(p-1)} \tag{4.4}$$

where the anti-BRS operator is now represented by

$$\bar{Q} = \omega^{ab} \bar{c}_a \partial_b - \frac{1}{2} \partial_f \omega^{ab} \bar{c}_a \bar{c}_b \frac{\partial}{\partial \bar{c}_f} \tag{4.5}$$

This differential operator has the following action on p-vector fields (3.3):

$$\bar{Q} V_{(p)}(\phi, c) = \frac{1}{(p+1)!} \left[(d V^{\flat}_{(p)})^{\sharp} \right]^{a_1 \cdots a_{p+1}} \bar{c}_{a_1} \cdots \bar{c}_{a_{p+1}} \tag{4.6}$$

This equation is the "Schroedinger picture" analogue of eq.(3.9) in the "Heisenberg picture" where $V_{(p)}$ was considered an operator rather than a state. Again we see that, up to the canonical isomorphisms $\sharp$ and $\flat$ between vector fields and forms, the operator $\bar{Q}$ acts like the exterior derivative "d". Therefore the solutions $V_{(p)}$ of eq.(4.4) are obtained by applying the isomorphism $\sharp$ to representatives of deRham cohomology classes. Furthermore, because (4.3) gives rise to a one-to-one map between solutions $V_{(p)}$ of (4.4) and solutions $\tilde{\varrho}^{(p)}$ of (4.1), we conclude that also the original cohomology problem (4.1) is isomorphic to deRham cohomology.

---

* We keep using $\lambda_a \equiv -i\partial_a$ for the bosonic variables.

5. SUPERSYMMETRY, GIBBS STATES AND KMS CONDITION.

In ref.[2] it was shown that every Hamiltonian system has a hidden supersymmetry generated by the supercharges Q_H and $\bar{Q}_H$ of Eq.(1.7). Since the supercharge $\bar{Q}_H$ is derived from the anti-BRS charge $\bar{Q}$, which we have now modified, we must check that the results of Ref.[2] continue to hold when ω^{ab} depends on ϕ. The new Hamiltonian supercharges, constructed according to (1.7), are given explicitly by

$$\begin{aligned} Q_H &= c^a(\partial_a - \beta\frac{\partial H}{\partial\phi^a}) \\ \bar{Q}_H &= \omega^{cd}\bar{c}_c(\partial_d + \beta\frac{\partial H}{\partial\phi^d}) - \frac{1}{2}\frac{\partial\omega^{gh}}{\partial\phi^f}c^f\bar{c}_g\bar{c}_h \end{aligned} \tag{5.1}$$

Calculating the anticommutator of these two supercharges, we obtain

$$\begin{aligned} [Q_H, \bar{Q}_H] &= 2\beta[\omega^{ab}\frac{\partial H}{\partial\phi^b}\partial_a + c^b\frac{\partial}{\partial\phi^b}(\omega^{ac}\frac{\partial H}{\partial\phi^c})\frac{\partial}{\partial c^a}] \\ &= 2i\beta\widetilde{\mathcal{H}} \end{aligned} \tag{5.2}$$

while their nilpotence is guaranteed by the nilpotence of Q and $\bar{Q}$, and by (1.7). Since their anticommutator closes on the super-Hamiltonian, $\bar{Q}_H$ and Q_H are genuine supersymmetry generators. Physical states which are invariant under this supersymmetry must be annihilated by both supercharges. If we are concerned[2] with calculating expectation values of scalar observables $A(\phi)$, we need only consider physical states given by 2n-form distributions

$$\tilde{\varrho}^{(2n)}(\phi, c) = \varrho(\phi)\, c^1 \cdots c^{2n} \tag{5.3}$$

because only then will the expectation value $\langle A\rangle_{\tilde{\varrho}} = \int d^{2n}\phi\, d^{2n}c\, A(\phi)\tilde{\varrho}(\phi, c)$ be nonvanishing. Any 2n-form will be annihilated by Q_H , since it involves multiplication by an extra ghosts. Invariance under the other supersymmetry generator $\bar{Q}_H$, on the other hand, leads to the nontrivial condition

$$\bar{Q}_H \tilde{\varrho}^{(2n)} = \frac{1}{(2n-1)!} \varepsilon_{a_1 \cdots a_{2n}} c^{a_2} \cdots c^{a_{2n}} (\partial_b + \beta \frac{\partial H}{\partial \phi^b})(\omega^{a_1 b} \varrho) = 0 \tag{5.4}$$

