ISTITUTO NAZIONALE DI FISICA NUCLEARE

Sezione di Trieste

E. Gozzi, M. Reuter and W.D. Thacker
ON THE TODA CRITERION

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E.Gozzi[*] , M.Reuter[†] and W.D.Thacker[*]

ABSTRACT

Using the path-integral formulation of *classical* mechanics, we present a further study of the Toda Criterion. This is an *approximate* criterion to detect *local* transitions from ordered to stochastic/ergodic motion or viceversa. We analyze the criterion by studying those minima of the classical path-integral weight that are not invariant under a *universal* supersymmetry present in any classical Hamiltonian system. This analysis relies on a theorem, that we previously proved, which says that systems which are in a phase with this supersymmetry un-broken are also in the ergodic phase, while systems which are in a phase characterized by ordered motion always have, in that phase, the supersymmetry broken. This study confirms that the Toda Criterion is neither a sufficient nor a necessary condition for the transition from ordered to stochastic motion. In the conclusions some ideas are put forward to find a true criterion based on our supersymmetry.

★ Dipartimento di Fisica Teorica, Universita' di Trieste, Italy and INFN sezione di Trieste.

† Institute for Theoretical Physics, Hannover University, Appelstrasse 2, D-3000 Hannover 1, Germany

∗ Dept. of Science and Mathematics, Parks College, Saint Louis University, Cahokia, Il 62206, USA

1. A REVIEW OF THE PATH-INTEGRAL APPROACH TO CLASSICAL MECHANICS.

In 1931, just after the introduction of quantum mechanics, Koopman[1] and von Neumann[2] proposed an *operatorial* approach to classical mechanics (CM) to better compare it with quantum mechanics (QM). This suggests that CM can also be formulated via path-integrals.

A first attempt, at least to our knowledge, to provide such a formulation of classical mechanics has been presented in a recent paper[3].

Let us now review this method. We start from Hamilton's equations:

$$\dot{\phi}^a(t) = \omega^{ab}\partial_b H\big(\phi(t)\big) \tag{1.1}$$

where $\phi^a \equiv \big(q^1, \cdots, q^n, p_1, \cdots, p_n\big)$, $a = 1, \cdots, 2n$, is a coordinate on a $2n$-dimensional phase-space $\mathcal{M}_{2n}$, H is the Hamiltonian and $\omega^{ab} = -\omega^{ba}$ is the standard symplectic matrix. Another important concept we shall need is that of a probability density function $\varrho(\phi^a, t)$ on phase space. The time evolution of these distributions is given by

$$\frac{\partial}{\partial t}\varrho(\phi^a, t) = -\big\{\varrho, H\big\} \equiv -\hat{L}\varrho(\phi^a, t) \tag{1.2}$$

Here we have introduced the Liouville operator $\hat{L} = -\partial_a H\omega^{ab}\partial_b$ which is the central element of the operatorial approach to classical mechanics[1],[2]. Equation (1.2) is formally solved as

$$\varrho(\phi, t) = e^{-\hat{L}t}\varrho(\phi, 0) \tag{1.3}$$

As there is an operatorial approach, there must exist also a corresponding path-integral formulation of classical mechanics. The simplest idea is to write a classical

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generating functional of the form:[‡]

$$Z_{cm} = \int \mathcal{D}\phi^a \tilde{\delta}[\phi^a - \phi^a_{cl}] \tag{1.4}$$

where ϕ^a_{cl} are the classical solutions of Hamilton's equations. The delta-functional is forcing the system to lie on its classical trajectories and it can be rewritten as:

$$\tilde{\delta}\left[\phi^a - \phi^a_{cl}\right] = \tilde{\delta}\left[\dot{\phi}^a - \omega^{ab}\partial_b H\right] det\left(\partial_t \delta^a_b - \omega^{ac}\partial_c\partial_b H\right) \tag{1.5}$$

