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REMARKS ON THE QUANTUM GROUP STRUCTURE OF THE RATIONAL $c < 1$ CONFORMAL THEORIES

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ABSTRACT

The rational $c < 1$ theories are reconsidered beyond the space of BRST states, allowing for intermediate states not contained in the Kac table. The intertwining properties of the screening charges Q_m, Q_{p-m} are used to derive linear relations for the general conformal blocks. The fusion rules are recovered on BRST states, combining these relations with previously obtained identities for the fusion matrices, due to the corresponding $\mathcal{U}_q(sl(2))$-invariant operators. The extended formulation is applied to give meaning for $q^p = 1$ to the quantum group covariant conformal correlations initiated by Moore and Reshetikhin. The correlations are manifestly covariant under the action of the $\mathcal{R}$- matrix and in the diagonal case they coincide with the averages of the screened vertices, recently proposed by Gómez and Sierra.

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1 Introduction

The representation theory of quantum groups [1,2] differs drastically from its classical analogue when the deformation parameter q is a root of unity, $q^p = 1$ [3-10]. Thus, while the embedding pattern of the Verma modules for generic q is governed by the finite Weyl group W of the complex semisimple algebra g, for $q^p = 1$ it is essentially parametrized by the infinite affine Weyl group $\hat{W}$ of $g^{(1)}$ [11]. The group $\hat{W}$ describes [12,13,14] the singular vectors of the Verma modules of the affine algebras $g^{(1)}$, underlying the RCFT. Comparing, for example, the diagrams depicting the embeddings of the Verma modules in the simplest case when $g = sl(2)$ (see Fig. 1a,b and Fig. 2), one sees that the only difference is the direction of some of the arrows. In particular, unlike the Virasoro (or $A_1^{(1)}$) modules, any $\mathcal{U}_q(sl(2))$ module is itself embedded in a bigger module.

The singular vectors of the Verma modules give rise to operators invariant under the (left) action of the algebra ; these operators, generated by the right action of the algebra, intertwine pairs of partially equivalent representations, which need not be highest weight representations. In particular the intertwining operators $\mathcal{D} = X_-^{2j+1}$ and $\underline{\mathcal{D}} = X_-^{p-2j-1}$ of $\mathcal{U}_q(sl(2))$ can be realized as finite difference operators in spaces C_j of functions of one complex variable [7]. The diagram on Fig. 1a then admits another interpretation, replacing the Verma modules with the functional spaces at the points; then the arrows indicate the action of the operators $\mathcal{D}$ and $\underline{\mathcal{D}}$.

The correspondence between singular vectors and intertwining operators has not been very useful for the representations of the Virasoro algebra, since there are no explicit general formulae for the singular vectors. This difficulty has been overcome using free fields (Fock spaces) realization [15,16,17,18]. The invariant operators , intertwining Fock spaces, are represented by the screening charges [17,18]. The diagram describing the action of these operators (see Fig. 3) is identical, including the direction of the arrows, with the diagram in the quantum group case, with $\mathcal{D}$, $\underline{\mathcal{D}}$ replaced by Q_m, Q_{p-m}, $1 \le m = 2j+1 \le p-1$.

The analogy between the quantum group generators and the screening operators first noticed in [3] has been further deepened in [19] where it has been shown that the elementary screening currents of the $A_{n-1}^{(1)}$ WZW models satisfy in a weak sense , under integration , the Serre identities for the negative (or positive, depending on the realization) root generators of $\mathcal{U}_q(sl(n))$.

This functorial equivalence of the intertwining operators (i.e., the operators invariant under the corresponding left action of the algebra) is essentially the source of all the striking similarities of the quantum groups and the RCFT - theories described by fields which are at most quantum group scalars.

In this paper we analyse the implications of the two sets of intertwining operators for the rational $c < 1$ conformal theories and for the related theories with an explicit action of the quantum group, initiated in [20] (see also [21,22,23,24]) and recently further developed in [25]. In Sect. 2 we reconsider the minimal theory in the initial big Fock spaces, thus allowing operators which do not leave invariant the "physical" space of BRST states. To do that one has to take into account intermediate states labelled by integers $m = 2j+1, m' = 2j'+1$, beyond the values described by the Kac table: $1 \le m \le p-1$, $1 \le m' \le p'-1$. They correspond to the triple of Fock spaces, depicted in the middle of Fig. 3. The existence of intertwining operators leads to linear relations for the n-point conformal blocks corresponding to the pairs $(j, \underline{j} = p - j - 1)$ (or, $(j, \bar{j} = -j - 1)$), $1 \leq 2j + 1 < p$, with j' fixed. These relations and the relations for the fusion matrix due to the invariant operators of $\mathcal{U}_q(sl(2))$ [7], are used to show that the minimal theory fusion rules (FR) are recovered on the BRST states and that they are not violated after braiding or fusing. This solves an old problem in the Coulomb gas approach of Dotsenko and Fateev (DF) [16], [26]. The consideration in Sect. 2 is alternative to the one followed in [18,27], which describes the minimal theory entirely in the space of BRST states. It is however more suitable when we turn in Sect. 3 to the quantum group covariant theory. Now all the states which are factorized out in the pure minimal theory have to be essentially resurrected, if one insists on the $\mathcal{R}$- covariance of the correlations . The interplay between the BRST charges and the quantum group invariant operators is once again used to give meaning to the correlations when $q^p = 1$. In Appendix B we show that the 4-point functions constructed in this way essentially coincide with the averages of the recently proposed new screened quantum group covariant vertices [25]. This sheds light on the relationship between the operator languages in [20] and [25]. In Sect. 4 we consider the general quantum group invariants, which can be realized using in particular the operators in [25]. Their transformation properties under the action of the braid group reflect the quantum group tensor product decomposition rules which are inconsistent in general with the FR upper bounds. The minimal theory correlation functions are recovered as an invariant subset, taking appropriate averages of the operators in [25]. Accordingly, the operator counterparts of the numerical "vertex - path" identities [28,29,30] - relating the quantum $\mathcal{R}$- matrix to the braid matrix, reduce in averages to the FR bounds. In Appendix C we extend the class of $\mathcal{R}$-covariant correlations, constructing chiral analogues of the non-diagonal local (and quasilocal) 4-point functions in [31,32]. Appendix A contains notation and useful formulae as well as a generalization beyond the thermal case of the correlations in Sect. 3, alternative to the approach in [25].

2 $\mathcal{U}_q(sl(2))$, Coulomb gas and fusion rules

Once we have an invariant operator, intertwining a pair of representations, we can expect relations for the corresponding group invariants. Indeed, such relations for the 3- and 4-point invariants were derived in [7] using the intertwining operator $\underline{\mathcal{D}} = X_-^{p-2j-1} : \mathcal{E}_{p-j-1} \to \mathcal{E}_j$, where $\mathcal{E}_j$ is a $2j+1$ - dimensional subspace of $\mathcal{C}_j$. The relation for the 3-point invariants (which reproduce the Clebsch - Gordan coefficients) imply furthermore a relation for the $q-6j$-symbols, $q = exp\ 2\pi\ ip'/p$, where p', p are coprime integers:

$$\left\{ \begin{matrix} j_1 & j_2 & j_5 \\ j_3 & j_4 & \underline{j_6} \end{matrix} \right\}_q = (-1)^{\underline{j_6} - j_6 + (p'-1)(j_2+j_4-j_1-j_3)} \sqrt{\frac{[2\underline{j_6}+1]_q}{[2j_6+1]_q}} \left\{ \begin{matrix} j_1 & j_2 & j_5 \\ j_3 & j_4 & j_6 \end{matrix} \right\}_q , \tag{1}$$

$$\underline{j} = p - j - 1, \quad [a]_q = \frac{q^{\frac{a}{2}} - q^{-\frac{a}{2}}}{q^{\frac{1}{2}} - q^{-\frac{1}{2}}} .$$

Here all j_i, except $\underline{j_6}$, are assumed to be regular, i.e., $2j_i + 1 < p, i = 1, 2, ...6$. Both sides of (1) are finite for triples $(j_1, j_2, j_5), (j_3, j_4, j_5)$, such that $j_1 + j_2 + j_5 + 1 < p$, $j_3 + j_4 + j_5 + 1 < p$, and they vanish identically if $2j_6 + 1 = p = 2\underline{j_6} + 1$. A similar identity without the sign factor in (1) results from the operator $\mathcal{D}$, with $\underline{j}$ replaced by $\bar{j} = -j - 1$. The relation (1) , derived in a purely quantum group framework, implies a corresponding relation for the minimal theory fusion matrix - represented, up to a mixing sign, by the product of two $q-6j$-symbols, with $q = exp2i\pi p'/p$, and $q' = exp2i\pi p/p'$ [32] (see App. A). It truncates the polynomial identities, inherited from the generic q case, to the bounds of the FRs.

The relations for the properly normalized Clebsch - Gordan coefficients just reflect the fact that the states $e^j_m(j_1,j_2)$ and $e^{\underline{j}}_m(j_1,j_2)$, $|m| \le j$, in the tensor product $\mathcal{E}_{j_1}\otimes\mathcal{E}_{j_2}$ of two regular representations, become identical, up to an overall constant, when $q^p = 1$. The same is true for the states $e^{j_1}_{m_1}(j,j_2)$ and $e^{j_1}_{m_1}(\underline{j},j_2)$. Unlike the contribution of the pairs $(j,\underline{j} = p-j-1)$ in the numerical polynomial equations, the representation states of course, cannot cancel; rather, to recover all linearly independent states in the tensor product one has to give up the conditions $X_+\ e^j_j = 0 = X_-^{2j+1}\ e^j_j$, thus getting an indecomposable representation $\mathcal{E}_{j,\underline{j}}$ of dimension $2j+1+2\underline{j}+1$ (see [3] for details).

We now turn to the implications of the analogues of the intertwining operators $\mathcal{D}$, $\underline{\mathcal{D}}$ for the minimal theory correlations. We are considering the BPZ minimal models with central charge $c_{p',p} = 1-6(p-p')^2/pp'$. In the free field representation the fields act on Fock spaces F_{α,α_0} generated by a free boson. These spaces are given the structure of Virasoro modules of dimension $\Delta_{n'n} = \Delta(\alpha_{n'n}) = \alpha_{n'n}\ (\alpha_{n'n} - 2\alpha_0)$, where $\alpha_{n'n} = \frac{1-n}{2}\alpha_- + \frac{1-n'}{2}\alpha_+$, $\alpha_-\ \alpha_+ = -1$, $\alpha_-^2 = p'/p = \delta$, and $2\alpha_0 = \alpha_+ + \alpha_-$ is the charge at infinity, determining a conjugate vacuum state $v^*_\alpha = v_{2\alpha_0-\alpha}$; the integers n', n will not be restricted for the time being.