Therefore the phase space density distribution ϱ must satisfy the equation:

$$(\partial_b + \beta \partial_b H)[\omega^{ab}(\phi)\varrho(\phi)] = 0 \tag{5.5}$$

In ref.[2] it was shown that supersymmetric invariant states satisfying (5.4) with constant ω^{ab} are precisely the Gibbs states. In case ω^{ab} are not constant, a solution of the equation above is

$$\varrho(\phi) = \frac{1}{2n} \omega_{ab} K^{ab} exp(-\beta H) \tag{5.6}$$

where K^{ab} is a constant matrix. Due to this constant matrix K^{ab}, the solutions (5.6) is not a scalar density as it should be if we want the $\tilde{\varrho}^{(2n)}$ of eqn.(5.3) to transform as a 2n-form.* The solutions of eq. (5.5) that are scalar densities are† :

$$\varrho(\phi) = k[det\{\omega_{ab}(\phi)\}]^{\frac{1}{2}} \; exp\{-\beta H(\phi)\} \tag{5.7}$$

where k is a constant. The origin of the determinant in eq. (5.7) is easy to understand. Recall that the Liouville measure on phase-space is given by the volume form $\omega^n \equiv (\frac{1}{2}\omega_{ab}(\phi) d\phi^a \wedge \phi^b)^n$, which translates into $K^n \equiv (\frac{1}{2}\omega_{ab} c^a c^b)^n$ in

---

* The state (5.6) is only *locally* a Gibbs state. In fact, thanks to the Darboux theorem, we can *locally* bring ω to the standard constant off-diagonal form, so that the factors in front of (5.6) becomes just over-all costants. Globally anyhow (5.6) is not a Gibbs state because the dependence on ϕ is not brought in only by H, as it should be in any Gibbs state, but also by ω.

† For the details of the derivation see the appendix C.

our formalism. Because of the identity*

$$K^n = n![det\{\omega_{ab}(\phi)\}]^{\frac{1}{2}}\ c^1c^2\cdots c^{2n} \tag{5.8}$$

the 2n-form obtained by inserting (5.7) into (5.3) can be written as

$$\tilde{\varrho}^{(2n)}(\phi,c) = k' exp\{-\beta H(\phi)\}\ K^n \tag{5.9}$$

Thus, using the proper (ϕ-dependent) volume form K^n, the supersymmetric invariant states are again characterized by a simple Boltzmann factor $exp(-\beta H)$ and so they are true Gibbs states. Note also that it is K^n, rather than $c^1c^2\cdots c^{2n}$, which is invariant under the Hamiltonian flow. In fact, using eq. (2.13) , it is easy to prove the slightly more general statement:

$$\widetilde{\mathcal{H}}K^m = 0,\ m = 1,2\cdots n \tag{5.10}$$

which embodies the conservation of Poincaré's integral invariants.

In this last part, we will follow ref.[2] to derive the KMS conditions[5] . Let us calculate, for two observables $A_1(\phi)$,$A_2(\phi)$, the following expression

$$\int d^{2n}\phi\ d^{2n}c\ A_1(\phi;0)[Q_H A_2(\phi;t)]\bar{Q}_H\tilde{\varrho}^{(2n)}(\phi,c) \tag{5.11}$$

If we requeire that the state $\tilde{\varrho}$ is of the form (5.3) and supersymmetric invariant, then the expression (5.11) above is zero. We will now check that, *even with the new* $\bar{Q}_H$ inserted in (5.11), this will lead to the KMS condition. Let us first

* See the appendix C for details.