We can now Fourier transform the delta function on the RHS of (1.5), using an auxiliary field λ_a , and exponentiate the determinant via anticommuting variables $c^a, \bar{c}_a$. Thus we arrive at

$$Z_{cm} = \int \mathcal{D}\phi^a \mathcal{D}\lambda_a \mathcal{D}c^a \mathcal{D}\bar{c}_a \; exp \; i \int \; dt \left\{\tilde{\mathcal{L}}\right\} \tag{1.6}$$

with the Lagrangian[♮]

$$\tilde{\mathcal{L}} = \lambda_a\left[\dot{\phi}^a - \omega^{ab}\partial_b H(\phi)\right] + i\bar{c}_a\left[\partial_t\delta^a_b - \omega^{ac}\partial_c\partial_b H(\phi)\right]c^b \tag{1.7}$$

Note that $\tilde{\mathcal{L}}$ contains only first order derivatives and therefore we can immediately read off the associated Hamilton function

$$\widetilde{\mathcal{H}} = \lambda_a\omega^{ab}\partial_b H + i\bar{c}_a\omega^{ac}\partial_c\partial_b H c^b \tag{1.8}$$

From the path-integral (1.6) we can compute easily the equal-time (anti-) commutators of ϕ^a, λ_a, c^a and $\bar{c}_a$ and, using standard techniques[4], we find that

$$\left\langle\left[\phi^a, \lambda_b\right]\right\rangle = i \; \delta^a_b, \left\langle\left[\bar{c}_b, c^a\right]\right\rangle = \delta^a_b \tag{1.9}$$

All other commutators vanish. In particular, ϕ^a and ϕ^b commute for all values of the indices a and b. In terms of the q's and p's (which were combined into

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‡ We neglect for the moment the source terms.
♮ For more details see ref. [3]

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ϕ^a) this means $\left\langle \left[q^i, p_j \right] \right\rangle = 0$ for all i and j. This shows very clearly that we are doing classical mechanics. The operator algebra (1.9) can be realized by differential operators

$$\lambda_a = -i \frac{\partial}{\partial \phi^a} \; , \; \bar{c}_a = \frac{\partial}{\partial c^a} \tag{1.10}$$

and multiplicative operators ϕ^a and c^a acting on functions $\tilde{\varrho}(\phi^a, c^a, t)$. Inserting the above operators into $\widetilde{\mathcal{H}}$ we obtain:

$$\widetilde{\mathcal{H}} = -i \omega^{ab} \partial_b H \partial_a + i \frac{\partial}{\partial c^a} \omega^{ac} \partial_c \partial_b H c^b \tag{1.11}$$

Looking at (1.11), it is clear that the Grassmannian part of $\widetilde{\mathcal{H}}$ gives zero if applied on distribution $\tilde{\varrho}$ that do not contain anticommuting variables, while the bosonic part is $(-i)$ times the Liouvillian: $\widetilde{\mathcal{H}}|_{(c=0)} = -i\hat{L}$.

This confirms that our path- integral is the right one to reproduce the operatorial approach of Koopman and von Neumann or, stated in a another way, we can say that the measure in the path-integral that produces the Liouville operator is just a Dirac delta. We did, somehow, the analog of what Feynman did for the Schroedinger operator: he asked himself what is the weight in a path-integral that produces the Schroedinger Kernel, and he found that it was $expiS$; for the Liouvillian, instead, it is just a Dirac delta.