For simplicity of notation we shall concentrate mainly on the thermal case, $\alpha = \alpha_{1n}$. The screened vertex operators are defined as [18]

$$V^r_{\alpha_{1n}}(z) = \int_{C_1} dt_1 ... \int_{C_r} dt_r\ V_{\alpha_{1n}}(z)\ \ V_{\alpha_-}(t_1)...V_{\alpha_-}(t_r)\ , \tag{2}$$

where $V_{\alpha_{1n}}$ is a Virasoro covariant vertex operator of dimension $\Delta(\alpha_{1n})$, mapping any Fock space F_β to $F_{\beta+\alpha_{1n}}$. The contours C_i are chosen to wind once around 0, starting and ending at z, in such a way that C_{i+1} is inside C_i. In expectation values the integrand in (2) is fixed by requiring that it is real (for real charges) for $z > t_1 > t_2 > .... > t_r$ on the real axis. The charge conservation condition [16] ensures that the vacuum expectation values of the products of the screened vertices in the presence of the background charge $2\alpha_0$ are invariant under projective transformations, and more generally, satisfy the Virasoro Ward identities. Deforming the contours one reproduces from the averages of the screened operators (2) the DF conformal blocks. In particular the 4-point correlations are recovered according to (see Fig. 4)

$$\langle\alpha_4{}^*|V^{s-k}_{\alpha_3}(1)V^{k-1}_{\alpha_2}(z)|\alpha_1\rangle_{2\alpha_0} = \beta^s_k(a,b,c;\delta)\ z^{2\alpha_1\alpha_2}(1-z)^{2\alpha_2\alpha_3}\ I^s_k(a,b,c;z), \tag{3}$$

$$\beta^s_k(a.b.c;\delta) = e^{i\pi[(a+c+(s-2)\delta)(s-1)-(k-1)c]}[s-k]_q!\ [k-1]_q!\ \prod_{j=0}^{k-2} s(a+j\delta)\ \prod_{j=0}^{s-k-1} s(d+j\delta)\ ,$$

where, following the notation of [16],

$$a = 2\alpha_-\alpha_1,\ c = 2\alpha_-\alpha_2,\ b = 2\alpha_-\alpha_3,\ d = 2\alpha_-(2\alpha_0-\alpha_4);\alpha_i = \alpha_{1n_i};\ s(a) = 2isin(\pi a),$$

$$s = j_1+j_2+j_3-j_4+1,\ k = 1,2,...,s,$$

and I_k^s is the multiple path-ordered contour integral [16] with k-1 contours running from 0 to z and s-k contours from 1 to infinity. In the general case the constant in front of $I_{k'k}^{s's}$ factorizes into $\beta_k^{s'}(a',b',c';\delta')\ \beta_k^s(a,b,c;\delta)$, where $a = 2\alpha_-\alpha_{n_1'n_1} = -a'\delta$, etc.

Felder proposed a BRST cohomological interpretation of the Coulomb gas realization of the minimal theory by introducing the screening charges Q_m and Q_{p-m}, $1 \leq m \leq p-1$.

$$Q_m = \int_1^{(+0)} dt_1 ... \int_1^{(+0)} dt_m\ V_{\alpha_-}(t_1)...V_{\alpha_-}(t_m). \tag{4}$$

The BRST charges Q_m and Q_{p-m} intertwine pairs of Fock spaces according to Fig. 3, i.e., in their range of definition they are invariant under the action of the Virasoro algebra. The irreducible Vir modules $\mathcal{H}_{m'm}$, $(1 \leq m^{(')} \leq p^{(')}-1)$ arise as the factors $\mathcal{H}_{m'm} = \mathrm{Ker}Q_m/\mathrm{Im}Q_{p-m}$ of the Fock spaces $F_{\alpha_{m'm}}$.

The screening charges can "float" from a vertex to a neighbouring one, if the intermediate states are consistent with their range of definition. In particular using the basic braiding relation for the vertices $V_\alpha(z_1)V_\beta(z_2) = \exp(2i\pi\epsilon\alpha\beta)V_\beta(z_2)V_\alpha(z_1)$, (see (3.20) of [18]), where $\epsilon = \pm 1$, depending on the direction of the path interchanging the two points, one has:

$$\begin{aligned} Pr_{n_4}\ V_{n_3}^{r_3+p-m}(z_3)\ V_{n_2}^{r_2}(z_2)\ Pr_{n_1} &= Pr_{n_4}\ V_{n_3}^{r_3}(z_3)\ Q_{p-m}V_{n_2}^{r_2}(z_2)\ Pr_{n_1} \\ &= e^{i\pi c(p-m)\epsilon}\ Pr_{n_4}\ V_{n_3}^{r_3}(z_3)\ V_{n_2}^{r_2+p-m}(z_2)\ Pr_{n_1}\ , \end{aligned} \tag{5}$$

if $n_1+n_2-2r_2-1 = 2p-m,\ n_4-n_3+2r_3+1 = m;\ 1 \leq n_i,\ m \leq p-1$.

Here Pr are projectors on the space $F_{\alpha_{n'n}}$. Let us rewrite this relation, which encodes the intertwining property of Q_{p-m} in a more transparent way, using the fact that when projected on $F_{\alpha_{1l}}$ the operator V_n^r reduces to a chiral vertex operator (CVO), $\begin{pmatrix} j\ j_1 \\ j_2 \end{pmatrix}$, denoting $n = 2j_2+1,\ l = 2j_1+1,\ n+l-2r-1 = 2j+1$,

$$e^{-i\pi c \underline{j_5}\epsilon} \begin{pmatrix} j_4\ \underline{j_5} \\ j_3 \end{pmatrix}_{z_3} \begin{pmatrix} \underline{j_5}\ j_1 \\ j_2 \end{pmatrix}_{z_2} = e^{-i\pi c j_5\epsilon} \begin{pmatrix} j_4\ j_5 \\ j_3 \end{pmatrix}_{z_3} \begin{pmatrix} j_5\ j_1 \\ j_2 \end{pmatrix}_{z_2}\ , \tag{6}$$

$$n_i = 2j_i+1, m = 2j_5+1, \underline{j_5} = p-j_5-1.$$

A similar identity with $\underline{j} = p-j-1$ replaced by $\bar{j} = -j-1$ is obtained using the operator Q_m. In terms of the DF n-point correlations, using (3), (6) reads

$$I_{\underline{k}}^s(a,b,c;z) = (-1)^{p'(2j_1+2j_4)+(j_2+j_4-j_1-j_3)} \frac{[\Delta_{12}^5]_q![\Delta_{35}^4]_q![\Delta_{45}^3]_q![j_3+j_4+j_5+1]_q!}{[\Delta_{34}^5]_q![\Delta_{15}^2]_q![\Delta_{25}^1]_q![j_1+j_2+j_5+1]_q!} I_k^s(a,b,c;z)\ , \tag{7}$$

where $\Delta_{il}^n = j_i+j_l-j_n$, $k = j_1+j_2-j_5+1$; $\underline{k} = j_1+j_2-\underline{j_5}+1$, $k, \underline{k} = 1,2,...,s$.

This relation simplifies for the normalized blocks $\tilde{I}_k^s$ (see App.A), useful in recovering the primary fields structure constants [16]:

$$\tilde{I}_{\underline{k}}^s(a,b,c;z) = (-1)^{(j_2+j_4-j_1-j_3)(p'+1)} \sqrt{\frac{[2\underline{j_5}+1]_q}{[2j_5+1]_q}}\ \tilde{I}_k^s(a,b,c;z)\ . \tag{8}$$

In general when I_k^s is replaced by $I_{k'k}^{s's}$ (7),(8) are modified by a factor $(-1)^{(k-\underline{k})(2j_2'+2j_5')}$. Note that both sides of (8) vanish identically if $2j_5+1 = p = 2\underline{j_5}+1$, or $2j_5+1 = 0 = 2\overline{j_5}+1$.

In deriving (7) we have assumed that both the regular j_5 and its partner $\underline{j_5} = p-j-1$ are allowed from the classical decomposition rules for the products $j_1 \otimes j_2$ and $j_3 \otimes j_4$. That implies that both j_5 and $\underline{j_5}$ violate the FR. In that case the integrals in (8) are finite, non-zero, unlike the corresponding averages for the Felder screened vertices, which vanish due to the constant β_k^s in (3), in agreement with the results in [18]. That means that the normalized DF correlations in (8) correspond to operators which do not preserve the "physical" BRST space $\oplus_m \mathcal{H}_m$. They create intermediate states in F_{2p-m} and in $\mathrm{Im}Q_{p-m} \subset F_m$, $(1 \leq m \leq p-1)$, which are compensated in the physical (local) 4-point functions taking into account (8). Similarly, intermediate states in $\mathrm{Im}Q_m \subset F_{-m}$ and in $F_m/\mathrm{Ker}Q_m$ appear. Note that one can extend (7), (8) in principle to arbitrary n-point functions. The point is that the normalization needed to recover the DF blocks I_k^s (or $\tilde{I}_k^s$) (cf. (3), (A.4)) can be attached to the bilocal chiral vertex operators $V_{\alpha,\alpha_1}^r(z,z_1)$, $V_{\alpha,\alpha_1}^r(z,0)|0\rangle = V_\alpha(z)|\alpha_1\rangle$. They are obtained by adding the vertex $V_{\alpha_1}(z_1)$ after the string of screening currents V_{α_-} on the r.h.s. of (2) (see, e.g., [25]).

The relation (7) (and the one obtained when $\underline{j}$ is replaced by $\overline{j} = -j-1$) shows that the set $\{I_k^s, k = 1,...s\}$ of basic integrals is not in general linearly independent. Furthermore it indicates a singularity of the integral on the r.h.s. of (7), as a function of the parameters a,b,c. Let us consider for simplicity the case when $j_1 = j_3$, $j_2 = j_4$. If the triple (j_1, j_2, j_5) is inconsistent with the FR and j_5 is regular, the primary field structure constant $D_{j_1 j_2}^{j_5}$, which can be recovered for coinciding arguments $z_{12} \to 0$, vanishes [16]. Hence, as in the example considered in [26], there should be a diverging constant in front of higher orders in z_{12}, leading altogether to a finite, non-zero descendent structure constant. Indeed , the corresponding primary field structure constant $D_{j_1 j_2}^{\underline{j_5}}$, recovered from the l.h.s is finite, non-zero. It is amusing to see that the quantum groups provide information about the analyticity properties of these generalized hypergeometric series.

