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R. Floreanini and L. Vinet

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Roberto Floreanini
Istituto Nazionale di Fisica Nucleare, Sezione di Trieste
Dipartimento di Fisica Teorica, Università di Trieste
Strada Costiera 11, 34014 Trieste, Italy

Luc Vinet(*)
Department of Physics
University of California
405 Hilgard Avenue
Los Angeles, CA 90024, USA

Abstract

The oscillator quantum algebra is shown to provide a group-theoretic setting for the q-Laguerre and q-Hermite polynomials.

(*) On leave from: Laboratoire de Physique Nucléaire, Université de Montréal, Montréal, Canada H3C 3J7

It is now being realized that quantum groups[1-5] and their representations might bear a relationship with the theory of basic or q-special functions[6-8] similar to the one between Lie theory and ordinary special functions.[9,10] Indeed, a few q-functions have already been identified as matrix elements of representations of quantum groups: the little q-Jacobi polynomials in $SU_q(2)$ representations,[11-13] the q-Bessel functions when the quantum group of motions in the plane is considered[14] and certain basic hypergeometric functions in the case of $SU_q(1,1)$.[15] Moreover, the Clebsh-Gordan coefficients of $SU_q(2)$ can be expressed in terms of q-Hahn polynomials,[16] while other q-special functions can be obtained from harmonic analysis on quantum spaces.[17] In the present letter, we establish the connection between the q-Laguerre and q-Hermite polynomials and the oscillator quantum group.

The oscillator quantum algebra $W_q(1)$ is generated by three elements a, $a^\dagger$ and N satisfying the defining relations:

$$\begin{aligned} &[N,a] = -a && [N,a^\dagger] = a^\dagger \\ &a\,a^\dagger - q^{\frac{1}{2}} a^\dagger a = q^{-\frac{N}{2}} && a\,a^\dagger - q^{-\frac{1}{2}} a^\dagger a = q^{\frac{N}{2}} \,. \end{aligned} \tag{1}$$

In the limit $q \to 1$, (1) reduce to the canonical commutation relations of the harmonic oscillator creation and annihilation operators. The algebra $W_q(1)$ and its generalizations[18-21] have found many applications. In particular, they have been used to construct oscillator realizations of the quantized envelopping algebras[22] and superalgebras[23-26] of type A, B, C and D. In terms of

$$k = q^{\frac{1}{2}(N+\frac{1}{2})}\,, \tag{2}$$

the relations (1) translate into

$$\begin{aligned} &k\,k^{-1} = k^{-1}k = 1 && k\,a\,k^{-1} = q^{-\frac{1}{2}}\,a \\ &[a,a^\dagger] = \frac{k+k^{-1}}{q^{\frac{1}{4}}+q^{-\frac{1}{4}}} && k\,a^\dagger\,k^{-1} = q^{\frac{1}{2}}\,a^\dagger\,. \end{aligned} \tag{3}$$

The algebra $W_q(1)$ is a *bona fide* quantum algebra as it can be endowed with a Hopf algebra structure[27] by taking the following definitions[28] of coproduct Δ: $W_q(1) \to W_q(1) \otimes W_q(1)$, antipode S: $W_q(1) \to W_q(1)$ and counit ε: $W_q(1) \to \mathbf{C}$:

$$\begin{aligned} &\Delta(k) = (k \otimes k)e^{-i\theta} && \Delta(N) = N \otimes 1 + 1 \otimes N + (\frac{1}{2} - \frac{2i\theta}{\ln q}) \\ &\Delta(a) = (a \otimes k^{\frac{1}{2}} + ik^{-\frac{1}{2}} \otimes a)e^{-i\theta/2} && \Delta(a^\dagger) = (a^\dagger \otimes k^{\frac{1}{2}} + ik^{-\frac{1}{2}} \otimes a^\dagger)e^{-i\theta/2} \\ \\ &S(k) = k^{-1}\,e^{2i\theta} && S(N) = -N - 2(\frac{1}{2} - \frac{2i\theta}{\ln q}) \\ &S(a) = -q^{-\frac{1}{4}}\,a && S(a^\dagger) = -q^{\frac{1}{4}}\,a^\dagger \\ \\ &\varepsilon(k) = e^{i\theta} && \varepsilon(N) = -(\frac{1}{2} - \frac{2i\theta}{\ln q}) \\ &\varepsilon(a) = \varepsilon(a^\dagger) = 0 && \varepsilon(1) = 1\,, \end{aligned} \tag{4}$$

with $\theta = \frac{\pi}{2} + 2\pi l$, $l \in \mathbf{Z}$.