The reader at this point may wonder why, to the get the Liouvillan, we had to cut off the Grassmannian part in (1.11). To understand that we have to explain which is the meaning of the full $\widetilde{\mathcal{H}}$ of eqn.(1.11). This is explained in detail in ref.[3] and we refer the reader to that paper for details. There it is explained that the Grassmannian variables c^a can be interpreted as "forms" i.e. $c^a = d\phi^a$, in the cotangent space to phase space, while the $\bar{c}_a$ are a basis in the tangent space to phase-space (i.e. they are a basis in the vector-field-space). The whole Cartan calculus on phase-space (exterior derivative, inner products, etc) has been translated in ref.[3], into a calculus based on these Grassmannian

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variables. It is then easy to prove[3] that $\widetilde{\mathcal{H}}$ is nothing else than the opposite of the lie-derivative[5] $l_{(dH)^\sharp}$ of the Hamiltonian flow$^\diamond$

$$\widetilde{\mathcal{H}} = -il_{(dH)^\sharp}$$

This lie-derivative generates the time-evolution not only of distributions in phase-space $\varrho(\phi^a)$ but also of general distributions $\widetilde{\varrho}(\phi^a, c^a)$ which are forms in phase-space. When we restrict this lie-derivative to act only on $\varrho(\phi^a)$, then it becomes just the Liouvillian.

In the spectrum of the Liouvillian operator are encoded many features of the dynamical system. The one we are most interested in here is that of ergodicity. It is characterized in the following way[7] : a system is ergodic if, at fixed energy, the eigenstate ϱ_o of $\widehat{L}$ with eigenvalue zero is non-degenerate. This means that, for a given energy, the equation

$$\widehat{L}\varrho_o = 0 \tag{1.12}$$

must have one and only solution for the system to be ergodic. Because of eq. (1.2), this solution ϱ_0 is time-independent and so it can only be function of the constants of motion $I_i(\phi^a), (i = 1, 2, \cdots, n)$, $\varrho_o = F(I_i)$ where $\{H, I_i\} = 0$. If the system is completely integrable, for instance, it has n constants of motion (including the energy). If, on the other side, the system is ergodic, the only analytic constant of motion is the energy $E = H(\phi^a)$, so that $\varrho_o = \varrho_o(E)$. For a fixed energy this is a non-degenerate function: it is just a constant[6] on the energy hyper-surface. If instead there exists a further constant of motion $I(\phi^a)$, we may build also the solution $\varrho_o = F(E, I(\phi))$, which is in general not constant for fixed E . Any F of the form above, normalizable on the energy hyper-surface,

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$\diamond$ Here we have used the notation[5] $(dH)^\sharp \equiv \omega^{ab}\partial_b H \partial_a$ for the Hamiltonian vector field generated by the gradient of H, and l_v denotes the lie-derivative along some vector field v . For the reader not familiar with this notation we recommend , as a quick way to learn[6] it, the appendix of ref.[6].

is a zero-eigenstate of the Liouvillian and therefore the system is not ergodic because all these are degenerate zero-eigenstates of the Liouvillian. Of course it is not necessary to have analytic constants of motion to lose ergodicity, it is enough (like for KAM systems) to have invariant[•] tori[∇] on the energy surface. The characteristic-functions of these tori will be the zero-eigenstates now of the Liouvillian degenerate with the constant function.

Let us now discuss the symmetries of $\tilde{S} = \int \tilde{\mathcal{L}} dt$. In ref.[3] we found that $\tilde{S}$ is invariant under the transformations generated by the following $ISp(2)$ charges:

$$
\begin{aligned}
Q &= ic^a \lambda_a, \\
\bar{Q} &= i\bar{c}_a \omega^{ab} \lambda_b \\
Q_g &= c^a \bar{c}_a \\
K &= \frac{1}{2}\omega_{ab} c^a c^b, \\
\bar{K} &= \frac{1}{2}\omega^{ab} \bar{c}_a \bar{c}_b
\end{aligned}
\tag{1.13}
$$

We showed in ref.[3] that the generators (1.13) have a deep geometrical meaning related to the symplectic geometry of our manifold. In particular the Q above is nothing else than the exterior derivative on phase-space, while K is the symplectic 2-form, and a similar interpretation can be given for all generators (see ref.[3] for details.) The generators (1.13) are not the only generators commuting[8] with $\tilde{\mathcal{H}}$. Let us consider[9] , for example, the quantities[†]

$$
\begin{aligned}
Q_H &= e^{\beta H} Q \, e^{-\beta H} \\
\bar{Q}_H &= e^{-\beta H} \bar{Q} \, e^{\beta H}
\end{aligned}
\tag{1.14}
$$

It is easy to verify[8] that

$$
\left[Q_H, Q_H \right] = \left[\bar{Q}_H, \bar{Q}_H \right] = 0
\tag{1.15}
$$

• Invariant under the hamiltonian flow.
∇ Or deformed tori.
† β is an arbitrary complex parameter.