Finally, combining the linear relations (7) with the relation (1) for the fusion matrices one gets following the notation of [16], (no summation in k,$\underline{\mathrm{k}}$),

$$\alpha_{i\underline{k}}^{(s)}(a,b,c;\delta)\ I_{\underline{k}}^s(b,a,c;1-z) = -\alpha_{ik}^{(s)}(a,b,c;\delta)\ I_k^s(b,a,c;1-z),$$
$$e^{i\pi\Delta_{\underline{k}}(a,b)}\alpha_{i\underline{k}}^{(s)}(c,b,a;\delta)\ I_{\underline{k}}^{(s)}(a,c,b;1/z) = -e^{i\pi\Delta_k(a,b)}\alpha_{ik}^{(s)}(c,b,a;\delta)\ I_k^s(a,c,b;1/z) \qquad (9)$$

where $\Delta_k(a,b) = \Delta_{j_6} = j_6(j_6+1)\delta - j$; $j_6 = j_1 + j_3 - k + 1$, $\underline{j_6} = j_1 + j_3 - \underline{k} + 1$ and $\alpha_{ik}^{(s)}(a,b,c;\delta)$ is the DF fusion (crossing) matrix,

$$I_i^s(a,b,c;z) = \sum_{k=0}^{s} \alpha_{ik}^s(a,b,c;\delta)\ I_k^s(b,a,c;1-z)\ ,$$

proportional to the $6j$-symbols (see (A.8)).

The same relations hold when $\underline{k}$ is replaced by $\overline{k} = j_1 + j_2 - \overline{j_6} + 1$.

The relations (7), (9) for the thermal case are easily generalized, taking into account all sign factors. Let $J_i = (j_i', j_i)$, and let $\{j_1, j_2, j_5, j_3, j_4\}$ be an admissible set, i.e., j_i, $i = 1,...,5$, are regular and the triples (j_1, j_2, j_5), (j_3, j_4, j_5) , obey the FR. In terms of the full normalized conformal blocks $\tilde{J}_{J_5}(\bar{z})$, which differ by a standard prefactor from the contour integrals $I^{s's}_{k'k}(z)$ (see (A.11)), the relations (9) say that the summation in the braiding relations

$$\tilde{J}_{J_5}(z_1,J_1;z_2,J_2;z_3,J_3;z_4,J_4)=e^{\epsilon i\pi(\Delta_{J_4}-\Delta_{J_1}-\Delta_{J_2}-\Delta_{J_3})}\sum_{J_6}\begin{Bmatrix} J_1 & J_2 \\ J_3 & J_4 \end{Bmatrix}_{J_5 J_6}\tilde{J}_{J_6}(z_3,J_3;z_2,J_2;z_1,J_1;z_4,J_4). \tag{10}$$

$$=\sum_{J_6} e^{\epsilon i\pi(\Delta_{J_1}+\Delta_{J_4}-\Delta_{J_5}-\Delta_{J_6})}\begin{Bmatrix} J_2 & J_1 \\ J_3 & J_4 \end{Bmatrix}_{KT}\tilde{J}_T(z_1,J_1;z_3,J_3;z_2,J_2;z_4,J_4) \tag{11}$$

effectively reduces to the FR bounds, since the contribution of each FR violating pair $(j,\underline{j})$ (and $(j,\bar{j})$) vanishes as a whole. Here $\{\ \}$ is the full fusion matrix (see (A.7)). Note that if only the first spin j of the pair $(j,\underline{j})$, or, $(j,\bar{j})$ appears and violates the FR (which can happen for particular combinations of $j_1,...,j_4$), then its contribution is identically zero.

This solves in general the problem of reconciling the DF realization with the fusion rules, discussed in [26]. The fusion transformations are derived in [16] assuming an analytic continuation to generic values of the parameters a,b,c, so that the basic blocks are linearly independent. This leads at the end to the appearance of terms violating both the upper and lower bounds of the FR . A careful derivation, using the standard contour deformation technique, would instead reproduce our relations (7), derived above directly from the intertwining property of the screening charges.

In the approach of [18,27] the general fusion matrix is expressed recursively by the elementary fusion matrix, describing the products $\frac{1}{2}\otimes j$. Accordingly the higher spins are thought to be obtained by subsequently fusing the elementary ones. In such an approach it is enough to ensure that the border points $2j+1=0 \pmod p$ do not appear as intermediate states (when the products of operators are applied on BRST states), so that the FR violating pairs $(j,\underline{j})$ and $(j,\bar{j})$ cannot be created. Then all other fusion matrix elements are defined to be zero. Unlike [27] we deal directly with the general spin correlations. Then there is no need to postulate that the fusion matrix elements vanish beyond the FR bounds - rather, we can adopt as in (1) the values obtained by analytic continuation from the generic q case.

The fact that the braiding transformations (10),(11), also hold with the classical decomposition bounds will be of special importance in our next consideration. Note that these transformations still have a sense when j_5 on the l.h.s. violates the FR. In this case the $6j$ - symbols develop singularities which are of the same type for both sides of (1). These singularities are compensated via L'Hôpital, taking into account (7) and the corresponding identities for the properly normalized $6j$ - symbols. Then any pair $(j,\underline{j})$, allowed by the classical upper bounds on the r.h.s. of (10),(11) gives a finite contribution.

3 $\mathcal{R}$-covariance versus fusion rules

In [20] (see also [22,23,24]) Moore and Reshetikhin (MR) have considered a theory with an explicit action of the quantum algebra $\mathcal{U}_q(sl(2))$, by replacing the conformal representation spaces $\mathcal{H}_j$ with the product $\mathcal{H}_j\otimes\mathcal{E}_j$.[1] They introduced vertex operators, covariant under

[1] Here and in what follows we use the notation $\mathcal{H}_j$ instead of $\mathcal{H}_m$, $1\leq m=2j+1\leq p-1$.

the action of $\mathcal{U}_q(sl(2))$ as linear combinations of CVO multiplied by q - Clebsch - Gordan coefficients, i.e.,

$$V_m^j(z) = \sum_{\substack{j_1, j_2 \\ m_1, m_2}} |m_1\rangle \begin{pmatrix} j_1 \, j_2 \\ j \end{pmatrix}_z \begin{bmatrix} j_1 & j & j_2 \\ m_1 & m & m_2 \end{bmatrix}_q \langle m_2| \tag{12}$$

which act in $\mathcal{H}= \oplus_j(\mathcal{H}_j \otimes \mathcal{E}_j)$. The sum in (12) is assumed to run over regular spins j_1, j_2, and such that the triple (j_1, j, j_2) is consistent with the FR. Here $|m_k\rangle$ is a normalized state in $\mathcal{E}_{j_k}$, and $\langle m|n\rangle = \delta_{mn}$. The order (j_1, j, j_2) in the Clebsch - Gordan coefficients and in the CVO need not be the same - we have preferred the symmetric choice.

The braid relations of the properly normalized correlations of the chiral vertex operators go over to transformations of the operators in (12) with the quantum $\mathcal{R}$- matrix. These transformations, which make the mixed correlations reminiscent of the local 2-dimensional n-point functions (i.e., they are symmetric "up to $\mathcal{R}$–matrices"), rely on numerical identities [28,29,30], connecting the vertex and path representations of the quantum $\mathcal{R}$ matrix:

$$\sum_{n_2, n_3, m} \begin{bmatrix} j_1 & j_3 & j \\ m_1 & n_3 & m \end{bmatrix}_q \begin{bmatrix} j & j_2 & j_4 \\ m & n_2 & m_4 \end{bmatrix}_q (\mathcal{R}_\epsilon^{j_3 j_2})_{m_3 \; m_2}^{n_3 \; n_2}$$

$$= \sum_{m_5, j_5} e^{i\pi\epsilon(\Delta_{j_1} + \Delta_{j_4} - \Delta_j - \Delta_{j_5})} \begin{Bmatrix} j_3 \, j_1 \, j \\ j_2 \, j_4 \, j_5 \end{Bmatrix}_q \begin{bmatrix} j_1 & j_2 & j_5 \\ m_1 & m_2 & m_5 \end{bmatrix}_q \begin{bmatrix} j_5 & j_3 & j_4 \\ m_5 & m_3 & m_4 \end{bmatrix}_q , \tag{13}$$

$$\Delta_j = \Delta(\alpha_{1\ 2j+1}) = j(j+1)\delta - j, \; \mathcal{R}_{-1}^{j_1 j_2} = (\mathcal{R}_1^{j_2 j_1})^{-1}.$$

Here $\mathcal{R}^{j_1 j_2} = (\pi^{j_1} \otimes \pi^{j_2}) \, \mathcal{R}$ represents the universal $\mathcal{R}$- matrix acting in the space $\mathcal{E}_{j_1} \otimes \mathcal{E}_{j_2}$ [30],

$$\mathcal{R}^{j_1 j_2} \, (e_{n_1}^{j_1} \otimes e_{n_2}^{j_2}) = \sum_{m_1, m_2} (\mathcal{R}^{j_1 j_2})_{m_1 m_2}^{n_1 n_2} \, e_{m_1}^{j_1} \otimes e_{m_2}^{j_2}. \tag{14}$$

The matrix elements are recalled in App. A.

The expression in front of the Clebsch - Gordan coefficients on the r.h.s. of (13) coincides up to a sign with the thermal braid matrix $\mathcal{B}_{23}(\epsilon)$ (cf. (11) and (A.7,8)). The $\mathcal{R}$- matrix can be written in terms of the $3j$-symbols as a sum of projectors

$$(\mathcal{R}_\epsilon^{j_1 j_2})_{m_1 m_2}^{n_1 n_2} = \sum_{j_5, m_5} e^{i\pi\epsilon(\Delta_{j_5} - \Delta_{j_1} - \Delta_{j_2})} \begin{bmatrix} j_2 & j_1 & j_5 \\ m_2 & m_1 & m_5 \end{bmatrix}_q \begin{bmatrix} j_1 & j_2 & j_5 \\ n_1 & n_2 & m_5 \end{bmatrix}_q . \tag{15}$$

The relation (13) with (15) inserted on the l.h.s. is equivalent , via the polynomial q - Racah identity, to the fundamental $6j$-symbols defining equality [30]:

$$\sum_m \begin{bmatrix} j_2 & j_3 & j \\ m_2 & m_3 & m \end{bmatrix}_q \begin{bmatrix} j_1 & j & j_4 \\ m_1 & m & m_4 \end{bmatrix}_q = \sum_{j_5, m_5} \begin{Bmatrix} j_3 \, j_2 \, j \\ j_1 \, j_4 \, j_5 \end{Bmatrix}_q \begin{bmatrix} j_1 & j_2 & j_5 \\ m_1 & m_2 & m_5 \end{bmatrix}_q \begin{bmatrix} j_5 & j_3 & j_4 \\ m_5 & m_3 & m_4 \end{bmatrix}_q . \tag{16}$$

Vice versa, one can recover the explicit expression (15), using (16) and the q - Racah identity.