For our present purposes, it will be convenient to introduce another pair of annihilation and creation operators A and $A^\dagger$ related to a and $a^\dagger$ in the following fashion:

$$A = q^{\frac{N}{4}} a \qquad A^\dagger = q^{\frac{(N-1)}{4}} a^\dagger . \tag{5}$$

It is immediate to verify that A and $A^\dagger$ satisfy

$$[N, A] = -A \qquad [N, A^\dagger] = A^\dagger , \tag{6a}$$

$$AA^\dagger - q A^\dagger A = 1 . \tag{6b}$$

Before proceeding further, let us collect a few results in q-analysis that will prove useful.[6] We shall denote by $(a;q)_n$ the q-shifted factorial:

$$(a;q)_n = \begin{cases} 1 , & n = 0 , \\ (1-a)(1-aq)\dots(1-aq^{n-1}) , & n = 1,2,\dots , \end{cases} \tag{7}$$

and shall take

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k) , \qquad |q| < 1 . \tag{8}$$

One can also define $(a;q)_n$ for arbitrary complex n by

$$(a;q)_n = \frac{(a;q)_\infty}{(aq^n;q)_\infty} . \tag{9}$$

These products satisfy various identities like for instance

$$q^{\frac{n(n-1)}{2}} (a^{-1}q^{1-n};q)_n = (-a^{-1})^n (a;q)_n . \tag{10}$$

We shall recall them whenever they will be required. Note also that $(q;q)_n/(1-q)^n \to n!$ as $q \to 1^-$. Of fundamental importance is Heine's q-binomial theorem which states that

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} z^n = \frac{(az;q)_\infty}{(z;q)_\infty} , \qquad |z| < 1, \ |q| < 1 . \tag{11}$$

Two q-exponential functions are obtained from the above formula. On the one hand, upon setting $a = 0$, one gets

$$e_q(z) = \sum_{n=0}^{\infty} \frac{1}{(q;q)_n} z^n = \frac{1}{(z;q)_\infty} , \qquad |z| < 1 , \tag{12}$$

while on the other, upon replacing z by $-z/a$ in (11), letting $a \to \infty$ and using (10), one finds

$$E_q(z) = \sum_{n=0}^{\infty} \frac{q^{\frac{n(n-1)}{2}}}{(q;q)_n} z^n = (-z;q)_\infty . \tag{13}$$

It is easy to see that $e_q(z)\,E_q(-z)=1$ and that

$$\lim_{q\to 1^-} e_q\big(z(1-q)\big)=\lim_{q\to 1^-} E_q\big(z(1-q)\big)=e^z\ . \tag{14}$$

Let T_z be the q-dilatation operator in the variable z, *i.e.*

$$T_z\,f(z)=f(qz)\ . \tag{15}$$

The q-difference operators D_z^+ and D_z^- are given by

$$D_z^+=z^{-1}(1-T_z)\ , \tag{16a}$$
$$D_z^-=z^{-1}(1-T_z^{-1})\ . \tag{16b}$$

Observe that $\frac{1}{(1-q)}D_z^+\ \to\ d/dz$ and $\frac{1}{(1-q^{-1})}D_z^-\ \to\ d/dz$ as $q\to 1$ and that the q-exponentials obey[29]

$$D_z^+\,e_q(z)=e_q(z)\ , \tag{17a}$$
$$D_z^-\,E_q(z)=-q^{-1}E_q(z)\ . \tag{17b}$$

We shall also need the q-integral

$$\int_0^a f(t)\,d_qt=a\,(1-q)\sum_{n=0}^{\infty}f(aq^n)\,q^n\ . \tag{18}$$

One checks that it is the inverse of D^+, in that $\int_0^a D_t^+f(t)\,d_qt=f(a)-f(0)$, $|q|<1$.

The q-gamma function is defined by

$$\Gamma_q(z)=\frac{(q;q)_\infty}{(q^z;q)_\infty}\,(1-q)^{1-z}\ ,\qquad |q|<1\ . \tag{19}$$

We have obtained a q-analog of Euler's formula for the gamma function which we were not able to find in the literature.