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and

$$\left[Q_H, \bar{Q}_H\right] = 2i\beta\widetilde{\mathcal{H}} \equiv 2\beta l_{(dH)^\sharp} \tag{1.16}$$

Here we have used, as before, the notation $(dH)^\sharp \equiv \omega^{ab}\partial_b H \partial_a$ for the Hamiltonian vector fields generated by the gradients of H. Furthermore l_v denotes the Lie derivative along some vector field v[10] [5]. Eq. (1.16) shows that the anticommutator[†] of Q_H and $\bar{Q}_H$ is the super-Hamiltonian and thus these operators, unlike Q and $\bar{Q}$, are genuine *supersymmetry* (susy) generators.

Up to now we have only shown that the action $\tilde{S}$ is invariant under the supersymmetry and under the graded symmetries mentioned above, but we have not addressed the question "Can these symmetries be spontaneously broken?"[11] The question to ask is: "How does the ground state transform? Is it invariant under the same symmetries of the action?"[*] For simplicity, let us assume that we are only interested in such observables "A" which do not depend on the ghosts: $A = A(\phi)$. Then in order to obtain a non-zero expectation value,

$$\langle A \rangle_t = \int d^{2n}\phi \, d^{2n}c \, A(\phi^a) \, \tilde{\varrho}(\phi^a, c^a, t) \tag{1.17}$$

the density $\tilde{\varrho}$ has to contain 2n ghosts: $\tilde{\varrho}(\phi^a, c^a, t) = \varrho(\phi^a, t) \, c^1 \cdots c^{2n}$. As a consequence, the classical average turns out to be of the standard form (without ghosts):

$$\langle A \rangle_t = \int d^{2n}\phi \, A(\phi) \, \varrho(\phi, t) \tag{1.18}$$

Let us now ask the question " What form does the $\tilde{\varrho}$ above has to have to be invariant under the supersymmetry ?" Since the above $\tilde{\varrho}$ contains a maximum number of ghosts it is trivially annihilated by Q_H: $Q_H\tilde{\varrho} = c^a\left(\partial_a - \beta\partial_a H\right)\tilde{\varrho}(\phi, c) = 0$.

[†] We indicate with the square brackets the graded commutators[3,4].

[*] We should remind the reader that supersymmetry can be spontaneously broken even in the case of finite degrees of freedom or finite volume[11].

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Invariance under $\bar{Q}_H$ requires

$$\bar{Q}_H \tilde{\varrho} = \frac{\partial}{\partial c^a} \omega^{ab} \big(\partial_a + \beta \partial_b H \big) \tilde{\varrho}(\phi, c) = 0 \qquad (1.19)$$

This equation can be satisfied only if

$$\varrho(\phi^a) = k e^{-\beta H(\phi^a)} \qquad (1.20)$$

where k is a constant. This result tells us that the supersymmetric invariant state has to be a Gibbs state *i.e.*, a state that depends only on the Hamiltonian. The parameter β naturally plays the role of $\frac{1}{KT}$ with T the temperature and K the Boltzmann constant. We feel that this result[8] is a striking surprise: it tells us that the canonical, or better the Gibbs distribution, is selected among all other possible states by the requirement of being invariant under the universal supersymmetry present for any dynamical system. It has been a challenge[12] for many years to find the physical reasons behind the Gibbs states, and many concepts like stability, locality, ergodicity, KMS conditions[‡], have been explored. It might be that this supersymmetry embodies all these features.