The summation over j_5 in (13), (15) , (16) runs according to the classical tensor product decomposition rules. For $q^p = 1$ the Clebsch - Gordan coefficients develop singularities. Let us assume that in (13), (15) , (16), all $j_i, i = 1,2,3,4$, and j are regular, i.e., $2j_i+1 < p, 2j+1 < p$, and furthermore, let the triples $(j_1, j_3, j), (j_2, j_4, j)$ in (13) (or $(j_2, j_3, j), (j_1, j_4, j)$ in (16)) be consistent with the FR. Then both sides of (13), (16) remain finite and the right hand sides include in general contributions beyond the upper bounds of the minimal theory FR (see [7] for a discussion of (16)). Indeed if $j = (p-1)/2$ appears on the r.h.s., it survives, since a zero in the fusion matrix is compensated by a singularity in the second Clebsch - Gordan coefficient. Each FR violating pair $(j, \underline{j} = p - j - 1)$ gives as a whole a finite, non-zero contribution, which corresponds to the indecomposable representation $\mathcal{E}_{j,\underline{j}}$. It can be computed via L'Hôpital - essentially the singularities of the standard normalization constant of the Clebsch - Gordan coefficients are compensated by the relations resulting from the intertwining operator $\underline{\mathcal{D}}$. For the same reasons the summation on the r.h.s of the equality (15) defining the $\mathcal{R}$- matrix is also finite and in general runs beyond the bounds of the FR. Let us see what are the implications of these observations for the 4-point functions.

Guided by (12) we construct explicitly these correlations using the normalized DF conformal blocks. Namely we define in the thermal case

$$F^{\vec{j}}_{\vec{m}}(z_1, z_2, z_3, z_4)=\sum_{j_5}(-1)^{\Delta^{j_5}_{12}} K^{j_5}_{\vec{m}}(j_1, j_2|j_3, j_4)\tilde{J}_{j_5}(z_1,j_1; z_2,j_2; z_3,j_3; z_4,j_4), \tag{17}$$

$$K^{j_5}_{\vec{m}}(j_1, j_2|j_3, j_4) = \sum_{l_1, m_5, l_4} \begin{bmatrix} 0 & j_1 & j_1 \\ 0 & m_1 & l_1 \end{bmatrix}_q \begin{bmatrix} j_1 & j_2 & j_5 \\ l_1 & m_2 & m_5 \end{bmatrix}_q \begin{bmatrix} j_5 & j_3 & j_4 \\ m_5 & m_3 & l_4 \end{bmatrix}_q \begin{bmatrix} j_4 & j_4 & 0 \\ l_4 & m_4 & 0 \end{bmatrix}_q, \tag{18}$$

$$\vec{m}= (m_1, m_2, m_3, m_4),$$

where $\tilde{J}_{j_5}$ is the conformal block of the previous section. However, in counterdistinction to what one would obtain strictly following (12), we will not restrict for $q^p = 1$ the sum in (17) to the FR bounds.

The first Clebsch - Gordan coefficient in (18) reduces simply to $\delta_{m_1 l_1}$. The choice of the sign in (17) is dictated from the explicit expression for the fusion matrix transforming the normalized DF blocks $\tilde{J}_{j_5}$; it can be absorbed in $\tilde{J}_{j_5}$, changing the normalization. Indeed with this choice we get using (13)

$$F^{j_1\ j_2\ j_3\ j_4}_{m_1 m_2 m_3 m_4}(z_1, z_2, z_3, z_4) = \sum_{n_2, n_3} (\mathcal{R}^{j_3 j_2}_{\epsilon})^{n_3 n_2}_{m_3 m_2}\ F^{j_1\ j_3\ j_2\ j_4}_{m_1 n_3 n_2 m_4}(z_1, z_3, z_2, z_4)\ , \tag{19}$$

and similarly,

$$F^{j_1\ j_2\ j_3\ j_4}_{m_1 m_2 m_3 m_4}(z_1, z_2, z_3, z_4) = \sum_{n_1, n_2} (\mathcal{R}^{j_2 j_1}_{\epsilon})^{n_2 n_1}_{m_2 m_1}\ F^{j_2\ j_1\ j_3\ j_4}_{n_2 n_1 m_3 m_4}(z_2, z_1, z_3, z_4)\ . \tag{20}$$

The last Clebsch - Gordan coefficient in (18), which reduces to a constant, is needed to ensure the corresponding relation with the last two arguments replaced. Combining (19), (20), one gets an analogous formula with z_1, z_3 replaced. The composite $\mathcal{R}^{j_1 j_3}$ - matrix satisfies the relation (compare with (10))

$$\sum_{m_1, m_3, m} \begin{bmatrix} j_3 & j_2 & j \\ m_3 & m_2 & m \end{bmatrix}_q \begin{bmatrix} j & j_1 & j_4 \\ m & m_1 & m_4 \end{bmatrix}_q (\mathcal{R}^{j_3 j_1}_{\epsilon})^{m_3\ m_1}_{n_3\ n_1} =$$

$$= e^{i\pi\epsilon(\Delta_{j_4}-\Delta_{j_1}-\Delta_{j_2}-\Delta_{j_3})} \sum_{m_5 j_5} \left\{ \begin{matrix} j_3 & j_2 & j \\ j_1 & j_4 & j_5 \end{matrix} \right\}_q \left[\begin{matrix} j_1 & j_2 & j_5 \\ n_1 & m_2 & m_5 \end{matrix} \right]_q \left[\begin{matrix} j_5 & j_3 & j_4 \\ m_5 & n_3 & m_4 \end{matrix} \right]_q . \quad (21)$$

We now come back to the problem discussed above. If we restrict for $q^p = 1$ the sum in the mixed functions (17) according to the upper bounds of the FR (in the cases when these bounds do not coincide with the classical ones), we will not be able to reproduce in general the $\mathcal{R}$-covariance condition (19). Indeed, as discussed above, the restricted numerical "vertex - path" identitities do not hold true.

This problem in the construction based on (13) was first noticed by the authors of [22]. The strategy followed in [24] is to use (12) (and hence (17)) with the sum restricted according to the FR , but to require the $\mathcal{R}$-covariance only on a subspace of the space $\mathcal{H}= \oplus_j \mathcal{H}_j \otimes \mathcal{E}_j$. This is equivalent to the $\mathcal{R}$- covariance of (17) in the cases when the classical and the fusion bounds coincide.

We adopt a different alternative, taking the sum in the 4-point functions (17) to run according to the classical bounds. Hence we admit in general terms violating the upper FR bounds. The arguments of the previous section can no longer be used to cancel these terms, since now the DF integrals are multiplied by the singular q-blocks (18). Thus the unphysical border point $j = (p-1)/2$ survives, reflecting the fact that the vertex operator $Pr_p\, V^r_{\alpha_{12}}\, Pr_{2j_4+1} \left[\begin{matrix} \frac{p-1}{2} & \frac{1}{2} & j_4 \\ m & m_3 & m_4 \end{matrix} \right]_q$ no longer vanishes identically on the states in $\mathcal{H}_{j_4} \otimes \mathcal{E}_{j_4}$. Similarly the contribution of the pairs $(j, \underline{j})$ can no longer be cancelled .[2] This is in agreement with the quantum group tensor product decomposition rules which do not coincide with the FR.

Our choice ensures that the $\mathcal{R}$-covariance is maintained for averages on the full space $\mathcal{H}= \oplus_j \mathcal{H}_j \otimes \mathcal{E}_j$. In particular (19) is valid for any j_1, j_2, j_3, j_4-regular. Although in this way we have to apply the "vertex-path" relation even in cases when both sides of (13) are divergent, no problem arises, since we actually use this relation always multiplied by the conformal blocks $\tilde{J}_j$ with the sum over j taken. That makes its contribution finite, combining once again the relations obtained from the intertwining operators $\underline{\mathcal{D}} = X_-^{p-2j-1}$ and Q_{p-2j-1}. (The relation (13) itself can be given meaning in that case by choosing an appropriate normalization of the Clebsch - Gordan coefficients.)

The appearance of unphysical intermediate states makes the correspondence of our correlations and the operator formalism in [20] rather heuristic , since this implies that, unlike [20,24], we allow operators which do not keep invariant the space $\mathcal{H}$. Furthermore the indecomposable representations $\mathcal{E}_{j,\underline{j}}$ will have to be taken properly into account to adapt the formulation in [20].

Recently Gomez and Sierra (GS) [25] have proposed new screened vertex operators acting in a Fock space, which differ from those in (2) by the choice of the contours. They are shown to provide a representation space for the (left) action of a quantum algebra, which reduces in the thermal case to the algebra $\mathcal{U}_q(sl(2))$, $q = exp(2\pi i p'/p)$, or, $\mathcal{U}_{q'}(sl(2))$, $q' = exp(2\pi i p/p')$.

[2]The contribution of these pairs can be made manifestly finite if (17) are rewritten in a different basis, so that the DF integrals $I^s_{k(j)}(z)$ are replaced via contour deformations by integrals with inhomogeneous behaviour for both $z = 0$ and $1 - z = 0$, while the coefficients in front of $I^s_{\underline{k}(j)}(z)$ become finite. This reflects the indecomposable character of the representation $\mathcal{E}_{j,\underline{j}}$.

The operators in [25] are manifestly $\mathcal{R}$-covariant and presumably they provide the proper operator language behind the correlations constructed here. We show in Appendix B that for $m_4 = -j_4$ our 4-point correlations can be recovered using the screened vertices of [25].

We end this Section with a remark concerning the $\mathcal{R}$- covariance beyond the thermal case. The general $\mathcal{R}$- matrix elements have been obtained in [25] as products of the thermal ones, times mixing phases. This reflects the fact that the quantum algebra in the general case does not reduce simply to the product of the two thermal algebras $\mathcal{U}_q$ and $\mathcal{U}_{q'}$, considered as Hopf algebras. On the other hand, with the knowledge of the explicit expression for the general fusion matrix it is straightforward to extend (17) and the $\mathcal{R}$- covariance relations (19), (20) beyond the thermal case, taking simply the product of the thermal Clebsch - Gordan coefficients. The resulting general $\mathcal{R}$- matrix factorizes up to an overall phase , depending only on the corresponding spins j_i, j_i' . Although the correlation functions obtained in this way coincide again up to a constant with those computed with the GS operators, they apparently describe a different quantum algebra. We leave the details to App. A.

