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q-OSCILLATOR REALIZATIONS OF THE QUANTUM SUPERALGEBRAS $sl_q(m,n)$ AND $osp_q(m,2n)$

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Abstract

Realizations of the quantum superalgebras corresponding to the $A(m,n)$, $B(m,n)$, $C(n+1)$ and $D(m,n)$ series are given in terms of the creation and annihilation operators of q-deformed Bose and Fermi oscillators.

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1. Introduction

Let $\mathcal{G}$ be a (simple) Lie algebra. The quantum Lie algebra[1−5] $\mathcal{G}_q$ is a deformation of the universal envelopping algebra of $\mathcal{G}$ which is endowed with a Hopf algebra structure.[6] This mathematical object is currently drawing a lot of attention, in part because of its connections with integrable systems and conformal field theories. The quantum algebra $\mathcal{G}_q$ can be characterized by giving its generators together with defining relations based on the Cartan matrix of $\mathcal{G}$.

The Weyl and Clifford algebras also admit quantum deformations[7] with q-analogues of the Bose, and respectively, Fermi oscillator operators as generators.[7−10] These quantized algebras have been used to construct oscillator realizations of the quantum algebras that correspond to all classical Lie algebras.[7] Here, we provide similar representations of the quantum Lie superalgebras associated to the unitary and the orthosymplectic series. Algebra homomorphisms from the quantized envelopping algebras of type $A(m,n)$, $B(m,n)$, $C(n+1)$ and $D(m,n)$ into the quantum Weyl superalgebra will be presented by expressing the generators of the quantum superalgebras as linears and bilinears in the creation and annihilation operators of q-bosons and q-fermions.

In Section 2 we review some results on the classification of contragredient Lie superalgebras. A general description of the quantum Lie superalgebras is given in Section 3. We introduce in Section 4 the q-analogue of the Bose and Fermi oscillators and present the quantized Weyl superalgebra. Section 5 comprises our main results, that is the q-oscillator realizations of the quantum Lie superalgebras $sl_q(m,n)$ and $osp_q(m,n)$. Unless stated otherwise, we shall stick to the conventions of Kac regarding superalgebras;[11−13] this means in particular, that we shall use non-symmetric Cartan matrices. We discuss in the Appendix the modifications that arise if one adopts instead, symmetric Cartan matrices.

2. Unitary and orthosymplectic Lie algebras

The Lie superalgebras $sl(m,n)$ and $osp(m,n)$ that respectively form the unitary and orthosymplectic series are in many ways similar to the classical Lie algebras. A superalgebra $\mathcal{G}$ of rank r belonging to either series can be characterized[11−13] by a Cartan matrix (a_{ij}) and a subset $\tau \subset I \equiv \{1,\ldots,r\}$ that identifies the odd generators. Unless $\mathcal{G}$ is an ordinary Lie algebra, in which case $\tau = \emptyset$, the set τ can actually be taken to consist of only one element.[11,12] Let $[\ ,\ ]$ stand for the graded product defined by $[x,y] = -(-)^{\deg x\,\deg y}[y,x]$ and $[x,[y,z]] = [[x,y],z] + (-)^{\deg x\,\deg y}[y,[z,x]]$, and denote as usual by $\mathrm{ad}\,x$ the adjoint operation $(\mathrm{ad}\,x)\,y = [x,y]$. The algebra $\mathcal{G}$ can be constructed from the $3r$ generators $\hat{e}_i$, $\hat{f}_i$ and $\hat{h}_i$, $i \in I$, which satisfy the relations[13]

$$\begin{aligned} [\hat{e}_i, \hat{f}_j] &= \delta_{ij}\,\hat{h}_i\,, & [\hat{h}_i, \hat{h}_j] &= 0\,, \\ [\hat{h}_i, \hat{e}_j] &= a_{ij}\,\hat{e}_j\,, & [\hat{h}_i, \hat{f}_j] &= -a_{ij}\,\hat{f}_j\,, \end{aligned} \tag{2.1}$$

and

$$(\mathrm{ad}\,\hat{e}_i)^{1-\tilde{a}_{ij}}\,\hat{e}_j = 0\,, \qquad (\mathrm{ad}\,\hat{f}_i)^{1-\tilde{a}_{ij}}\,\hat{f}_j = 0\,, \qquad i \neq j\,, \tag{2.2}$$

with

$$\deg \hat{h}_i = 0 \; ; \qquad \deg \hat{e}_i = \deg \hat{f}_i = 0 \; , \quad i \notin \tau \; ; \qquad \deg \hat{e}_i = \deg \hat{f}_i = 1 \; , \quad i \in \tau \; ,$$

and $(\tilde{a}_{ij})$ the matrix which is obtained from the non-symmetric Cartan matrix (a_{ij}) by substituting -1 for the strictly positive elements in the rows with 0 on the diagonal entry. In the case of Lie algebras the matrices (a_{ij}) and $(\tilde{a}_{ij})$ coincide and equation (2.2) reduce to the standard Serre relations.[14]