Let us now proceed to the proof of the theorem[8] which says that, if the system is in the phase with the supersymmetry (1.16) unbroken, then it is in the ergodic phase, while systems which are in a phase characterized by ordered motion have, in that phase, the supersymmetry always broken. The proof goes as follows: let us assume that our system has another global constant of motion $I(\phi)$ besides the energy $H(\phi)$ so the system is not ergodic. Then, besides $\varrho_o(H)$, any normalizable function of $I(\phi)$, $\varrho'_o(I)$, would be an eigenfunction of $\hat{L}$ with zero eigenvalue and the corresponding $\tilde{\varrho}'_o(\phi, c) = \varrho'_o c^1 \cdots c^{2n}$ would be a "ground state" of the super-Hamiltonian[♮] [◊] : $\widetilde{\mathcal{H}} \tilde{\varrho}'_o = 0.$. Since both

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‡ in ref.[8] we derived the KMS condition from this supersymmetry.

♮ As the super-hamiltonian is the lie-derivative of the Hamiltonian flow, this equation just indicates that the state is invariant under the Hamiltonian flow because it is built out of constants of motion.

◊ Even if the superHamiltonian is un-bounded below, so that the state at zero eigenvalue is not the ground state, the considerations we will present are still correct, as it is proved in details in ref.[8].

$\tilde{\varrho}_0(H)$ and $\vec{\varrho}_0'(I)$ are ground states, not only ergodicity is spoiled but also supersymmetry. In fact, $\vec{\varrho}_0'(I)$ is not supersymmetric invariant, *i.e.*, $\bar{Q}_H \vec{\varrho}_0' \neq 0$ just because the solutions of the equation $\bar{Q}_H \tilde{\varrho} = 0$ are only states of the form $\tilde{\varrho} = k e^{-\beta H} c^1 \cdots c^{2n}$ and $\vec{\varrho}_o'(I)$ is not of this form. So for systems with more than one constant of motion, the supersymmetry is broken. In fact the ground state could be any normalizable function of them[•]

$$\vec{\varrho}_o'' = F\left[\tilde{\varrho}_o, \vec{\varrho}_0'\right] = \tilde{F}\left[H, I, c\right] \tag{1.21}$$

and this is not supersymmetric invariant! Therefore integrable systems or systems with few integrals of motion besides the energy, have this supersymmetry spontaneously broken. Furthermore the system has no "stability" reasons to choose the supersymmetric ground state instead of the most general state (1.21): both of them are ground states of the theory because both of them satisfy the operatorial condition $\widetilde{\mathcal{H}}\tilde{\varrho} = 0$. In general it might be that we are not able to find analytic constants of motion, but still the system is non-ergodic presenting some invariant[▽] surfaces, of measure different from zero, on the energy manifold : KAM[7] systems are of this kind. In this case the ϱ_o' associated to this surfaces could be built out of the characteristic function of this surfaces[8] and it is easy to see that they are not supersymmetric invariant. This proves the second part of the theorem, that is: "If the system is in an "ordered phase" then, in that phase, the supersymmetry is broken.

The first part of the theorem is trivial. In fact, if we are in the phase with the supersymmetry unbroken, it means that the only state at zero eigenvalue of the super-Hamiltonian is the canonical Gibbs state, that implies that there are no other constant of motion beside the energy so the system is ergodic. In fact, if there were other constants of motion[†] , then we could build, as we explained

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• This is due to the fact that in $\hat{L}$ we have only first-order derivatives.

▽ Invariant under the Hamiltonian flow.

† Or invariant surfaces of measure different from zero on the energy surface.

before, other states at zero of the super-Hamiltonian and these states would not be supersymmetric invariant contradicting, in this way, the hypothesis that the system was in a phase with the supersymmetry unbroken.