4 Quantum group invariants

One can invert (17) using the normalization relation for the Clebsch - Gordan coefficients:

$$(-1)^{j_1+j_2-j}\tilde{J}_j(\vec{z}) = \sum_{m_i,m,l_4} \begin{bmatrix} j_1 & j_2 & j \\ m_1 & m_2 & m \end{bmatrix}_q \begin{bmatrix} j & j_3 & j_4 \\ m & m_3 & l_4 \end{bmatrix}_q \begin{bmatrix} j_4 & j_4 & 0 \\ l_4 & m_4 & 0 \end{bmatrix}_q F_{\vec{m}}^{\vec{j}}(\vec{z}). \tag{22}$$

The r.h.s. of (22) is thought of as the limit for $q \to q_0$, $q_0^p = 1$, of the expression extended to generic values of q . Similarly, the local 2-dimensional (scalar) functions are recovered as

$$\sum_{m_1,m_2,m_3,m_4} F^{j_1\ j_2\ j_3\ j_4}_{m_1 m_2 m_3 m_4}(\vec{z})\ F^{j_1\ j_2\ j_3\ j_4}_{m_1 m_2 m_3 m_4}(\bar{\vec{z}}) = \sum_j \tilde{J}_j(\vec{z})\ \tilde{J}_j(\bar{\vec{z}})\ . \tag{23}$$

The summation on the r.h.s. of (23) reduces to the FR bounds taking into account the results of Section 2.

The inverted formula (22) expresses the conformal 4-point chiral correlation as a $\mathcal{U}_q(sl(2))$-invariant. The invariants of the algebra $\mathcal{U}_q(sl(2))$ under the action of Δ^{n-1} in $\mathcal{E}_{j_1} \otimes ...\mathcal{E}_{j_n}$ were realized in [7] as n-point functions of complex variables $u_1, ..., u_n$, obtained by invariant pairing of the basic 3-point kernels, related to the Clebsch - Gordan coefficients. The monomials $u^{j+m}/\sqrt{[j+m]_q![j-m]_q!}$ correspond to the states e^j_m . The invariants read (see also [33] for n=2,3, and [34])

$$S_n^{\{a\}}(j_1,..,j_n) = \sum_{m_i,m_{a_i}} \begin{bmatrix} 0 & j_1 & a_1 \\ 0 & m_1 & m_{a_1} \end{bmatrix}_q \begin{bmatrix} a_1 & j_2 & a_2 \\ m_{a_1} & m_2 & m_{a_2} \end{bmatrix}_q \begin{bmatrix} a_2 & j_3 & a_3 \\ m_{a_2} & m_3 & m_{a_3} \end{bmatrix}_q \cdots \tag{24}$$

$$\cdot \begin{bmatrix} a_{n-2} & j_{n-1} & a_{n-1} \\ m_{a_{n-2}} & m_{n-1} & m_{a_{n-1}} \end{bmatrix}_q \begin{bmatrix} a_{n-1} & j_n & 0 \\ m_{a_{n-1}} m_n & & 0 \end{bmatrix}_q e^{j_1}_{m_1} \otimes e^{j_2}_{m_2} \ldots \otimes e^{j_n}_{m_n} ,$$

$$\{a\} = \{a_1 = j_1, a_2, a_3, ..., a_{n-2}, a_{n-1} = j_n\}.$$

In the second quantized version of [35] the variable u is replaced by an operator, generating together with the finite difference operator D_u a q-deformed Heisenberg algebra. The GS vertex operators $e^j_m(z)$ provide yet another realization of the states e^j_m and of the invariants (24).[3]

For generic q the invariants in $\{ \mathcal{E}_{j_{i_1}} \otimes ... \otimes \mathcal{E}_{j_{i_n}} \}$ for a_{n-1}- fixed span a representation of the braid group $\mathbb{B}_N$, $n = N - 1$, generated by $g_i = I \otimes ... R^{i\ i+1} ... \otimes I$, $i = 1, 2, ..N - 1$, where R=P $\mathcal{R}$ and P is the permutation operator. This can be seen using (14) and the "vertex - path" representation (13) of the matrix $\mathcal{R}^{j_i j_{i+1}}$ acting in $\mathcal{E}_{j_i} \otimes \mathcal{E}_{j_{i+1}}$. Consider the restricted subset of invariants for j_i, $i = 1, ..., n$ - regular, labelled by restricted paths {a}, i.e., all a_i are regular and furthermore, any triple (a_{i-1}, j_i, a_i) is consistent with the fusion rules. The action of $\mathbb{B}_N$ is well defined on this subset for $q^p = 1$, since (13) is well defined, as discussed above. However, it does not keep the subset invariant. On the other hand if we take, under the same restrictions, the vacuum expectation value of both sides of (24), when the states are realized by the GS screened operators, we reproduce the conformal blocks, as in (22) for n=4. They provide a restricted set, invariant under the action of $\mathbb{B}_N$. Taking the average has the effect of automatically truncating all summations violating the bounds of the FR. This implies in particular that the "vertex-path" identities can be given meaning with summation restricted according to the FR. Indeed, consider for admissible $\{j_1, j_2, j_5, j_3, j_4\}$ instead of the numerical relations (13) the operator identity obtained by multiplying both sides of (13) with $\begin{bmatrix} j_4 & j_4 & 0 \\ m_4 & -m_4 & 0 \end{bmatrix}_q$ $e^{j_1}_{m_1}(z_1) e^{j_2}_{m_2}(z_2) ... e^{j_4}_{-m_4}(z_4)$ and summing over m_i . The vacuum expectation value reduces the r.h.s. to the bounds of the FR, reflecting the fact that the FR are fulfilled on the space of BRST states.

For $j_i = 1/2$, $i = 1, ..., n - 1 = N$ one can consider the corresponding representations of the centralizer algebra of $\mathcal{U}_q(sl(2))$ in the space $\mathcal{E}_{\frac{1}{2}}^{\otimes N}$. The centralizer is the Temperley - Lieb - Jones algebra [36], isomorphic to a factor of a Hecke algebra $H_N(q)$ of type A_N . The irreducible representations of the factor algebras for $q^p = 1$ realized in [37] in terms of a restricted set of Young tableaux, are equivalent to the representations provided by the related conformal blocks [38]. To interpret these algebras as the centralizers of the quantum algebra one needs the restricted tensor product, obeying the bounds of the FR [6] (see also [28], [3]). It is clear now that instead of imposing the rather artificial restricted product on the general invariants, one can realize them by the products of the independently defined operators in [25]. Taking the expectation values keeps invariant the restricted set of invariants. More generally, given the quantum group covariant operators $e^{j',j}_{m',m}(z)$, the full minimal theory can be recovered.

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[3] By multiplying both sides of (17) with $\prod_{i=1}^4 u_i^{j_i+m_i} \sqrt{[j_i+m_i]_q![j_i-m_i]_q!}$ and summing over m_i one gets a linear combination of two types of invariants, which can be interpreted as the 4-point function of $\mathcal{U}_q$ -invariant conformal fields $\phi_{j_i}(z_i, u_i)$. Examples of such 4-point correlations, in cases when the classical and FR bounds coincide, were considered in [23], using the formalism developed in [7].

5 Appendix A: Notation and useful formulae

We start by recalling the relation for the 4-point $\mathcal{U}_q(sl(2))$-invariant kernels in [7]. In terms of the properly normalized Clebsch - Gordan coefficients it reads (see (18) for notation)

$$\tilde{K}^{p-j-1}_{\underline{m}}(j_1, j_2|j_3, j_4) = (-1)^{(p'+1)(j_2+j_4-j_1-j_3+\underline{j}-j)} \sqrt{\frac{[2\underline{j}+1]_q}{[2j+1]_q}}\, \tilde{K}^{j}_{\underline{m}}(j_1, j_2|j_3, j_4), \qquad (A.1)$$

$$\underline{j} = p - j - 1,\ 1 \leq 2j + 1 < p,$$

where

$$\tilde{K}^{\underline{j}}_{\underline{m}}(j_1, j_2|j_3, j_4) = c_j\, K^{\underline{j}}_{\underline{m}}(j_1, j_2|j_3, j_4),\quad , c_{\underline{j}} = (-1)^{p'(\underline{j}-j)}\, c_j\,,$$

$$c_j = c_j(j_1, j_2)\, c_j(j_3, j_4)\,,\ c_j(j_1, j_2) = \sqrt{[j_1 + j_2 - j]_q![j_1 + j_2 + j + 1]_q!}\,.$$

In deriving the relations (A.1), (7), (8), one has to use that $(p'-1)(p-1)=0$ mod 2, and

$$\frac{[\Delta_1 + p]_q!}{[\Delta_2 + p]_q!} = \frac{[\Delta_1]_q!}{[\Delta_2]_q!}(-1)^{p'(\Delta_1-\Delta_2)},\quad [\Delta]_q![p-1-\Delta]_q!(-1)^{(p'+1)\Delta} = [p-1]_q!. \qquad (A.2)$$

In (8) we use the normalized DF integrals

$$\tilde{I}^{s's}_{k'k}(a,b,c;z) = \sqrt{X^{s'}_{k'}(a',b',c';\delta')\, X^{s}_{k}(a,b,c;\delta)}\, I^{s's}_{k'k}(a,b,c;z)\,, \qquad (A.3)$$

$$X^s_k(a,b,c;\delta) = \prod_{1}^{k-1} s(j\delta) \prod_{1}^{s-k} s(j\delta)$$

$$\cdot \prod_{j=0}^{k-2} \frac{s(a+j\delta)s(c+j\delta)}{s(a+c+(k-2+j)\delta)} \prod_{j=0}^{s-k-1} \frac{s(b+j\delta)s(d+j\delta)}{s(b+d+(s-k-1+j)\delta)}, \qquad (A.4)$$

$$k, k' = 1, 2, ..., s,$$

and the normalized fusion (crossing) matrix,

$$\tilde{\alpha}^s_{ik}(a,b,c;\delta) = \left(\frac{X^s_i(a,b,c;\delta)}{X^s_k(b,a,c;\delta)}\right)^{\frac{1}{2}} \alpha^s_{ik}(a,b,c;\delta)\,, \qquad (A.5)$$

$$\tilde{I}^s_i(a,b,c;z) = \sum_{k=0}^{s} \tilde{\alpha}^s_{ik}(a,b,c;\delta)\, \tilde{I}^s_k(b,a,c;1-z)\,. \qquad (A.6)$$

Let $A = (j'_1, j_1)$, $C = (j'_2, j_2)$, $B = (j'_3, j_3)$, $D = (j'_4, j_4)$, $J = (j'_5, j_5)$, $T = (j'_6, j_6)$. The general fusion matrix factorizes [16] to

$$\left\{\begin{matrix} A & C \\ B & D \end{matrix}\right\}_{JT} = \tilde{\alpha}^s_{ik}(a,b,c;\delta)\, \tilde{\alpha}^{s'}_{i'k'}(a',b',c';\delta')\,. \qquad (A.7)$$

It is expressed by the $6j$-symbols according to [32]

$$\tilde{\alpha}^{s}_{ik}(a,b,c;\delta) = (-1)^{(i-1)(1+2j_2'+2j_3')+(k-1)(1+2j_1'+2j_2')}$$
$$\cdot(-1)^{(s-1)(2j_4')} \left\{ \begin{matrix} j_1(a) & j_2(c) & j_5 \\ j_3(b) & j_4(d) & j_6 \end{matrix} \right\}_q , \qquad (A.8)$$

where, a,c,b are given by $2\alpha_-\alpha_i = -2j_i\delta + 2j_i', i = 1,2,3$, resp.; $d = (2j_4+2)\delta - 2j_4' - 2$; $s = j_1 + j_2 + j_3 - j_4 + 1,\ \ j_5 = j_1 + j_2 - i + 1,\ \ j_6 = j_2 + j_3 - k + 1,\ \ J = (j_5', j_5)$.