Following the established notation[11,12], we put

$$\begin{aligned}
A(m,n) &= sl(m+1,n+1) \; , && m,n \geq 0 \; , \quad m \neq n \; , \\
A(m,m) &= sl(m+1,m+1)/\{\lambda\, \mathbf{1}_{2m+2}\} \; , && m > 0 \; , \quad \lambda \in \mathbf{C} \; , \\
B(m,n) &= osp(2m+1,2n) \; , && m \geq 0 \; , \quad n > 0 \; , \\
C(n+1) &= osp(2,2n) \; , && n > 0 \; , \\
D(m,n) &= osp(2m,2n) \; , && m \geq 2 \; , \quad n > 0 \; .
\end{aligned}$$

We give below the Cartan matrix (a_{ij}), the set τ and the rank r, which are associated to the superalgebras belonging to these series.[12,13] In each case, we also specify a set of rational numbers d_i, $i = 1, \ldots, r$, such that: $d_i\, a_{ij} = d_j\, a_{ji}$. These numbers d_i will enter in the defining relations of the quantum superalgebras (see next section). In what follows

$$\mathcal{A}_n = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & & & \\ & & \ddots & & \\ & & & 2 & -1 \\ & & & -1 & 2 \end{pmatrix} , \tag{2.3}$$

stands for the $n \times n$ Cartan matrix of the rank n ordinary Lie algebra A_n.

- $A(m,n)$

$$(a_{ij}) = \begin{pmatrix} \mathcal{A}_m & & & & \\ & & -1 & & \\ & -1 & 0 & 1 & \\ & & -1 & & \\ & & & & \mathcal{A}_n \end{pmatrix} , \tag{2.4}$$

$$\tau = \{m+1\} \qquad\qquad r = m+n+1 \; , \tag{2.5}$$

$$d_i = (\underbrace{1, \ldots, 1}_{m+1}, \underbrace{-1, \ldots, -1}_{n}) \; . \tag{2.6}$$

When $m = n$, the algebra generated by the elements $\hat{e}_i$, $\hat{f}_i$ and $\hat{h}_i$, $i = 1, \ldots, 2m+1$, has a one-dimensional center[12] which consists of the element $\hat{c} \equiv (\hat{h}_1 - \hat{h}_{2m+1}) + 2(\hat{h}_2 - \hat{h}_{2m}) + \ldots + m(\hat{h}_m - \hat{h}_{m+2}) + (m+1)\hat{h}_{m+1}$. The identification with $A(m,m)$ is achieved once this center has been factored out. This is the only case where such a situation occurs.[11]

- $B(m,n)$

$$(a_{ij}) = \begin{pmatrix} \mathcal{A}_{n-1} & & & & & \\ & & -1 & & & \\ & -1 & 0 & 1 & & \\ & & -1 & & & \\ & & & & \mathcal{A}_{m-1} & \\ & & & & & -1 \\ & & & & -2 & 2 \end{pmatrix}, \tag{2.7}$$

$$\tau = \{n\} \qquad\qquad r = m + n\ , \tag{2.8}$$

$$d_i = (\underbrace{1, \ldots, 1}_{n}, \underbrace{-1, \ldots, -1}_{m-1}, -\frac{1}{2})\ . \tag{2.9}$$

- $B(0,n)$

$$(a_{ij}) = \begin{pmatrix} \mathcal{A}_{n-1} & & \\ & & -1 \\ & -2 & 2 \end{pmatrix}, \tag{2.10}$$

$$\tau = \{n\} \qquad\qquad r = n\ , \tag{2.11}$$

$$d_i = (\underbrace{1, \ldots, 1}_{n-1}, \frac{1}{2})\ . \tag{2.12}$$

- $C(n+1)$

$$(a_{ij}) = \begin{pmatrix} 0 & 1 & & & \\ -1 & & & & \\ & & \mathcal{A}_{n-1} & & \\ & & & & -2 \\ & & & -1 & 2 \end{pmatrix}, \tag{2.13}$$

$$\tau = \{1\} \qquad\qquad r = n+1 \ , \tag{2.14}$$

$$d_i = (1, \underbrace{-1, \dots, -1}_{n-1}, -2) \ . \tag{2.15}$$

• $D(m,n)$

$$(a_{ij}) = \begin{pmatrix} \mathcal{A}_{n-1} & & & & & & \\ & & -1 & & & & \\ & -1 & 0 & 1 & & & \\ & & -1 & & & & \\ & & & & \mathcal{A}_{m-1} & & -1 \\ & & & & & & 0 \\ & & & & -1 & 0 & 2 \end{pmatrix} , \tag{2.16}$$

$$\tau = \{n\} \qquad\qquad r = m+n \ , \tag{2.17}$$

$$d_i = (\underbrace{1, \dots, 1}_{n}, \underbrace{-1, \dots, -1}_{m}) \ . \tag{2.18}$$