So far we have argued that unbroken supersymmetry implies ergodicity. It is important to note that the converse of this statement is not true. In fact, since the energy is not fixed in $\mathbf{Z}_{cm}$, we could take $e^{-\beta H}$, as well as $F(H), G(H) \cdots$ as acceptable ground states for an ergodic system, where $F(H)$, $G(H) \cdots$ are arbitrary normalizable functions. Clearly all these functions are constants on the energy hypersurface and thus proportional to each other, but not outside. Since in general $F, G \cdots$ are not supersymmetric invariant, we conclude that ergodicity is possible even if our supersymmetry (1.16) is spontaneously broken, i.e., unbroken supersymmetry is not *necessary* for ergodicity, but only sufficient. We will analyze, in a forthcoming publication, the symmetries of $\mathbf{Z}_{cm}$ at fixed energy, that is with a $\delta(H - E)$ inserted, and we will discuss which unbroken symmetry is a necessary condition for ergodicity. At the same level we can say that ordered motion is only a *sufficient* condition for the breaking of supersymmetry, but not a necessary one.

2. TODA CRITERION

In 1974 M.Toda[13] proposed a criterion to detect transitions from ordered to stochastic motion. It goes as follows: Let us suppose that our Hamiltonian has the form:

$$H = \frac{1}{2} \sum_{i=1}^{n} p_i^2 + U(q) \tag{2.1}$$

so that the equation of motion are

$$\frac{d^2 q}{dt^2} = -\frac{\partial U}{\partial q_i}, i = 1 \cdots, n \tag{2.2}$$

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The equation for the first variation[†] is

$$\frac{d^2 \delta q_i}{dt^2} = -\sum W_{ij} \delta q_j \qquad (2.3)$$

where

$$W_{ij} = \frac{\partial^2 U}{\partial q_i \partial q_j} \qquad (2.4)$$

Toda thought that the transition to stochastic motion would occur when there is exponential divergence of nearby trajectories. This will take place when the matrix W_{ij} develops negative eigenvalues. That means we have to find the points in which

$$\|W\| = 0 \qquad (2.5)$$

The critical energies E_c, at which the transition occur, are then defined as the energy contour $U(q) = E_c$ which touches the surface (2.5) . If we insert the (2.5) into the Hamiltonian via a lagrangian multiplier κ and minimize the overall H,[14] we get:

$$\frac{\partial U}{\partial q_i} = \kappa \frac{\partial \|W\|}{\partial q_i}, \quad i = 1, \cdots n \qquad (2.6)$$

This equation together with (2.5) gives the critical values of the energy. Toda[13] found in this manner a good approximate value of the critical energy for the Henon-Heiles[15] model and an explanation of why the equal mass Toda model never presents critical energy and transition to stochasticity.

We will now present a study of this criterion based on our path-integral. Let us remember what we proved before : Ordered motion is a *sufficient* condition to have supersymmetry broken, while supersymmetry un-broken is a *sufficient* condition to have ergodicity. So we could try to detect what are the parameters

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† This equation controls the behavior of nearby trajectories.

we have to change to go from a supersymmetry-broken phase to a supersymmetry unbroken phase. This of course will not be in one to one correspondence with the transition ordered to stochastic phase, because the conditions above are only *sufficient* conditions. The interplay between the susy broken/unbroken phases , ordered/ergodic phases and the relative *sufficient/necessary* inferences are indicated in the scheme below.

$$
\begin{array}{ccc}
ordered\ motion & & ergodic\ motion \\
\Downarrow & & \Uparrow \\
broken\ susy & \longrightarrow & unbroken\ susy
\end{array}
\tag{2.7}
$$

From this picture it is clear that looking at the transition broken susy/unbroken susy is not equivalent to looking at the transition ordered motion/ergodic motion because the two arrows on the right and left-hand-side are only pointing in one direction (up or down) and not in both. As it is clear from the discussion on ergodicity, contained in the previous section, *the arrows would be pointing in both directions if we had decided to work at constant energy* that means inserting in the Z_{cm} of (1.6) a $\delta(H - E)$.[*] .If , as we believe, under this condition the upper and lower level of (2.7) are equivalent, then let us proceed to find the transitions susy broken to susy unbroken.