write (5.11) explicitly

$$
\begin{aligned}
&\int d^{2n}\,\phi\;\; d^{2n}c\; A_1(\phi;0)c^a[(\partial_a-\beta\partial_aH)A_2(\phi;t)]\cdot\\
&\qquad\cdot[\omega^{eb}(\partial_b+\beta\partial_bH)\frac{\partial}{\partial c^e}-\frac{1}{2}\frac{\partial\omega^{fg}}{\partial\phi^b}c^b\frac{\partial}{\partial c^f}\frac{\partial}{\partial c^g}]\cdot\\
&\qquad\cdot\tilde{\varrho}^{2n}(\phi,c)=\\
&=\int d^{2n}\,\phi\;\;\varrho(\phi)[\omega^{ab}\frac{\partial A_1(\phi;0)}{\partial\phi^a}\frac{\partial A_2(\phi;t)}{\partial\phi^b}+\beta\omega^{ab}\frac{\partial H}{\partial\phi^a}\frac{\partial A_1(\phi;0)}{\partial\phi^b}A_2(\phi;t)]
\end{aligned}
\tag{5.12}
$$

thus leading to the classical KMS condition[5]

$$
\langle\{A_1(0),H\}A_2(t)\rangle_\varrho=\frac{1}{\beta}\langle\{A_1(0),A_2(t)\}\rangle_\varrho
$$

or equivalently

$$
\left\langle\dot{A}_1(0)A_2(t)\right\rangle_\varrho=\frac{1}{\beta}\langle\{A_1(0),A_2(t)\}\rangle_\varrho
\tag{5.13}
$$

In the above calculation an extra term from $\bar{Q}_H$ cancels with a term coming from the derivative of ω^{ab} leading to the result previously obtained[2] for a constant symplectic form.

6. CONCLUSIONS

A lot of work remains to be done using this formalism. In particular we would like to see what is the form that supersymmetric invariant states have if they are not in the 2n-ghost sector. For sure they will not be Gibbs state but their form might be as universal and important as the Gibbs states. Second we would like to see what the KMS condition becomes for these new states of ghost number $\neq 0$.

Regarding the KMS condition, we would also like to see if it can be derived in the same way for systems with an infinite number of degrees of freedom, where this condition was originally proposed.

Lastly, we would like to make contact between our formalism and the KMS-functionals and non-commutative-geometry formalism proposed by D.Kastler and A.Jaffe[8]

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APPENDIX A

In this appendix we will compile some information regarding canonical transformations. We do this to make the paper self-contained even if all we are going to write is by now standard knowledge[6] .

Let an arbitrary infinitesimal transformation of ϕ be of the form

$$\phi'^a = \phi^a - \epsilon^a(\phi) \tag{A1}$$

The symplectic form ω, being a 2-form, is invariant under arbitrary coordinate transformations:

$$\begin{aligned} \omega'_{ab}(\phi')d\phi'^a \wedge d\phi'^b &= \omega_{ef}(\phi)d\phi^e \wedge d\phi^f \\ \omega'_{ab}(\phi')\frac{\partial\phi'^a}{\partial\phi^e}\frac{\partial\phi'^b}{\partial\phi^f}d\phi^e \wedge d\phi^f &= \omega_{ef}(\phi)d\phi^e \wedge d\phi^f \end{aligned} \tag{A2}$$

From (A2) we get immediately

$$\omega'_{ab}(\phi') = \frac{\partial\phi^e}{\partial\phi'^a}\frac{\partial\phi^f}{\partial\phi'^b}\omega_{ef}(\phi) \tag{A3}$$

Let us remember[6] that the Lie derivative $l_\epsilon\omega$ of ω along the vector field with components ε^e, is defined as

$$l_\varepsilon \omega_{ab} = \omega'_{ab}(\phi) - \omega_{ab}(\phi)$$

Using (A3) and (A1) it is easy to find that

$$\omega'_{ab}(\phi) = \omega_{ab}(\phi) + \varepsilon^e \partial_e \omega_{ab} + \partial_a \varepsilon^e \omega_{eb} + \partial_b \varepsilon^e \omega_{ae} + O(\varepsilon^2) \tag{A4}$$

so

$$l_\varepsilon \omega_{ab} = \varepsilon^e \partial_e \omega_{ab} + (\partial_a \varepsilon^e) \omega_{eb} + (\partial_b \varepsilon^e) \omega_{ae} \tag{A5}$$