3. Quantum Lie superalgebras

Let $\mathcal{G}$ be a rank r superalgebra belonging to the unitary or the orthosymplectic series, described in the previous section. Let $q \in \mathbf{C}\backslash\{0\}$ be the deformation parameter which we shall sometimes write $q = e^{\eta/2}$. We shall also use $q_i = q^{d_i}$, with d_i the numbers, given in the previous section, that symmetrize the Cartan matrix (a_{ij}), and shall assume $q_i^4 \neq 1$. The quantum superalgebras $\mathcal{G}_q$ of the universal envelopping algebra of $\mathcal{G}$ is again generated by $3r$ elements e_i, f_i and h_i, $i \in I$, which satisfy[10]

$$\begin{aligned} &[e_i, f_j] = \delta_{ij} \, \frac{\sinh(\eta \, d_i \, h_i)}{\sinh(\eta \, d_i)} \ , \qquad && [h_i, h_j] = 0 \ , \\ &[h_i, e_j] = \, a_{ij} \, e_j \ , && [h_i, f_j] = - \, a_{ij} \, f_j \ , \end{aligned} \tag{3.1}$$

with

$$\deg h_i = \ 0 \ ; \qquad \deg e_i = \deg f_i = \ 0 \ , \quad i \notin \tau \ ; \qquad \deg e_i = \deg f_i = \ 1 \ , \quad i \in \tau \ ,$$

and further obey certain generalized Serre relations which will be specified. It is convenient to introduce the quantities $k_i = q_i^{h_i}$ in terms of which the defining relations (3.1) become:

$$\begin{aligned} k_i\,k_i^{-1} &= k_i^{-1}\,k_i = 1\ , & k_i\,k_j &= k_j\,k_i\ , \\ k_i\,e_j\,k_i^{-1} &= q_i^{a_{ij}}\,e_j\ , & k_i\,f_j\,k_i^{-1} &= q_i^{-a_{ij}}\,f_j\ , \\ [e_i, f_i] &= \delta_{ij}\,\frac{k_i^2 - k_i^{-2}}{q_i^2 - q_i^{-2}}\ . \end{aligned} \tag{3.2}$$

The quantum superalgebra $\mathcal{G}_q$ is endowed with a Hopf algebra structure.[6] The action of the coproduct $\Delta : \mathcal{G}_q \to \mathcal{G}_q \otimes \mathcal{G}_q$, antipode $S : \mathcal{G}_q \to \mathcal{G}_q$ and counit $\varepsilon : \mathcal{G}_q \to \mathbf{C}$ on the generators is as follows:[10]

$$\begin{aligned} \Delta(h_i) &= h_i \otimes 1 + 1 \otimes h_i & \Delta(k_i) &= k_i \otimes k_i\ , \\ \Delta(e_i) &= e_i \otimes k_i + k_i^{-1} \otimes e_i & \Delta(f_i) &= f_i \otimes k_i + k_i^{-1} \otimes f_i\ , \\ \\ S(h_i) &= -h_i & S(k_i) &= k_i^{-1}\ , \\ S(e_i) &= -q_i^{a_{ii}}\,e_i & S(f_i) &= -q_i^{a_{ii}}\,f_i\ , \\ \\ \varepsilon(h_i) &= \varepsilon(e_i) = \varepsilon(f_i) = 0 & \varepsilon(1) &= 1\ . \end{aligned} \tag{3.3}$$

One can define the q-analogue ad_q of the adjoint operation by[15,10]

$$\mathrm{ad}_q = (\mu_L \otimes \mu_R)(\mathrm{id} \otimes S)\Delta\ , \tag{3.4}$$

with id the identity operator and μ_L, μ_R the left and right (graded) multiplications: $\mu_L(x)y = xy$, $\mu_R(x)y = (-)^{\deg x \deg y}\,yx$. The quantum Serre relations are most simply expressed in terms of the following rescaled generators,[15]

$$\mathcal{E}_i = e_i\,k_i^{-1} \qquad\qquad \mathcal{F}_i = f_i\,k_i^{-1}\ . \tag{3.5}$$

They then take a form similar to (2.2) and read

$$(\mathrm{ad}_q\,\mathcal{E}_i)^{1-\tilde{a}_{ij}}\,\mathcal{E}_j = 0\ , \qquad (\mathrm{ad}_q\,\mathcal{F}_i)^{1-\tilde{a}_{ij}}\,\mathcal{F}_j = 0\ , \qquad i \neq j\ , \tag{3.6}$$

The defining system for the generators of $\mathcal{G}_q$ is thus completed by adding these generalized Serre relations to Eq.(3.1) or Eq.(3.2).

Let us record for reference, the explicit forms that conditions (3.6) take for $sl_q(m,n)$ and $osp_q(m,2n)$. One has, always with $i \neq j$,

$\tilde{a}_{ij} = 0$:

$$e_i\,e_j - (-1)^{\deg e_i \deg e_j}\,e_j\,e_i = 0\ ; \tag{3.7}$$

$\tilde{a}_{ij} = -1$:

for $\deg e_i = 0$,

$$e_i^2\, e_j - 2\cosh(\eta\, d_i)\, e_i e_j e_i + e_j\, e_i^2 = 0\ , \tag{3.8a}$$

for $\deg e_i = 1$,

$$e_i^2\, e_j - (\cosh(2\eta\, d_i) - \sinh(2\eta\, d_i))\; e_j\, e_i^2 = 0\ ; \tag{3.8b}$$