We look, first of all, at the minima of the action (1.7). As it is usually done[11] in the search for the minima of supersymmetric theories, in first approximation we shall neglect the derivative terms and the Grassmannian part of (1.7).

$$
\bar{\mathcal{L}} \approx \lambda_a \omega^{ab} \frac{\partial H}{\partial \phi^b}
\tag{2.8}
$$

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[*] This condition would produce a dependence on E in the Z_{cm} *i.e.* $Z_{cm}(E)$, and so also the pattern of susy broken/unbroken would depend on E, that means it can be broken at some energy and unbroken at other energies[a]

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The minimization gives:

$$\frac{\delta \int \tilde{\mathcal{L}}}{\delta \phi^k} = \lambda_a \omega^{ab} \frac{\partial^2 H}{\partial \phi^b \partial \phi^k} = 0$$

$$\frac{\delta \int \tilde{\mathcal{L}}}{\delta \lambda_k} = \omega^{kb} \frac{\partial H}{\partial \phi^b} = 0$$

(2.9)

Remembering that the $\|\omega\| \neq 0$, the solutions of (2.9) can be divided in three classes. The first one is:

$$\begin{cases} \lambda_a \neq 0 \\ det \, \omega^{ab} \frac{\partial^2 H}{\partial \phi^b \partial \phi^k} = 0 \Rightarrow det \, \frac{\partial^2 H}{\partial \phi^b \partial \phi^k} = 0 \\ \frac{\partial H}{\partial \phi^b} = 0 \end{cases}$$

(2.10)

and the second and third ones are:

$$\begin{cases} \lambda_a = 0 \\ det \, \frac{\partial^2 H}{\partial \phi^b \partial \phi^k} \begin{cases} = & 0, \\ \neq & 0 \end{cases} \\ \frac{\partial H}{\partial \phi^b} = 0 \end{cases}$$

(2.11)

These two sets of solutions (2.10),(2.11) corresponds, respectively, to susy spontaneously broken and unbroken. Let us in fact remember[8] the susy transformation on $\bar{c}_a$, that can be derived from the generators (1.14)[†]

$$\delta \bar{c}_a = \epsilon (\lambda_a + \beta \frac{\partial H}{\partial \phi^a})$$

(2.12)

From the third conditions of eqs. (2.10),(2.11), we have $\frac{\partial H}{\partial \phi^a} = 0$, we can then

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† ϵ is one of the two infinitesimal anticommuting parameter of the supersymmetry transformations.

re-write (2.12) as

$$\delta \bar{c}_a = \epsilon \lambda_a \qquad (2.13)$$

Let us now look at the first set of solutions (2.10). There $\lambda_a \neq 0$, and so we have by (2.13) that[*] :

$$\langle \delta \bar{c}_a \rangle \neq 0 = \left\langle \left[Q_H, \bar{c}_a \right] \neq 0 \right\rangle \qquad (2.14)$$

This last condition implies that the "ground state" is not annihilated by Q_H and so the supersymmetry is spontaneously broken. Using the same tools it is straightforward to see that the second set of solutions (2.11), with $\lambda_a = 0$, has the supersymmetry un-broken.