The *canonical transformations* (2.16) are those that leave invariant the coefficients of the symplectic two form:

$$l_\varepsilon \omega_{ab} = 0 \tag{A6}$$

If we remember the abstract definition of theLie-derivative* [6]

$$l_\varepsilon \omega = i_\varepsilon d\omega + d i_\varepsilon \omega \tag{A7}$$

and the fact that $d\omega = 0$, we get from (A7) and (A6) that $d(i_\varepsilon \omega) = 0$. This means that $i_\varepsilon \omega$ is a closed form and locally $i_\varepsilon \omega = dG$. In components this can be written as

$$\varepsilon^h \omega_{hf} = \partial_f G$$

this gives eq.(2.17):

$$\varepsilon^e = (\partial_f G) \omega^{fe} \tag{A8}$$

With the explicit form for ε we can now find the transformations (2.18). Let us remember[1] that the λ and $\bar{c}$ are a basis of $T_\phi \mathcal{M}$, while the c are a basis

* With i_ε we indicate the interior product with the vector field ε.

for $T^*_\phi \mathcal{M}$ †.

$$
\begin{aligned}
\lambda'_a &= \frac{\partial \phi^b}{\partial \phi'^a}\lambda_b = \left[\delta^b_a + \partial_a(\omega^{bc}\partial_c G)\right]\lambda_b \\
\bar{c}'_a &= \frac{\partial \phi^b}{\partial \phi'^a}\bar{c}_b = \left[\delta^b_a + \partial_a(\omega^{bc}\partial_c G)\right]\bar{c}_b \\
c'^a &= \frac{\partial \phi'^a}{\partial \phi^b}c^b = \left[\delta^a_b - \partial_b(\omega^{ac}\partial_c G)\right]c^b \\
H'(\phi') &= H(\phi) - \partial_t G
\end{aligned}
\tag{A9}
$$

The last transformation derives from the well-known requirement[6] that Hamilton's equation of motion keep the same form under canonical transformations. Using the transformations above with functions G which do not explicitly dependent on t, it is easy to see that

$$
\widetilde{\mathcal{H}}'(\phi') = \widetilde{\mathcal{H}}(\phi) + ic^b\bar{c}_a\omega^{fc}(\partial_c H)\partial_b\partial_f(\omega^{ae}\partial_e G) + O(G^2)
$$

Using this, let us now calculate $\widetilde{\mathcal{L}}'(\phi')$:

$$
\begin{aligned}
\widetilde{\mathcal{L}}' &= \frac{\partial \phi^b}{\partial \phi'^a}\lambda_b\frac{\partial \phi'^a}{\partial \phi^c}\dot{\phi}^c - i\dot{c}^a\bar{c}_a - i\frac{\partial^2\phi'^a}{\partial\phi^f\partial\phi^e}\dot{\phi}^f c^e\frac{\partial\phi^b}{\partial\phi'^a}\bar{c}_b - \widetilde{\mathcal{H}}' = \\
&= \lambda_a\dot{\phi}^a - i\dot{c}^a\bar{c}_a + i\frac{\partial^2(\omega^{ac}\partial_c G)}{\partial\phi^f\partial\phi^e}\dot{\phi}^f c^e\bar{c}_a - \widetilde{\mathcal{H}} - ic^b\bar{c}_a\omega^{fe}\partial_e H\frac{\partial^2(\omega^{ac}\partial_c G)}{\partial\phi^b\partial\phi^f} \\
&= \lambda_a\dot{\phi}^a - i\dot{c}^a\bar{c}_a - \widetilde{\mathcal{H}} + ic^b\bar{c}_a\frac{\partial^2(\omega^{ac}\partial_c G)}{\partial\phi^f\partial\phi^b}(\dot{\phi}^f - \omega^{fe}\partial_e H)
\end{aligned}
\tag{A10}
$$

The last piece in the equation above is zero because of the equation of motion, so we get that

$$
\widetilde{\mathcal{L}}'(\phi') = \widetilde{\mathcal{L}}(\phi)
$$

† $\bar{c}_a$ are covariant while c^a are controvariant.