$\tilde{a}_{ij} = -2$:

for $\deg e_i = 0$,

$$e_i^3\, e_j - (1 + 2\cosh(2\eta\, d_i))\left(e_i^2 e_j e_i - e_i e_j e_i^2\right) - e_j\, e_i^3 = 0\ , \tag{3.9a}$$

for $\deg e_i = 1$,

$$e_i^3\, e_j + (1 - 2\cosh(2\eta\, d_i))\left((-1)^{\deg e_j}\, e_i^2 e_j e_i + e_i e_j e_i^2\right) + (-1)^{\deg e_j}\, e_j\, e_i^3 = 0\ . \tag{3.9b}$$

In deriving these equations one should recall that $q_i^{a_{ij}} = q_j^{a_{ji}}$. Substituting $e_k \rightarrow f_k$ and $\eta \rightarrow -\eta$ in the above relations, one obtains the corresponding conditions on the generators f_k.

4. q-Analogues of the Bose and Fermi oscillators

Let s and t be two positive integers. The Weyl superalgebra, here denoted by $W(s,t)$, is generated by the annihilation and creation operators of s Bose and t Fermi oscillators. The q-deformation of $W(s,t)$ is obtained by introducing the quantum analogues of these oscillators.[7]

The annihilation, creation and number operators b_i, $b_i^\dagger$ and N_i, $i = 1,\ldots,s$, of bosonic q-oscillators are taken to satisfy,

$$b_i\, b_i^\dagger - q^2\, b_i^\dagger\, b_i = q^{-2N_i} \qquad b_i\, b_i^\dagger - q^{-2}\, b_i^\dagger\, b_i = q^{2N_i}\ , \tag{4.1}$$

$$[N_i, b_j] = -\delta_{ij}\, b_i \qquad [N_i, b_j^\dagger] = \delta_{ij}\, b_i^\dagger\ , \tag{4.2}$$

and for $i \neq j$

$$[b_i, b_j] = [b_i^\dagger, b_j^\dagger] = [b_i, b_j^\dagger] = 0 \qquad [N_i, N_j] = 0\ , \tag{4.3}$$

with $\deg b_i = \deg b_i^\dagger = \deg N_i = 0$.

Similarly, the annihilation, creation and number operators, ψ_i, $\psi_i^\dagger$ and M_i, $i = 1,\ldots,t$, of fermionic q-oscillators are defined through,

$$\psi_i\, \psi_i^\dagger + q^2\, \psi_i^\dagger\, \psi_i = q^{2M_i} \qquad \psi_i\, \psi_i^\dagger + q^{-2}\, \psi_i^\dagger\, \psi_i = q^{-2M_i}\ , \tag{4.4}$$

$$[M_i, \psi_j] = -\delta_{ij}\, \psi_j \qquad [M_i, \psi_j^\dagger] = \delta_{ij}\, \psi_j^\dagger\ , \tag{4.5}$$

$$\{\psi_i, \psi_j\} = 0 \qquad \{\psi_i^\dagger, \psi_j^\dagger\} = 0\ , \tag{4.6a}$$

and for $i \neq j$,

$$\{\psi_i, \psi_j^\dagger\} = 0 \qquad\qquad [M_i, M_j] = 0 \ , \tag{4.6b}$$

with $\deg \psi_i = \deg \psi_i^\dagger = 1$, $\deg M_i = 0$, and $\{x, y\} = xy + yx$. It is further assumed that bosonic and fermionic operators commute,

$$[b_i, \psi_j] = [b_i, \psi_j^\dagger] = [b_i^\dagger, \psi_j] = [b_i^\dagger, \psi_j^\dagger] = 0 \ , \tag{4.7a}$$

$$[N_i, \psi_j] = [N_i, \psi_j^\dagger] = [M_i, b_j] = [M_i, b_j^\dagger] = [N_i, M_j] = 0 \ . \tag{4.7b}$$

The algebra $W_q(s,t)$ generated by the operators b_i, $b_i^\dagger$, N_i, $i = 1, \ldots, s$, and ψ_j, $\psi_j^\dagger$, M_j, $j = 1, \ldots, t$, subjected to equations (4.1)–(4.7), will be referred to as q-analogue of the Weyl superalgebra $W(s,t)$. The second conditions in (4.1) and (4.4) are sometimes omitted,[8–10] their presence amounts to requiring the invariance[16] of the defining system under $q \to q^{-1}$. Note that equations (4.1) are equivalent to

$$b_i\, b_i^\dagger = \frac{q^{2(N_i+1)} - q^{-2(N_i+1)}}{q^2 - q^{-2}} \qquad\qquad b_i^\dagger\, b_i = \frac{q^{2N_i} - q^{-2N_i}}{q^2 - q^{-2}} \ , \tag{4.8}$$

and (4.4) to

$$\psi_i\, \psi_i^\dagger = \frac{q^{2(1-M_i)} - q^{-2(1-M_i)}}{q^2 - q^{-2}} \qquad\qquad \psi_i^\dagger\, \psi_i = \frac{q^{2M_i} - q^{-2M_i}}{q^2 - q^{-2}} \ . \tag{4.9}$$

When $q = 1$, equations (4.1)–(4.7) reduce to the canonical commutation and anticommutation relations of ordinary bosonic and fermionic annihilation and creation operators. We shall denote by $\hat{b}_i$, $\hat{b}_i^\dagger$, $\hat{\psi}_i$ and $\hat{\psi}_i^\dagger$ the classical relatives of b_i, $b_i^\dagger$, ψ_i and $\psi_i^\dagger$; note that $N_i \to \hat{N}_i = \hat{b}_i^\dagger\, \hat{b}_i$, and $M_i \to \hat{M}_i = \hat{\psi}_i^\dagger\, \hat{\psi}_i$ as $q \to 1$.