We can say that the "order parameter" to detect transition from susy broken to unbroken is $\langle \lambda \rangle$. In statistical mechanics this parameter is known as the "response function". Let us now see if, instead of λ_a, we could use the $\| \frac{\partial^2 H}{\partial \phi^b \partial \phi^k} \|$ as an order parameter: *that would be the Toda criterion*[‡] applied to the supersymmetry. It is clear from the solutions above, (2.10) and (2.11), that we cannot do that. In fact, while Susy broken (eq. (2.10)) is characterized by $\| \frac{\partial^2 H}{\partial \phi^b \partial \phi^k} \| = 0$, Susy unbroken (eq. (2.11)) can have both $\| \frac{\partial^2 H}{\phi^b \partial \phi^k} \| = 0$ and $\| \frac{\partial^2 H}{\partial \phi^b \partial \phi^k} \| \neq 0$. So if we have a transition from $\| \frac{\partial^2 H}{\phi^b \partial \phi^k} \| = 0$ to $\| \frac{\partial^2 H}{\partial \phi^b \partial \phi^k} \| \neq 0$ this could be a transition from the first solution (2.10)(susy broken) to the second one (2.11) (susy unbroken), but it could also be a transition inside the two sets of solutions contained in (2.11). This would mean that in this case we are staying in the phase of unbroken susy despite the fact the $\| \frac{\partial^2 H}{\partial \phi^b \partial \phi^k} \|$ has changed from zero to different from zero.

This confirms that the Toda criterion cannot be used as a criterion for the transition susy unbroken/susy broken and similarly, if the upper and lower level

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[*] We use the expectation value $\langle * \rangle$ just because all these variations are intended under the sign of the path-integral (1.6).

[‡] Note that ,for Hamiltonians of the form (2.1), the $\| \frac{\partial^2 H}{\partial \phi^b \partial \phi^k} \|$ is the same as (2.4).

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of (2.7) are equivalent[b], it cannot be used as a criterion for the transition ordered/stochastic motion. *The order parameter to use should be instead the $\langle \lambda_a \rangle$.*

The fact that the Toda criterion was not good enough to detect the transition ordered/stochastic motion had been already pointed out by many authors[16]. These authors found counterexamples in which the Toda criterion is not satisfied. Casati[16] for example found a case (Toda model with un-equal mass) that has stochastic behaviour but the determinant of Toda is always positive. This would correspond for us to the solution (2.11): we have susy unbroken (ergodicity) even if the determinant is positive. Benettin et al.[16] found a case in which the determinant is negative (so it crossed zero) even if there is only ordered motion.[o]

This would mean in our case Susy Broken ($\lambda_a \neq 0$) and det.=0. This in not included in our first solution (2.10). The reason might be that, as Benettin et al. says, the case they studied gives positive det. if they choose action-angle variables, and that would bring the case into our first solution (2.10). Or it might be that our solutions, that are based on the approximate lagrangian (2.8), are also approximate, so that the second line in the first solution (2.10) is only approximately zero. The minimum to study would be those of a sort of "effective action" on the line of the statistical approach to Chaos[17] to which our approach is similar. Or, third case, it might be that, once we insert the constraint of constant energy in Z_{cm}, the symmetry in (2.7) that is in one to one correspondence with the transition ordered/stochastic, is not anymore the susy but some other symmetry with an "order parameter" different from $\langle \lambda_a \rangle$. All this is under study because, as everybody knows, it is still an open problem to find the order parameter (or more) that detects the transition ordered/stochastic motion.

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b As presumably they are at constant energy

o M.Berry[18] suggested another example : the case of a circular potential $U(r)$ in 2-dimension with a range of r with $U''(r) < 0$. Toda predicts chaos, but all such systems are integrable.

The goal of this paper has been that of testing our path-integral approach and its symmetries with one of the first criterion[13] proposed in the history of dynamical systems. We are pleased that the test has shown (without looking at counter-examples[16]) the approximate status of the criterion, confirming in this way what had already been recognized[16] by many, using counterexamples. Next we would like to confront our approach with more advanced criteria like the ones using Liaupunov exponents, Kolmogoroff entropy and topological entropy[18] . To do that we have first to express all these quantities into our own language. The first results seems to indicate that they can all be built out of ghosts condensate and similar things on the line of ref.[19][19] . This work is in progress.

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