The explicit expression in (A.8) was derived in [32] comparing the Racah identities for the 6j-symbols and for the fusion matrix. There remains an arbitrariness of sign, which is fixed by the known particular values of α^{s}_{ik} , computed in [16]. The first sign factors could be absorbed changing the normalization of the blocks (A.3), however, the last factor, which is trivial in the thermal case, cannot be distributed. It is actually important in generalizing the $\mathcal{R}$-covariance relations in Sect. 3 beyond the thermal case. We define

$$F^{\vec{J}}_{\vec{M}}(z_1, J_1, ..., z_4, J_4) \doteq \sum_{J_5} K^{J_5}_{\vec{M}}(J_1\ J_2|J_3\ J_4)\tilde{J}_{J_5}(z_1, J_1, ..., z_4, J_4)\ , \qquad (A.9)$$

$$K^{J_5}_{\vec{M}}(J_1\ J_2|J_3\ J_4) = (-1)^{\Delta^{5}_{12}(1+2j_2'+2j_3')+\Delta^{5'}_{1'2'}(1+2j_2+2j_3)} K^{j_5}_{\vec{m}}(j_1,j_2|j_3,j_4) K^{j_5'}_{\vec{m}'}(j_1',j_2'|j_3',j_4') \quad (A.10)$$

$$\vec{M} = (M_1, M_2, M_3, M_4),\ M = (m', m),\ J = (j', j)\ .$$

where $K^{j_5}_{\vec{m}}$ is defined in (18) and $\tilde{J}_{J_5}$ is represented up to a prefactor by the normalized DF integral (A.3),

$$\tilde{J}_{J_5}(\vec{z}) = f(\{\alpha_i, z_{ik}\}) z^{2\alpha_1\alpha_2}(1-z)^{2\alpha_2\alpha_3}\ \tilde{I}^{s's}_{i'i}(a,b,c;z),\ z = \frac{z_{12}z_{34}}{z_{13}z_{24}}, \qquad (A.11)$$

$$f(\{\alpha_i, z_{ik}\}) = \frac{z_{13}^{\Delta_4-\Delta_1-\Delta_2-\Delta_3} z_{14}^{\Delta_4-\Delta_1-\Delta_2-\Delta_3}}{z_{24}^{2\Delta_2}\ z_{34}^{\Delta_3+\Delta_4-\Delta_1-\Delta_2}},$$

with the same identification of the parameters a,b,c, as above. The relations (19),(20), are generalized with a $\mathcal{R}$-matrix given by

$$\mathcal{R}^{J_1J_2}_{\epsilon} = e^{-2i\pi\epsilon(j_1'j_2+j_1j_2')}\ \mathcal{R}^{j_1j_2}_{\epsilon}\ \mathcal{R}^{j_1'j_2'}_{\epsilon}. \qquad (A.12)$$

where $\mathcal{R}^{j_1j_2}$ (or $\mathcal{R}^{j_1'j_2'}$) is the thermal $\mathcal{R}$- matrix with matrix elements

$$(\mathcal{R}^{j_1j_2})^{n_1n_2}_{m_1m_2} = \frac{(1-q^{-1})^{m_1-n_1}}{[m_1-n_1]!}\left(\frac{[j_1-n_1]_q![j_1+m_1]_q![j_2+n_2]_q![j_2-m_2]_q!}{[j_1+n_1]_q![j_1-m_1]_q![j_2-n_2]_q![j_2+m_2]_q!}\right)^{1/2}$$
$$\cdot q^{(n_1n_2+m_1m_2+m_1(n_2+1)+n_1(m_2-1))/4}\delta_{n_1+n_2,m_1+m_2}. \qquad (A.13)$$

The additional Clebsch - Gordan coefficient $\begin{bmatrix} j_4 & j_4 & 0 \\ l_4 & m_4 & 0 \end{bmatrix}_q$ in (18) is simply

$$\begin{bmatrix} j & j & 0 \\ l & m & 0 \end{bmatrix}_q = (-1)^{j+m} q^{\frac{-m}{2}} [2j+1]_q^{-1/2} \, \delta_{l+m,0}.$$

It is needed as in (17), (18) to ensure the $\mathcal{R}$- covariance when z_3 is replaced by z_4. One has to use along with (15) the symmetry relation of the Clebsch - Gordan coefficients [30]:

$$\begin{bmatrix} j_5 & j_3 & j_4 \\ m_5 & m_3 & l_4 \end{bmatrix}_q = (-1)^{\Delta^4_{35}+j_3-m_3} q^{-\frac{m_3}{2}} \sqrt{\frac{[2j_4+1]_q}{[2j_5+1]_q}} \begin{bmatrix} j_4 & j_3 & j_5 \\ l_4-m_3 & m_5 \end{bmatrix}_q . \qquad (A.14)$$

6 Appendix B: Relating the 4-point correlations

In this Appendix we shall recover the factorized expression (17) starting from a 4-point correlation of GS operators. In order to keep the notation simple we shall consider only thermal operators; in particular $\delta = \alpha_-^2$, $q = exp(2\pi i \delta)$.

The screened vertex operators in [25] are defined according to

$$E^r_{\alpha_{1,2j+1}}(z) = \int_{i\infty}^{(+z)} dt_1 ... \int_{i\infty}^{(+z)} dt_r \; V_{\alpha_-}(t_1)...V_{\alpha_-}(t_r) \; V_{\alpha_{1,2j+1}}(z) \, , \qquad (B.1)$$

where $2j+1 < p$, and the contours around z are ordered as depicted in Fig. 5, so that t_1 runs along a contour which contains all the other contours. They can be identified with the states e^j_m, $m = r - j$, of a $2j+1$ -dimensional representation of $\mathcal{U}_q(sl(2))$.

We shall compute the 4-point correlation of the operators $E^{r_i}_{\alpha_i}$ with $\alpha_i = \alpha_{1,2j_i+1}$, $\alpha_4 = 2\alpha_0 - \alpha_{1,2j_4+1}$, arbitrary $r_i, i = 1,2,3$, and $r_4 = 0$. The charge conservation condition implies that $m \equiv m_1 + m_2 = j_4 - m_3$. Reducing the contours in (B.1) to contours from z to infinity and path-ordering the integrals with $|t_1| > |t_2| ... > |t_r| > |z|$ creates a constant $(q^{-\frac{1}{2}} - q^{\frac{1}{2}})^{j-m} \mathcal{E}(j,m)$,

$$\mathcal{E}(j,m) = [2j]_q! q^{\mathcal{F}(j,m)/2} \frac{[j-m]_q!}{[j+m]_q!} , \qquad \mathcal{F}(j,m) = (j-m)(j+m+1). \qquad (B.2)$$

Deforming the contours to the right we can write the 4-point function as a linear combination of path-ordered integrals $\mathcal{M}_{rut}$ along the real axis (see (Fig. 6)):

$$\langle \alpha_4 | E^{r_3}_{\alpha_3}(1) E^{r_2}_{\alpha_2}(z) E^{r_1}_{\alpha_1}(0) | 0 \rangle_{2\alpha_0} = z^{2\alpha_1\alpha_2} (1-z)^{2\alpha_2\alpha_3} (q^{-\frac{1}{2}} - q^{\frac{1}{2}})^{s-1} \prod_{i=1}^{3} \mathcal{E}(j_i, m_i)$$

$$\cdot \sum_{x=0}^{j_1-m_1} \mathcal{C}(x) \sum_{y=0}^{j_1+j_2-m} \mathcal{D}(x+y) \mathcal{M}_{x,y,s-1-x-y}(z) \, , \qquad (B.3)$$

where $s = j_1 + j_2 + j_3 - j_4 + 1$, and

$$\mathcal{C}(x) = q^{(j_2+m_2)(j_1-m_1-x)/2} \begin{bmatrix} j_1+j_2-m-x \\ j_2-m_2 \end{bmatrix}_q , \quad \mathcal{D}(x+y) = q^{(j_3+m_3)(j_1+j_2-m-x-y)/2} \begin{bmatrix} s-1-x-y \\ j_3-m_3 \end{bmatrix}_q . \qquad (B.4)$$

In particular $M_{k-1,0,s-k}(z) = I_k^s(a,b,c;z)$. Following a contour deformation argument similar to the one in [16] we can deform all the integrals from z to 1 in M_{rut} into integrals from 0 to z and from 1 to ∞, and express M_{rut} as a linear combination of DF integrals I_k^s. First we deform one $[z,1]$ contour getting the recursion relation:

$$M_{rut} = -\frac{[r+1]_q s(a+r\delta)}{[u]_q s(a+c+(2r+u-1)\delta)} M_{r+1,u-1,t} \tag{B.5}$$

$$-\frac{[t+1]_q s(a+b+c+(2u+2r+t-2)\delta)}{[u]_q s(a+c+(2r+u-1)\delta)} M_{r,u-1,t+1}.$$

One can solve this recursion obtaining

$$M_{x,y,s-1-x-y} = \sum_{k=x+1}^{y+x+1} \mathcal{A}(x,y,k) I_k^s, \tag{B.6}$$

where

$$\mathcal{A}(x,y,k) = (-1)^y \begin{bmatrix} s-k \\ x+y-k+1 \end{bmatrix}_q \begin{bmatrix} k-1 \\ x \end{bmatrix}_q$$

$$\cdot \prod_0^{x+y-k} \frac{s(a+b+c+(s+x+y-3-i)\delta)}{s(a+c+(k+x+y-2-i)\delta)} \prod_0^{k-x-2} \frac{s(a+(x+i)\delta)}{s(a+c+(k+x-2-i)\delta)} . \tag{B.7}$$

For the r.h.s. of (B.3) (neglecting the prefactor and the constant in front of the sum) we get, after regrouping the sum:

$$\sum_{k=1}^{s} I_k^s(a,b,c;z)\, \mathcal{P}_k = \sum_{k=1}^{s} I_k^s(z) \sum_{x=0}^{j_1-m_1} \mathcal{C}(x) \sum_{y=k-x-1}^{j_1+j_2-m-x} \mathcal{D}(x+y)\mathcal{A}(x,y,k) . \tag{B.8}$$