The defining relations of the q-Weyl superalgebra can be realized by expressing the q-oscillator operators in terms of their classical analogues. For the bosonic operators take[9]

$$b_i = \sqrt{\frac{f(\hat{N}_i + 1)}{\hat{N}_i + 1}}\ \hat{b}_i \qquad b_i^\dagger = \sqrt{\frac{f(\hat{N}_i)}{\hat{N}_i}}\ \hat{b}_i^\dagger \qquad N_i = \hat{N}_i \ , \tag{4.10}$$

with

$$f(\hat{N}_i) = \frac{q^{2\hat{N}_i} - q^{-2\hat{N}_i}}{q^2 - q^{-2}} = \frac{\sinh(\eta\, \hat{N}_i)}{\sinh \eta} \ . \tag{4.11}$$

(Notice that q has to be real or a pure phase, *i.e.* η has to be real or purely imaginary, for b_i and $b_i^\dagger$ in (4.10) to be hermitian conjugates). For the fermionic operators set

$$\psi_i = \hat{\psi}_i \qquad\qquad \psi_i^\dagger = \hat{\psi}_i^\dagger \qquad\qquad M_i = \hat{M}_i \ . \tag{4.12}$$

It is easy to check that equations (4.1)–(4.7) are verified under such identifications. For instance, since $\hat{M}_i^2 = \hat{M}_i$, one has $q^{2\hat{M}_i} = (1 - \hat{M}_i) + q^2 \hat{M}_i = \hat{\psi}_i\, \hat{\psi}_i^\dagger + q^2\, \hat{\psi}_i^\dagger\, \hat{\psi}_i$.

5. q-Oscillator representations of quantum superalgebras

We shall now construct q-oscillator representations of the quantum superalgebras $sl_q(m,n)$ and $osp_q(m,2n)$. We shall provide explicit expressions for the corresponding generators as linears and bilinears in q-deformed bosonic and fermionic oscillator operators. We shall successively consider the quantum superalgebras $A_q(m,n)$, $B_q(m,n)$, $C_q(n+1)$ and $D_q(m,n)$ associated to the $A(m,n)$, $B(m,n)$, $C(n+1)$ and $D(m,n)$ Lie superalgebra series described in Section 2.

Let us observe first that the quantum algebra corresponding to the classical Lie algebra A_n admits the following four representations[7,17]

$\pi^{(1)}_{A_n}$:

$$e_k = b_k^\dagger b_{k+1} \qquad f_k = b_{k+1}^\dagger b_k \qquad h_k = N_k - N_{k+1}\ , \quad k = 1,\ldots,n\ . \tag{5.1}$$

$\pi^{(2)}_{A_n}$:

$$\begin{aligned}
e_{n-2k+1} &= i\, b^\dagger_{n-2k+1} b^\dagger_{n-2k+2} \\
f_{n-2k+1} &= i\, b_{n-2k+1} b_{n-2k+2} && k = 1,\ldots,[\![n/2]\!]\ , \\
h_{n-2k+1} &= N_{n-2ik+1} + N_{n-2k+2} + 1\ , \\
\\
e_{n-2k} &= i\, b_{n-2k} b_{n-2k+1} \\
f_{n-2k} &= i\, b^\dagger_{n-2k} b^\dagger_{n-2k+1} && k = 0,\ldots,[\![(n-1)/2]\!]\ . \\
h_{n-2k} &= -(N_{n-2k} + N_{n-2k+1} + 1)\ ,
\end{aligned} \tag{5.2}$$

$\pi^{(3)}_{A_n}$:

$$e_k = \psi_k^\dagger \psi_{k+1} \qquad f_k = \psi_{k+1}^\dagger \psi_k \qquad h_k = M_k - M_{k+1}\ , \quad k = 1,\ldots,n\ . \tag{5.3}$$

$\pi^{(4)}_{A_n}$:

$$\begin{aligned}
e_{n-2k+1} &= i\, \psi^\dagger_{n-2k+1} \psi^\dagger_{n-2k+2} \\
f_{n-2k+1} &= i\, \psi_{n-2k+1} \psi_{n-2k+2} && k = 1,\ldots,[\![n/2]\!]\ , \\
h_{n-2k+1} &= M_{n-2k+1} + M_{n-2k+2} - 1\ , \\
\\
e_{n-2k} &= i\, \psi_{n-2k} \psi_{n-2k+1} \\
f_{n-2k} &= i\, \psi^\dagger_{n-2k} \psi^\dagger_{n-2k+1} && k = 0,\ldots,[\![(n-1)/2]\!]\ . \\
h_{n-2k} &= -(M_{n-2k} + M_{n-2k+1} - 1)\ ,
\end{aligned} \tag{5.4}$$

The symbol $[\![x]\!]$ stands for the integer part of x. Equivalent representations are obtained upon exchanging e_i and f_i, and letting $h_i \to -h_i$.