LEMMA: The function $\Gamma_q(z)$ admits the following integral representation

$$\Gamma_q(z)=\int_0^{\frac{1}{1-q}} t^{z-1}\,E_q\left(-q(1-q)\,t\right)\,d_qt\ ,\quad |q|<1\ . \tag{20}$$

The proof can be easily obtained by successively using (18), (13), (9) and (12).

The connection between $W_q(1)$ and the q-Laguerre and q-Hermite polynomials will be obtained by considering two different realizations of this quantum algebra. Our analysis will closely follow the one given in Ref.[10] for the ordinary (undeformed) case. The first representation is constructed on the space of entire functions in the complex variable z. It will be convenient to use the polar coordinates $z = \rho\, e^{i\theta}$. We shall endow this space with the following inner product

$$\begin{aligned}\langle\varphi|\psi\rangle &= \int_{\mathcal{D}} \overline{\varphi(z)}\,\psi(z)\,d_q\mu(z) \\ &\equiv \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{\frac{1}{1-q}} d_q\rho^2\; E_q\left(-q(1-q)\rho^2\right)\overline{\varphi(\rho e^{i\theta})}\,\psi(\rho e^{i\theta})\,.\end{aligned} \tag{21}$$

An orthonormal basis is then provided by the monomials

$$|n\rangle = \frac{1}{\sqrt{\Gamma_q(n+1)}}\, z^n\,. \tag{22}$$

Indeed, one can check that: $\langle n|m\rangle = \delta_{m,n}$. Also note that $\Gamma_q(n+1) = (q;q)_n/(1-q)^n$. A representation of $W_q(1)$ in this space is obtained by setting

$$N = z\frac{d}{dz}\,, \qquad A = \frac{1}{1-q}D_z^+\,, \qquad A^\dagger = z\,. \tag{23}$$

It is easy to see that the relations (6) are identically obeyed under this identification. In this model the action of the generators on the basis states is readily computed; one finds

$$N\,|n\rangle = n\,|n\rangle \qquad A\,|n\rangle = \sqrt{\frac{1-q^n}{1-q}}\,|n-1\rangle \qquad A^\dagger\,|n\rangle = \sqrt{\frac{1-q^{n+1}}{1-q}}\,|n+1\rangle\,. \tag{24}$$

The operators A and $A^\dagger$ are seen to be the hermitian conjugate one of the other when q is real.

In analogy[29] with ordinary Lie theory, we introduce the operator

$$U(\alpha,\beta,\gamma) = E_q\left(\alpha(1-q)A^\dagger\right)E_q\left(\beta(1-q)A\right)E_q\left(\gamma(1-q)N\right)\,, \tag{25}$$

which in the realization (23) becomes

$$U(\alpha,\beta,\gamma) = E_q\left(\alpha(1-q)z\right)E_q\left(\beta D_z^+\right)E_q\left(\gamma(1-q)z\frac{d}{dz}\right)\,. \tag{26}$$

We define the matrix elements $U_{kn}(\alpha,\beta,\gamma)$ through

$$U(\alpha,\beta,\gamma)\, z^n = \sum_{k=0}^{\infty} U_{kn}(\alpha,\beta,\gamma)\, z^k\,. \tag{27}$$

[Note that we are using the unnormalized basis: $\{z^n = \sqrt{\Gamma_q(n+1)}|n\rangle\}$.] We shall show that the elements $U_{kn}(\alpha,\beta,\gamma)$ can be expressed in terms of the q-Laguerre polynomials which are usually presented as follows[30]

$$L_k^{(\lambda)}(x;q) = \frac{(q^{\lambda+1};q)_k}{(q;q)_k}\sum_{l=0}^{k}\frac{(q^{-k};q)_l\, q^{\frac{l(l-1)}{2}}\,(1-q)^l\left(q^{k+\lambda+1}x\right)^l}{(q^{\lambda+1};q)_l\,(q;q)_l}\,. \tag{28}$$