The sum over y gives:

$$\sum_y \mathcal{D}\mathcal{A} = \begin{bmatrix} s-k \\ j_3-m_3 \end{bmatrix} \begin{bmatrix} k-1 \\ x \end{bmatrix} (-1)^{j_1+j_2-m-x} e^{i\pi(j_1+j_2-m-k+1)(a+c+(j_1+j_2-m+k-2)\delta)}$$

$$\cdot \prod_0^{k-x-2} \frac{s(a+(x+i)\delta)}{s(a+c+(k+x-2-i)\delta)} \prod_0^{j_1+j_2-m-k} \frac{s(b+(s-k-1-i)\delta)}{s(a+c+(2k+i)\delta)} , \tag{B.9}$$

using a standard formula for the q-hypergeometric function:

$$\sum_{t=0}^{l} e^{i\pi(B-(l-1)\delta)t}(-1)^t \begin{bmatrix} l \\ t \end{bmatrix}_q \prod_0^{t-1} \frac{s(A+B+i\delta)}{s(A+i\delta)} = (-1)^l e^{i\pi l(A+B)} \prod_0^{l-1} \frac{s(B-i\delta)}{s(A+i\delta)} . \tag{B.10}$$

With (B.9) the sum over x in (B.8) becomes

$$\mathcal{P}_k = \sum_x \mathcal{C} \sum_y \mathcal{D}\mathcal{A} = \begin{bmatrix} s-k \\ j_3-m_3 \end{bmatrix}_q e^{i\pi(j_1+j_2-m-k+1)(a+c+(j_1+j_2-m+k)\delta)} \prod_0^{j_1+j_2-m-k} \frac{s(b+(s-k-i)\delta)}{s(a+c+(2k-2+i)\delta)}$$

$$\cdot\sum_{x=0}^{j_1-m_1}\begin{bmatrix}k\\x\end{bmatrix}_q(-1)^{j_1+j_2-m-x}\prod_0^{k-x}\frac{s(a+(x+i)\delta)}{s(a+c+(k+x+i)\delta)}\begin{bmatrix}j_1+j_2-m-x\\j_2-m_2\end{bmatrix}_q q^{-\frac{1}{2}(j_2-m_2)(j_1-m_1-x)}e^{-i\pi c(j_1-m_1-x)}$$

$$=(-1)^{j_2-m_2}q^{-\frac{1}{2}\mathcal{F}_5}\frac{[2j+1]_q[j_3+m_3]_q![j+m]_q![\Delta_{12}^5]_q![\Delta_{35}^4]_q!}{[j_2-m_2]_q![j_3-m_3]_q![j-m]_q![\Delta_{15}^2]_q![\Delta_{34}^5]_q!}\mathcal{S}\,, \tag{B.11}$$

where $j\equiv j_5=j_1+j_2-k+1,\ m\equiv m_5=m_1+m_2,\ \Delta_{ab}^c=j_a+j_b-j_c,\ \mathcal{F}_5=\mathcal{F}(j_5,m_5)$,

and

$$\mathcal{S}=\sum_{r\geq 0}\frac{(-1)^r q^{\frac{1}{2}r(j_2+m_2)}[j_2-m_2+r]_q![j_1+m_1+r]_q!}{[r]_q![j_1-m_1-r]_q![j_2-j+m_1+r]_q![j_2+j+m_1+1+r]_q!}\,. \tag{B.12}$$

This sum is proportional to the Clebsch - Gordan coefficient $\begin{bmatrix}j_1\ j_2\ j_5\\m_1\ m_2\ m\end{bmatrix}_q$, namely,

$$\begin{bmatrix}j_1\ j_2\ j_5\\m_1\ m_2\ m\end{bmatrix}_q=\left([2j_5+1]_q\frac{[j_5+m]_q![j_1-m_1]_q![\Delta_{12}^5]_q![\Delta_{25}^1]_q![j_1+j_2+j_5+1]_q!}{[j_5-m]_q![j_1+m_1]_q![j_2+m_2]_q![j_2-m_2]_q![\Delta_{15}^2]_q!}\right)^{\frac{1}{2}}$$

$$.q^{\frac{1}{4}(\mathcal{F}_1+\mathcal{F}_2-\mathcal{F}_5)}(-1)^{j_1-m_1}\mathcal{S}\,. \tag{B.13}$$

To obtain (B.13) one has to use the Racah-Fock form [30,39] of the Clebsch - Gordan coefficient and an identity for the generalized q-hypergeometric function ${}_3\Phi_2$ [40]:

$${}_3\Phi_2(-n,\alpha,\beta;\gamma,\delta|q,q)=\prod_0^n\frac{(1-q^{\gamma-\alpha+j})}{(1-q^{\gamma+j})}q^{n\alpha}{}_3\Phi_2(-n,\alpha,\delta-\beta;\alpha-n+1-\gamma,\delta|q,q^{1+\beta-\gamma})\,, \tag{B.14}$$

with $n=j_1-m_1,\alpha=j_1+m_1+1,\beta=-j_5+m,\delta=j_2-j+m_1+1,\gamma=-j_2-j_5+m_1$.

The Clebsch - Gordan coefficient $\begin{bmatrix}j_5\ j_3\ j_4\\m\ m_3\ j_4\end{bmatrix}_q$ is simply

$$\begin{bmatrix}j_5\ \ j_3\ \ j_4\\m\ m_3\ j_4\end{bmatrix}_q=q^{\frac{1}{4}(\mathcal{F}_3-\mathcal{F}_5)}(-1)^{j_5-m}\left(\frac{[j_5+m]_q![j_3+m_3]_q![\Delta_{35}^4]_q![2j_4+1]_q}{[j_5-m]_q![j_3-m_3]_q![\Delta_{34}^5]_q![\Delta_{45}^3]_q![j_3+j_4+j_5+1]_q!}\right)^{\frac{1}{2}}. \tag{B.15}$$

Putting everything together we finally obtain

$$\begin{bmatrix}j_1\ j_2\ j_5\\m_1\ m_2\ m\end{bmatrix}_q\begin{bmatrix}j_5\ j_3\ j_4\\m\ m_3\ j_4\end{bmatrix}_q(-1)^{k-1}\sqrt{X_k^s(a,b,c;\delta)}=(q^{\frac{1}{2}}-q^{-\frac{1}{2}})^{s-1}\sqrt{\mathcal{E}_1...\mathcal{E}_4}\ \mathcal{P}_k\,, \tag{B.16}$$

$$k=j_1+j_2-j_5+1,$$

with $\mathcal{E}_i$ given in (B.2) and X_k^s - in (A.4). Taking into account (A.3) one sees that (B.16) reproduces (17) for $m_4=-j_4$ up to a constant. Indeed, changing the normalization of the operators in (B.1) according to

$$\tilde{E}^{r}_{\alpha_{1,2j+1}} = (-1)^{j-m}(\mathcal{E}(j,m))^{-1/2} E^{r}_{\alpha_{1,2j+1}}, \; r = j-m, \tag{B.17}$$

we obtain

$$F^{j_1\; j_2\; j_3\; j_4}_{m_1 m_2 m_3 -j_4}(\vec{z}) = \begin{bmatrix} j_4 & j_4 & 0 \\ -m_4 & m_4 & 0 \end{bmatrix}_q \langle 0|\tilde{E}_{2\alpha_0-\alpha_{1,2j_4+1}}(z_4)\tilde{E}^{r_3}_{\alpha_{1,2j_3+1}}(z_3)\tilde{E}^{r_2}_{\alpha_{1,2j_2+1}}(z_2)\tilde{E}^{r_1}_{\alpha_{1,2j_1+1}}(z_1)|0\rangle_{2\alpha_0}. \tag{B.18}$$

In particular for $2j_1+2j_2 \leq p-2,\ 2j_3+2j_4 \leq p-2$, we can identify the l.h.s of (B.18) with the MR 4-point function $\langle\langle V^{j_1}_{m_1}(z_1)...V^{j_4}_{-j_4}(z_4)\rangle\rangle$, where both the conformal and the quantum group average are taken. (The inversed order of the operators is due to the opposite choice of the right and left Fock vacuum states in (17).)

This suggests that in general we can identify our correlations with the expectation values of the products of some Virasoro and $\mathcal{U}_q$ - covariant fields $e^j_m(z)$,

$$F^{j_1\; j_2\; j_3\; j_4}_{m_1 m_2 m_3 m_4}(\vec{z}) = \langle 0|e^{j_1}_{m_1}(z_1)...e^{j_4}_{m_4}(z_4)|0\rangle. \tag{B.19}$$

Their n-point functions admit integral representations realized by the Fock space averages of the screened vertices of [25]. In (B.18) e^j_m is represented either by $E^{j-m}_{\alpha_{1,2j+1}}$, or by $\begin{bmatrix} j & j & 0 \\ -m & m & 0 \end{bmatrix}_q E^{j+m}_{2\alpha_0-\alpha_{1,2j+1}}$. However, it does not mean that the GS correlations respect the symmetry $\alpha \rightarrow 2\alpha_0-\alpha$ (see App. C). The operators $e^j_m(z)$ provide the $\mathcal{U}_q(sl(2))$- invariants discussed in Sect. 4.

The computation can be easily generalized beyond the thermal case. Similarly to the factorization of the fusion matrix (A.7), the coefficient in the l.h.s. of (B.8) with I^s_k replaced by $I^{s's}_{k'k}$ factorizes to $\mathcal{P}_{k'}(a',b',c')\ \mathcal{P}_k(a,b,c)$, with $\mathcal{P}_k(a,b,c)$ recovered from (B.11). This creates an additional sign factor, if compared with (A.9,10), reflecting the fact that the general Clebsch - Gordan coefficients in [25] factorize to the thermal ones only up to mixing signs. One can combine the two approaches to find the corresponding generalization of (17) beyond the thermal case for arbitrary $m_i,\ m'_i,\ i=1,2,3,4$.

7 Appendix C: Non-diagonal solutions of the $\mathcal{R}$- covariance condition

We can use the results for the non-diagonal local minimal theory correlations [31,32] to construct more general solutions of the $\mathcal{R}$ -covariance relations. We shall assume that p is even, and hence p' is odd, and for simplicity we shall consider correlations of only two different fields. One can generalize (17) according to

$$F^{(N)}_{\underline{m}}(\vec{z};\vec{\bar{j}};\vec{j}) = \sum_{\bar{j}_5,j_5}(-1)^{\Delta^5_{12}} K^{\bar{j}_5}_{\underline{m}}(\bar{j}_1\ \bar{j}_2|\bar{j}_3\ \bar{j}_4) N_{\bar{j}_5,j_5}(\hat{j}_1,\hat{j}_2,\hat{j}_3,\hat{j}_4) J_{j_5}(z_1,j_1;...,z_4,j_4)\ , \tag{C.1}$$

where $\hat{j} = (j,\bar{j})$. The constants $N_{\bar{j},j}$ are assumed to satisfy the restrictions coming from requiring locality of the corresponding non-diagonal functions [32]

$$S^{(N)}(z_1,\hat{j}_1,...,z_4,\hat{j}_2) = \sum_{\bar{j},j} \tilde{J}_j(z_1,j_1;...,z_4,j_4)\ N_{\bar{j},j}(\hat{j}_1,\hat{j}_2,\hat{j}_1,\hat{j}_2)\ \tilde{J}_{\bar{j}}(\overline{z_1},\overline{j_1};...,\overline{z_4},\overline{j_4})\ . \quad \text{(C.2)}$$

These conditions ensure as well the $\mathcal{R}$-covariance of (C.1), assuming as in Sect.3 that the summation is not restricted to the FR bounds. One can recover the local function (C.2) from (C.1) and (17), in the same way as the scalar combination (23) is recovered.