Under the standard inner product on the Hilbert space of oscillator states, $\pi^{(1)}_{A_n}$, $\pi^{(3)}_{A_n}$ and $\pi^{(4)}_{A_n}$ are unitary, while $\pi^{(2)}_{A_n}$ is antiunitary. Upon suitably combining these representations, realizations of the quantum superalgebras $sl_q(m,n)$ and $osp_q(m,2n)$ will be obtained.

- $A_q(m,n)$

In this case we can form four algebra homomorphisms of $A_q(m,n)$ into $W_q(n+1,m+1)$. For instance, we can take the first m generators (e_i, f_i, h_i) to be realized as in $\pi^{(3)}_{A_m}$ and the last n ones given as in $\pi^{(1)}_{A_n}$. Explicitly, this provides the following unitary representation of $A_q(m,n)$,

$$\begin{array}{llll}
e_k = \psi_k^\dagger \psi_{k+1} & f_k = \psi_{k+1}^\dagger \psi_k & h_k = M_k - M_{k+1}\ , & k = 1,\ldots,m\ , \\
e_{m+1} = \psi_{m+1}^\dagger b_2 & f_{m+1} = \psi_{m+1} b_2^\dagger & h_{m+1} = M_{m+1} + N_2\ , & \\
e_{m+l} = b_l^\dagger b_{l+1} & f_{m+l} = b_{l+1}^\dagger b_l & h_{m+l} = N_l - N_{l+1}\ , & l = 2,\ldots,n+1\ .
\end{array} \tag{5.5}$$

This construction has been sketched in Ref.[10].

One can also join the representation $\pi^{(3)}_{A_m}$ with the representation $\pi^{(2)}_{A_n}$ (or its equivalent under $e_i \leftrightarrow f_i$, $h_i \to -h_i$) using for e_{m+1}, f_{m+1}, and h_{m+1} the expressions given in (5.5). One has then,

$$\begin{array}{llll}
e_k = \psi_k^\dagger \psi_{k+1} & f_k = \psi_{k+1}^\dagger \psi_k & h_k = M_k - M_{k+1}\ , & k = 1,\ldots,m\ , \\
e_{m+1} = \psi_{m+1}^\dagger b_1 & f_{m+1} = \psi_{m+1} b_1^\dagger & h_{m+1} = M_{m+1} + N_1\ , & \\
 & & & \\
e_{m+2} = i\, b_1^\dagger b_2^\dagger & f_{m+2} = i\, b_1 b_2 & h_{m+2} = N_1 + N_2 + 1\ , & \\
e_{m+3} = i\, b_2 b_3 & f_{m+3} = b_2^\dagger b_3^\dagger & h_{m+3} = -(N_2 + N_3 + 1)\ , &
\end{array} \tag{5.6}$$

and so until the index $n+m+1$ is reached. The representation of $A_q(m,n)$ thus obtained is not unitary anymore. However, it becomes unitary when the symmetric Cartan matrix $(a^s_{ij}) = (d_i\, a_{ij})$ is adopted (see Appendix).

The representation $\pi^{(4)}_{A_m}$ can similarly be attached to either representation $\pi^{(1)}_{A_n}$ or representation $\pi^{(2)}_{A_n}$ to form two additional representations of $A_q(m,n)$. The first one is unitary, while the second one becomes unitary once the rescalings associated to the use of the symmetric Cartan matrix have been performed. When $m = n$, the center $c = (h_1 - h_{2m+1}) + 2(h_2 - h_{2m}) + \ldots + m(h_m - h_{m+2}) + (m+1)h_{m+1}$ should be factored out.

From the four representations that we have just described one can obtain four additional homomorphisms of $A_q(m,n)$ in $W_q(m+1,n+1)$ by exchanging in an obvious fashion the bosonic and fermionic operators.

• $B_q(m,n)\ ,\quad m>0$

Four algebra homomorphisms of $B_q(m,n)$ into $W_q(n,m)$ are obtained by combining $\pi^{(1)}_{A_{n-1}}$ or $\pi^{(2)}_{A_{n-1}}$ with $\pi^{(3)}_{A_{m-1}}$ or $\pi^{(4)}_{A_{m-1}}$. A unitary representation follows from using $\pi^{(1)}_{A_{n-1}}$ and $\pi^{(3)}_{A_{m-1}}$. This is the only one that we shall describe explicitly; the others are similarly constructed. Set

$$\begin{array}{llll} e_k = b^\dagger_k\, b_{k+1} & f_k = b^\dagger_{k+1}\, b_k & h_k = N_k - N_{k+1}\ , & k=1,\dots,n-1\ ,\\ e_n = \psi_1\, b^\dagger_n & f_n = \psi^\dagger_1\, b_n & h_n = M_1 + N_n\ , & \\ e_{n+l} = \psi^\dagger_l\, \psi_{l+1} & f_{n+l} = \psi^\dagger_{l+1}\psi_l & h_{n+l} = M_l - M_{l+1}\ , & l=1,\dots,m-1\ ,\\ e_{m+n} = (-1)^M \psi^\dagger_m & f_{m+n} = \psi_m (-1)^M & h_{n+m} = 2M_m - 1\ . & \end{array} \tag{5.7}$$

where $M=\sum_{i=1}^m M_i$. It is not difficult to check that the defining relations of $B_q(m,n)$ are then satisfied. Note that a Klein operator enters in the expression of e_{m+n} and f_{m+n}.