APPENDIX B

In this appendix we present details of the calculations done to obtain the expression (3.1) for $\bar{Q}$, and to check that $[Q,\bar{Q}]=0$.

As we said in the paper, we use the 5th relation of equation (1.6) to derive $\bar{Q}$, that is

$$[\bar{K},Q]=\bar{Q}$$

Writing the charges in the operatorial form (1.4), we get

$$\begin{aligned}
[\bar{K},Q] &= \frac{1}{2}[\omega^{ab}\frac{\partial}{\partial c^a}\frac{\partial}{\partial c^b}, c^e\frac{\partial}{\partial \phi^e}] \\
&= \frac{1}{2}\Big(\omega^{ab}\frac{\partial}{\partial \phi^e}\frac{\partial}{\partial c^a}\frac{\partial}{\partial c^b}c^e - \frac{\partial}{\partial \phi^e}\omega^{ab}c^e\frac{\partial}{\partial c^a}\frac{\partial}{\partial c^b}\Big) \\
&= \frac{1}{2}\Big(\omega^{ab}\frac{\partial}{\partial \phi^b}\frac{\partial}{\partial c^a} - \omega^{ab}\frac{\partial}{\partial \phi^a}\frac{\partial}{\partial c^b}\Big) - \frac{1}{2}\frac{\partial \omega^{ab}}{\partial \phi^e}c^e\frac{\partial}{\partial c^a}\frac{\partial}{\partial c^b} \\
&= \omega^{ab}\frac{\partial}{\partial c^a}\frac{\partial}{\partial \phi^b} - \frac{1}{2}\frac{\partial \omega^{ab}}{\partial \phi^e}c^e\frac{\partial}{\partial c^a}\frac{\partial}{\partial c^b} \doteq \bar{Q}
\end{aligned} \tag{B1}$$

Let us now show that

$$[Q,\bar{Q}]=0$$

$$\begin{aligned}
[Q,\bar{Q}] &= [c^a\frac{\partial}{\partial \phi^a}, \omega^{be}\frac{\partial}{\partial c^b}\frac{\partial}{\partial \phi^e} - \frac{1}{2}\frac{\partial \omega^{gh}}{\partial \phi^f}c^f\frac{\partial}{\partial c^g}\frac{\partial}{\partial c^h}] \\
&= c^a\frac{\partial}{\partial \phi^a}\omega^{be}\frac{\partial}{\partial c^b}\frac{\partial}{\partial \phi^e} + \omega^{be}\frac{\partial}{\partial c^b}\frac{\partial}{\partial \phi^e}c^a\frac{\partial}{\partial \phi^a} + \\
&\quad - \frac{1}{2}c^a c^f\frac{\partial}{\partial c^g}\frac{\partial}{\partial c^h}\frac{\partial}{\partial \phi^a}\frac{\partial \omega^{gh}}{\partial \phi^f} - \frac{1}{2}\frac{\partial \omega^{gh}}{\partial \phi^f}\frac{\partial}{\partial \phi^a}c^f\frac{\partial}{\partial c^g}\frac{\partial}{\partial c^h}c^a \\
&= \frac{\partial \omega^{be}}{\partial \phi^a}\frac{\partial}{\partial \phi^e}c^a\bar{c}_b - \frac{1}{2}\frac{\partial \omega^{gh}}{\partial \phi^f}\frac{\partial}{\partial \phi^h}c^f\bar{c}_g + \frac{1}{2}\frac{\partial \omega^{gh}}{\partial \phi^f}\frac{\partial}{\partial \phi^g}c^f\bar{c}_h \\
&= c^a\bar{c}_b\Big(\frac{\partial \omega^{be}}{\partial \phi^a}\frac{\partial}{\partial \phi^e} - \frac{1}{2}\frac{\partial \omega^{be}}{\partial \phi^a}\frac{\partial}{\partial \phi^e} - \frac{1}{2}\frac{\partial \omega^{be}}{\partial \phi^a}\frac{\partial}{\partial \phi^e}\Big) \\
&=0
\end{aligned} \tag{B2}$$