Let us illustrate the construction by an example, corresponding to the (A,D) -type [41] local correlations of [31,32]. Consider (C.1) with $\hat{j}_1 = \hat{j}_3$, $\hat{j}_2 = \hat{j}_4$, $\bar{j}_1 = j_1$, $\bar{j}_2 = \sigma(j_2) = \frac{p}{2} - j - 1$. Then the constants $N_{\bar{j}}$ are given by

$$N_{\overline{j_5},j_5}(\hat{j}_1,\hat{j}_2,\hat{j}_1,\hat{j}_2) = (-1)^{s(\hat{j}_5)}\ \delta_{\overline{j_5},\sigma(j_5)} = (-1)^{s(\hat{j}_2)+s(\hat{j}_5)} N_{\overline{j_5},j_5}(\hat{j}_2,\hat{j}_1,\hat{j}_1,\hat{j}_2)\ ,$$

$$N_{\bar{j},j}(\hat{j}_1,\hat{j}_1,\hat{j}_2,\hat{j}_2) = (-1)^{s(\hat{j}_2)+j}\ \delta_{\bar{j},j},\ s(\hat{j}) = \Delta_j - \Delta_{\bar{j}}\ , \quad \text{(C.3)}$$

$$2j_1 = 0 \bmod 2,\ 2j_5+1,\ 2j_2+1 = \frac{p}{2} \bmod 2\ .$$

Note that given j_1, j_2 , there are two non-diagonal combinations $F^{(N)}(\vec{\bar{j}};\vec{j})$ and $F^{(N)}(\vec{j}\ ;\ \vec{\bar{j}}\ )$, which are not identical in general. For $\alpha_1 = \alpha_3 = \alpha_{1,2j_1+1}$ the corresponding correlations can be recovered by the GS operators choosing

$$\alpha_2 = 2\alpha_0 - \alpha_4 = 2\alpha_0 - \alpha_{1,2j_2+1}\ , \quad \text{(C.4)}$$

or,

$$\alpha_2 = 2\alpha_0 - \alpha_4 = 2\alpha_0 - \alpha_{1,p-2j_2-1}\ . \quad \text{(C.5)}$$

Let us stress that while in (17) the upper bounds of the fusion rules are not respected in general, in the non-diagonal generalization (C.1) the lower bounds in the summation over j_5 , entering $\tilde{J}_{j_5}$, can be violated as well.

We recall that the scalar and the (A,D) -type (thermal) local correlations admit a 2-dimensional, volume integral representation [31], equivalent to (C.2). This is not so for the exceptional (A,E)-type correlations, which can be realized only by the factorized linear combination (C.2). Similarly, the GS operators, which are analogues of the local 2-dimensional operators, cannot reproduce the corresponding (A,E)-type counterparts described by (C.1).

One can consider as well chiral analogs of the local correlations containing fermion fields, just taking over the corresponding constants $N_{\bar{j},j}$ [32]. Accordingly some of the $\mathcal{R}$- covariance relations will be modified by a minus sign. In exactly the same way one can construct also the analogues of the quasilocal 4-point functions with Z_2 statistics [31,32], which generalize to all minimal values of $c < 1$ the order - disorder Ising model correlations.

These results can be extended beyond the thermal case, either generalizing (A.9,10), or, the non-thermal version of [25].

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FIGURE CAPTIONS

Fig. 1*a* The diagram describing the embeddings of the $\mathcal{U}_q(sl(2))$Verma modules M_j (2j - integer) for $q^p = 1$. The arrows point to the embedded modules. The middle point is chosen to correspond to M_j with $1 \leq 2j+1 \leq p-1$. The same diagram describes the action of the finite difference operators $\mathcal{D}$ and $\underline{\mathcal{D}}$ in the spaces of functions $\mathcal{C}_j$.

Fig. 1*b* A different picture of the same diagram; $m = 2j+1$. The horizontal arrows correspond to compositions of embeddings depicted on Fig. 1a.

Fig. 2 The diagram of embeddings of the Virasoro Verma modules $M_{m',m}$ [13]. The standard parametrization is recovered identifying $M_{m',m}$ with $M_{p'-m',p-m}$ and $M_{m'+kp',m+kp}$, k - integer. The same diagram describes the singular vectors of the $A_1^{(1)}$ - Verma modules [14]. The arrows correspond to the action on the weights of the elements of the affine Weyl group $\hat{W}$, generated by w, w_0 .

Fig. 3 The complex of Fock spaces $F_{m',m}$, $m^{(')} = 2j^{(')}+1$ [18]. The arrows correspond to the screening charges Q_m, Q_{p-m} .

Fig. 4 The contours in the integrals representing the 4-point Felder' correlation on the l.h.s. of (3).

Fig. 5*a* The contours in the multiple integral representing the GS screened vertex $E^{j-m}_{\alpha_{1,2j+1}}(z)$.

Fig. 6*a* The path-ordered integrals describing three screened vertices sitting at the points 0,z,1.

Fig. 6*b* The deformed contours of the path-ordered integrals M_{rut} in (B.3).

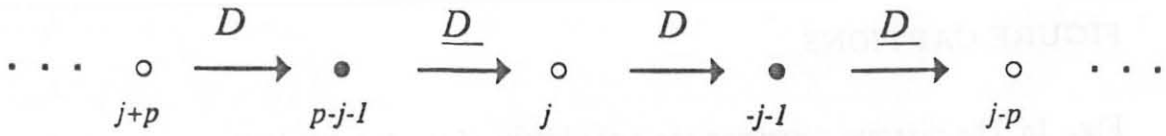

Fig. 1a

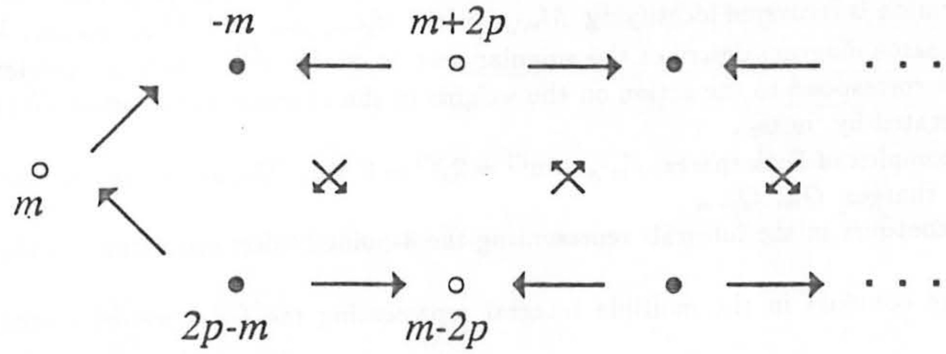

Fig. 1b

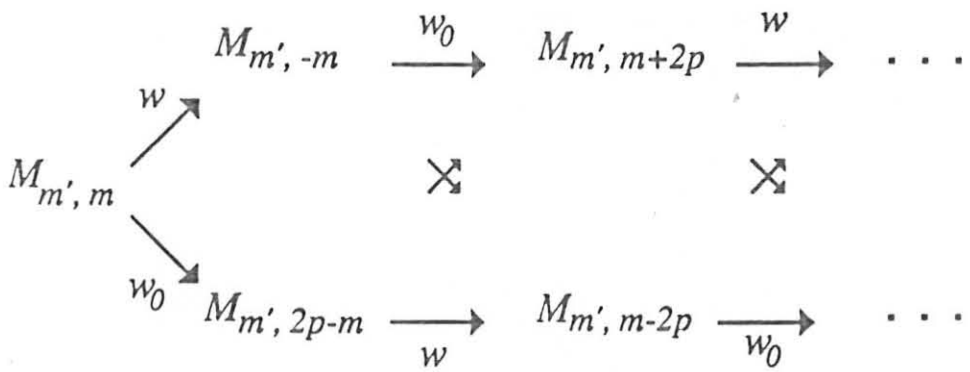

Fig. 2

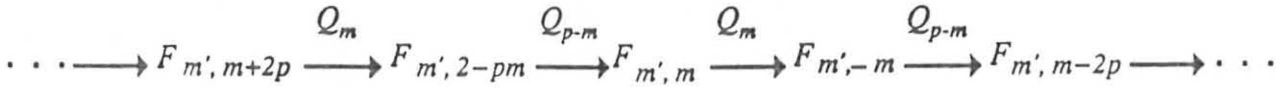

Fig. 3

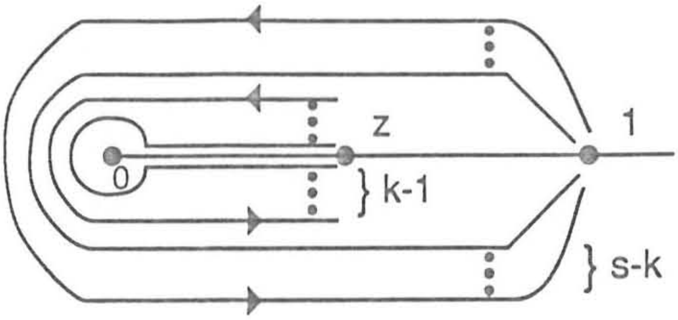

Fig. 4

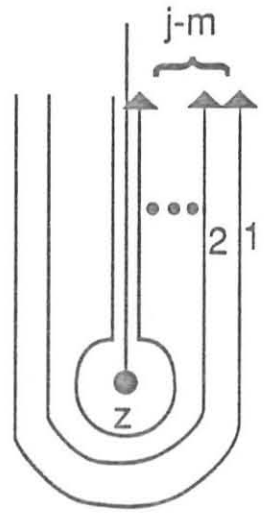

Fig. 5

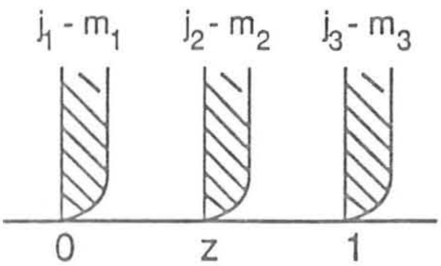

Fig. 6a

Fig. 6b