Let us point out that different homomorphisms of $B_q(m,n)$ into $W(n,m)$ can be obtained by exchanging the b's and the ψ's. However, one then needs to use a set τ with more than one element. For $\tau=\{n,m+n\}$ in particular, a representation of $B_q(m,n)$ is obtained through combining $\pi^{(3)}_{A_{n-1}}$ and $\pi^{(1)}_{A_{m-1}}$ as follows:

$$\begin{array}{llll} e_k = \psi^\dagger_k\, \psi_{k+1} & f_k = \psi^\dagger_{k+1}\, \psi_k & h_k = M_k - M_{k+1}\ , & k=1,\dots,n-1\ ,\\ e_n = b_1\, \psi^\dagger_n & f_n = b^\dagger_1\, \psi_n & h_n = N_1 + M_n\ , & \\ e_{n+l} = b^\dagger_l\, b_{l+1} & f_{n+l} = b^\dagger_{l+1} b_l & h_{n+l} = N_l - N_{l+1}\ , & l=1,\dots,m-1\ ,\\ e_{m+n} = (-1)^N b^\dagger_m & f_{m+n} = b_m (-1)^N & h_{n+m} = 2N_m + 1\ . & \end{array} \tag{5.8}$$

where $N=\sum_{i=1}^m N_i$.

• $B_q(0,n)$

The representations of $B_q(0,n)$ only require q-bosons. Homomorphisms of $B_q(0,n)$ into $W_q(n,0)$ can be constructed from either $\pi^{(1)}_{A_{n-1}}$ or $\pi^{(2)}_{A_{n-1}}$. In the first case one has

$$\begin{array}{llll} e_k = b^\dagger_k\, b_{k+1} & f_k = b^\dagger_{k+1}\, b_k & h_k = N_k - N_{k+1}, & \text{for } k=1,\dots,n-1\ ,\\ e_n = b^\dagger_n & f_n = b_n & h_n = 2N_n + 1\ . & \end{array} \tag{5.9}$$

This representation is unitary. The other one has (e_k,f_k,h_k), $k=1,\dots,n-1$, as in $\pi^{(2)}_{A_{n-1}}$, with (e_n,f_n,h_n) as in (5.9). These bosonic realizations of $osp_q(1,2n)$ were given in Ref.[17].

• $C_q(n+1)$

We have two homomorphisms of $C_q(n+1)$ in $W_q(n,1)$. There is one which is constructed out of the representation $\pi^{(2)}_{A_{n-1}}$ given in (5.2) when n is odd, or, when n is even, out of the equivalent representation obtained from the substitution $e_i \leftrightarrow f_i$ and $h_i \to -h_i$. It is explicitly defined by

$$\begin{aligned} e_1 &= \psi_1 b_1 & f_1 &= \psi_1^\dagger b_1^\dagger & h_1 &= N_1 - M_1 + 1\,, \\ e_2 &= i b_1^\dagger b_2^\dagger & f_2 &= i b_1 b_2 & h_2 &= N_1 + N_2 + 1\,, \\ e_3 &= i b_2 b_3 & f_3 &= i b_2^\dagger b_3^\dagger & h_3 &= -(N_2 + N_3 + 1)\,, \end{aligned} \tag{5.10}$$

and so on, till:

$$e_{n+1} = \frac{i}{2\cosh\eta} b_n^2 \qquad f_{n+1} = \frac{i}{2\cosh\eta} (b_n^\dagger)^2 \qquad h_{n+1} = -(N_n + \frac{1}{2})\,, \quad \text{for } n \text{ even},$$

$$e_{n+1} = \frac{i}{2\cosh\eta} (b_n^\dagger)^2 \qquad f_{n+1} = \frac{i}{2\cosh\eta} b_n^2 \qquad h_{n+1} = N_n + \frac{1}{2}\,, \quad \text{for } n \text{ odd}.$$

This representation becomes unitary when referred to the symmetric Cartan matrix $(a^s_{ij}) = (d_i\, a_{ij})$ (see Appendix).