In the process, we shall obtain from (27) a generating function for these q-orthogonal polynomials. Using the identities

$$(q^{k+1};q)_{n-k} = \frac{(q;q)_n}{(q;q)_k}\,, \tag{29}$$

$$(q^{-k};q)_n = (-1)^n q^{-kn} q^{\frac{n(n-1)}{2}} \frac{(q;q)_k}{(q;q)_{k-n}}\,, \tag{30}$$

one easily recast the expression (28) for the q-Laguerre polynomials in the form

$$L_k^{(\lambda)}(x;q) = \sum_{l=0}^{k} \frac{(q;q)_{k+\lambda}}{(q;q)_{k-l}\,(q;q)_{l+\lambda}\,(q;q)_l}\, q^{l(l+\lambda)}\,(1-q)^l\,(-x)^l\,. \tag{31}$$

In the limit $q \to 1^-$, $L_k^{(\lambda)}(x;q)$ tends to the ordinary Laguerre polynomials $L_k^{(\lambda)}(x)$.

Clearly, $E_q\left(\gamma(1-q)zd/dz\right)\, z^n = E_q\left(\gamma(1-q)n\right)\, z^n$. From the definition of E_q and with the help of (29), one shows that

$$E_q\left(\beta D_z^+\right)\, z^n = \sum_{m=0}^{n} \frac{(q;q)_n}{(q;q)_{n-m}\,(q;q)_m}\, q^{\frac{m(m-1)}{2}} \beta^m z^{n-m}\,. \tag{32}$$

One thus obtains

$$\begin{aligned} &E_q\bigl(\alpha(1-q)z\bigr)\, E_q\left(\beta D_z^+\right)\, z^n \\ &= \sum_{l=0}^{\infty}\sum_{m=0}^{n} \frac{(q;q)_n}{(q;q)_{n-m}\,(q;q)_m\,(q;q)_l}\, q^{\frac{m(m-1)+l(l-1)}{2}}\, \alpha^l \beta^m\,(1-q)^l\, z^{n-m+l} \\ &= \sum_{k=0}^{\infty} q^{\frac{(n-k)(n-k-1)}{2}} \beta^{n-k}\, z^k \left[\sum_{l=0}^{k} \frac{(q;q)_n\, q^{l(l+n-k)}\,(1-q)^l}{(q;q)_{k-l}\,(q;q)_{n-k+l}\,(q;q)_l} \left(\frac{\alpha\beta}{q}\right)^l\right]. \end{aligned} \tag{33}$$

Comparing with (31), one thus finds that

$$U_{kn}(\alpha,\beta,\gamma) = E_q\bigl(\gamma(1-q)n\bigr)\, q^{\frac{(n-k)(n-k-1)}{2}}\, \beta^{n-k}\, L_k^{(n-k)}\left(-\frac{\alpha\beta}{q};q\right). \tag{34}$$

Now, using (30), the q-binomial formula and (9), the right-hand side of Eq.(32) can be summed to give[29]

$$E_q\left(\beta D_z^+\right)\, z^n = \left(-\frac{\beta}{z};q\right)_n z^n\,. \tag{35}$$

One therefore also gets

$$U(-\alpha,q,0)\, z^n = E_q\left(-\alpha(1-q)z\right)\, \left(-\frac{q}{z};q\right)_n z^n\,. \tag{36}$$

Combining (36) and (34), one finally obtains from (27) the following generating function:

$$E_q\left(-\alpha(1-q)z\right)\,\left(-\frac{q}{z};q\right)_n z^n = \sum_{k=0}^{\infty} q^{\frac{(n-k)(n-k+1)}{2}}\, L_k^{(n-k)}(\alpha;q)\, z^k\,. \tag{37}$$

This is the q-analog of the relation[31,10]

$$e^{-\alpha z}\,(z+1)^n = \sum_{k=0}^{\infty} L_k^{(n-k)}(\alpha)\, z^k\,, \tag{38}$$

for ordinary Laguerre polynomials, to which it reduces in the limit $q \to 1^-$.