In the same way, but with a much longer calculation, it is possible also to prove that

$$[\bar{Q},\bar{Q}]=0$$

and that

$$[\bar{Q},\widetilde{\mathcal{H}}]=0$$

In both calculations crucial use is made of the identity

$$\omega^{ae}\frac{\partial\omega^{bc}}{\partial\phi^{e}}+\omega^{be}\frac{\partial\omega^{ca}}{\partial\phi^{e}}+\omega^{ce}\frac{\partial\omega^{ab}}{\partial\phi^{e}}=0 \tag{B3}$$

It is thanks to this identity that the many more terms present here ,in comparison to the few terms present in the case[1] of constant ω^{ab}, get either cancelled or balanced with other terms to produce the same algebra as before[1] .

APPENDIX C

In this appendix we would like to present the details of the derivation of the two equations: (5.8), (5.9). K is the symplectic 2-form:

$$K=\frac{1}{2}\omega_{ab}c^{a}c^{b}=\frac{1}{2}\omega_{ab}d\phi^{a}\wedge d\phi^{b} \tag{C1}$$

Let us write the volume form $\omega^{n}=K^{n}$ as

$$K^{n}=F(\phi)\delta(c)\equiv F(\phi)c^{1}c^{2}\cdots c^{2n} \tag{C2}$$

and now let us determine the function $F(\phi)$. From the expression above we have

$$\begin{aligned}F(\phi)&=\int d^{2n}c\,K^{n}=(n)!\int d^{2n}c\,e^{K}\\&=n!\int d^{2n}c\,e^{c^{a}\omega_{ab}c^{b}}=n!pf(\omega)=\pm n![det(\omega_{ab})]^{\frac{1}{2}}\end{aligned} \tag{C3}$$

This is eqn. (5.8), where $pf(\omega)$ is the Pfaffian of ω_{ab}, whose square is well known to coincide with $det(\omega_{ab})$.

Let us now solve eq.(5.5).

$$(\partial_b + \beta\partial_b H)[\omega^{ab}(\phi)\varrho(\phi)] = 0 \tag{C4}$$

A solution, presented in (5.6) , can be obtained identifying the quantity in the square brackets of (C4) that is $\omega^{ab}(\phi)\varrho(\phi)$, with a constant K^{ab} multiplied by $exp(-\beta H)$

$$\omega^{ab}(\phi)\varrho(\phi) = K^{ab} exp(-\beta H)$$

which leads to (5.6):

$$\varrho(\phi) = \frac{1}{2n}\omega_{bd}K^{db} exp(-\beta H)$$

This solution, anyhow, is not covariantly correct in the sense that K^{db} being a constant does not transform as the components of a 2-vector under canonical transformations. Another manner to solve eq. (C4) is by making the ansatz $\varrho(\phi) = f(\phi) exp(-\beta H)$ and inserting it into (C4) to determine $f(\phi)$. The equation that one gets is:

$$\omega^{ab}\frac{\partial_b f}{f} + \partial_b \omega^{ab} = 0$$

We can proceed to solve it as follows :

$$\begin{aligned}
\delta^b_e \frac{\partial_b f}{f} + \omega_{ea}\partial_b\omega^{ab} &= 0 \\
\frac{\partial_e f}{f} \equiv \partial_e(lnf) &= +\,\omega_{ea}\partial_b\omega^{ba} \\
&= \omega^{bf}\partial_b\omega_{fe} \\
&= \frac{1}{2}\omega^{bf}[\partial_b\omega_{fe} + \partial_f\omega_{eb}] \\
&= \frac{1}{2}\omega^{bf}\partial_e\omega_{fb} \\
&= \frac{1}{2}tr(\omega^{-1}\partial_e\omega) \\
&= \frac{1}{2}\partial_e trln(\omega) \\
&= \partial_e ln\left(det\omega\right)^{\frac{1}{2}}
\end{aligned} \tag{C5}$$

where we used the fact that $d\omega = 0$. So we get:

$$f(\phi) = k[det\omega_{ab}(\phi)]^{\frac{1}{2}}$$

which generates the solution (5.7) and consequently it gives rise to the *covariantly correct* solution (5.9). k in the equation above is a constant.

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