The other representation of $C_q(n+1)$ in $W_q(n,1)$, uses $\pi^{(1)}_{A_{n-1}}$ and is defined as follows

$$\begin{aligned} e_1 &= \psi_1^\dagger b_1 & f_1 &= \psi_1 b_1^\dagger & h_1 &= M_1 + N_1\,, \\ e_{k+1} &= b_k^\dagger b_{k+1} & f_{k+1} &= b_{k+1}^\dagger b_k & h_{k+1} &= N_k - N_{k+1}\,, \quad k = 1,\ldots,n-1\,, \\ e_{n+1} &= \frac{i}{2\cosh\eta}(b_n^\dagger)^2 & f_{n+1} &= \frac{i}{2\cosh\eta} b_n^2 & h_{n+1} &= N_n + \frac{1}{2}\,. \end{aligned} \tag{5.11}$$

- $D_q(m,n)$

Two homomorphisms of $D_q(m,n)$ into $W_q(m,n)$ are obtained upon combining $\pi^{(1)}_{A_{n-1}}$ or $\pi^{(2)}_{A_{n-1}}$ with $\pi^{(3)}_{A_{m-1}}$. The first produces the following unitary realization:

$$\begin{aligned} e_k &= b_k^\dagger b_{k+1} & f_k &= b_{k+1}^\dagger b_k & h_k &= N_k - N_{k+1}\,, & k &= 1,\ldots,n-1\,, \\ e_n &= \psi_1 b_n^\dagger & f_n &= \psi_1^\dagger b_n & h_n &= N_n + M_1\,, \\ e_{n+l} &= \psi_l^\dagger \psi_{l+1} & f_{n+l} &= \psi_{l+1}^\dagger \psi_l & h_{n+l} &= M_l - M_{l+1}\,, & l &= 1,\ldots,m-1\,, \\ e_{m+n} &= \psi_m^\dagger \psi_{m-1}^\dagger & f_{m+n} &= \psi_{m-1}\psi_m & h_{m+n} &= M_{m-1} + M_m - 1\,. \end{aligned} \tag{5.12}$$

For $D_q(m,1)$, the form of this q-oscillator representation had been conjectured in Ref.[16]. A second realization is formed by taking the first $n-1$ generators (e_k, f_k, h_k) as in representation $\pi^{(2)}_{A_{n-1}}$ keeping the remaining generators as in (5.12). Finally, two new homomorphisms of $D_q(m,n)$ into $W_q(m,n)$ can be obtained from the representations just described by letting $b_i \leftrightarrow \psi_i$, $b_i^\dagger \leftrightarrow \psi_i^\dagger$, $N_i \to M_i - 1$ and $M_i \to N_i + 1$ in all the generators, except for e_{m+n} and f_{m+n}, which are realized as $e_{m+n} = i\, b_{m-1}^\dagger\, b_m^\dagger$, $f_{m+n} = i\, b_{m-1}\, b_m$.

Appendix. Conversion to symmetric Cartan matrices

Two Cartan matrices $A = (a_{ij})$ and $A' = (a'_{ij})$ are equivalent[13] if there exists a matrix D such that $\det D \neq 0$ and $A' = D\,A$. Using this freedom, we can symmetrize the Cartan matrices of the basic Lie superalgebras. In fact, let $D_{ij} = d_i\,\delta_{ij}$ with d_i the components of the vector given in Section 2; the symmetric Cartan matrices $A^s = (a^s_{ij})$ of Ref.[18] are related to those listed in Section 2 by $A^s = D\,A$.

We here indicate how various formulas translate when one chooses to describe quantum superalgebras with (a^s_{ij}) instead of (a_{ij}). Let E_i, F_i and H_i, $i = 1, \ldots, r$, be the elements that generate the quantum superalgebra characterized by (a^s_{ij}) and τ. They satisfy the defining relations[10]

$$\begin{aligned} [E_i, F_j] &= \delta_{ij}\,\frac{\sinh(\eta h_i)}{\sinh \eta}\,, \qquad & [H_i, H_j] &= 0\,, \\ [H_i, E_j] &= a^s_{ij}\,E_j\,, & [H_i, F_j] &= -\,a^s_{ij}\,F_j\,. \end{aligned} \tag{A.1}$$

$$\deg H_i = 0\,: \qquad \deg E_i = \deg F_i = 0\,, \quad i \notin \tau\,; \qquad \deg E_i = \deg F_i = 1\,, \quad i \in \tau\,,$$

together with the Serre relations (3.6), still involving the Cartan matrix (a_{ij}) and the rescaled generators $\mathcal{E}_i = E_i\,e^{-\frac{\eta}{2}H_i}$, $\mathcal{F}_i = F_i\,e^{-\frac{\eta}{2}H_i}$.

This set of generators is straightforwardly related to the set e_i, f_i and h_i, $i = 1, \ldots, r$, that satisfy (3.1) and (3.6). One has

$$E_i = \sqrt{\frac{\sinh(\eta\,d_i)}{\sinh \eta}}\,e_i \qquad F_i = \sqrt{\frac{\sinh(\eta\,d_i)}{\sinh \eta}}\,f_i \qquad H_i = d_i\,h_i\,. \tag{A.2}$$

When d_i is negative, E_i and F_i will no longer be hermitian conjugate, if e_i and f_i were. Conversely, as indicated in Section 5, there might be cases where one needs to use (a^s_{ij}) and the generators E_i, F_i and H_i for certain representations to be unitary.

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