The q-Hermite polynomials $H_n(w;q)$ are defined by[32–34]

$$H_n(w;q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q w^k\,, \tag{39}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is the q-binomial coefficient

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k\,(q;q)_{n-k}}\,. \tag{40}$$

Since[13]

$$(x+y)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^k\, x^{n-k}\,, \tag{41}$$

for any two indeterminates x and y satisfying $xy = q\,yx$, we can write[33]

$$H_n(w;q) = (w + T_w)^n \cdot 1\,. \tag{42}$$

These polynomials obey the following orthogonality relation[34]

$$\frac{1}{2\pi i}\oint_{|w|=1} H_m\!\left(-\frac{\overline{w}}{\sqrt{q}};q\right)\, H_n\!\left(-\frac{w}{\sqrt{q}};q\right)\, f(w)\,\frac{dw}{w} = q^{-n}\,(q;q)_n\,\delta_{m,n}\,, \tag{43}$$

with

$$f(w) = \sum_{k=-\infty}^{\infty} q^{k^2/2}\, w^k\,, \qquad |w| = 1\,. \tag{44}$$

Note that $f(w) = \vartheta_3\left(\frac{1}{2\pi i}\ln q, \frac{1}{2\pi i}\ln w\right)$.

The relation between the q-Hermite polynomials and the oscillator quantum algebra is established from the following representation of $W_q(1)$ on functions of w. [See also Ref.[19].] Take

$$A = \frac{1}{1-q}\,\frac{1}{w}\,\left(1 - \sqrt{w}\,T_w\right) \qquad A^\dagger = w\left(1 - \sqrt{\frac{q}{w}}\,T_w\right)\,, \tag{45a}$$

$$N = \frac{\ln[1-(1-q)A^\dagger A]}{\ln q} \, . \tag{45b}$$

It is again easily verified that the defining relations (6) are identically satisfied by these operators. For instance, it is immediate to see from the above definition of N and using $AA^\dagger - q\,A^\dagger A = 1$ that $[q^N, A] = (1-q)q^N A$ and to conclude that $[N, A] = -A$.

Now one checks that

$$\phi_0(w) = \left[f\left(\frac{w}{\sqrt{q}}\right)\right]^{\frac{1}{2}} , \tag{46}$$

satisfy $A\,\phi_0 = 0$, that is

$$\phi_0(w) = \sqrt{w}\,\phi_0(qw) \, . \tag{47}$$

The basis states $\phi_n(w)$, corresponding to the vectors z^n in the first representation are obtained by repeated application of $A^\dagger$ on this ground state. One notes using (47) that $A^\dagger\,\phi_0(w)\,\mathcal{F}(w) = \phi_0(w)\left[-\sqrt{q}(-\frac{w}{\sqrt{q}} + T_w)\right]\mathcal{F}(w)$ for any arbitrary function $\mathcal{F}$, to find with the help of (42) that

$$\phi_n(w) = \left(A^\dagger\right)^n \phi_0(w) = \left(-\sqrt{q}\right)^n \phi_0(w)\; H_n(-\frac{w}{\sqrt{q}}; q) \, . \tag{48}$$

In this representation of the oscillator quantum algebra, the basis vectors are thus expressed in terms of the q-Hermite polynomials. The orthogonality relation that they satisfy is inferred from (43). Furthermore, one verifies that A, $A^\dagger$ and N, as given in (45), act on the (unnormalized) basis states $\phi_n(w)$ exactly as their homologues (23) do on vectors z^n:

$$N\,\phi_n(w) = n\,\phi_n(w) \, , \qquad A\,\phi_n(w) = \left(\frac{1-q^n}{1-q}\right)\phi_{n-1}(w) \, , \qquad A^\dagger\,\phi_n(w) = \phi_{n+1}(w) \, . \tag{49}$$

It follows that $\phi_n(w)$ will transform like z^n under the action of $U(\alpha,\beta,\gamma)$:

$$\begin{aligned} &E_q\left(\alpha(1-q)A^\dagger\right) E_q\left(\beta(1-q)A\right) E_q\left(\gamma(1-q)N\right)\ \phi_n(w) \\ &\qquad = \sum_{k=0}^{\infty} U_{kn}(\alpha,\beta,\gamma)\ \phi_k(w) \\ &\qquad = \sum_{k=0}^{\infty} E_q\left(\gamma(1-q)n\right) q^{\frac{(n-k)(n-k-1)}{2}}\ \beta^{n-k}\ L_k^{(n-k)}(-\frac{\alpha\beta}{q}; q)\ \phi_k(w) \, . \end{aligned} \tag{50}$$

The left-hand side of this equation can be evaluated straightforwardly and after some simplifications and the redefinitions $\bar{\alpha} = \alpha/\sqrt{q}$, $\bar{\beta} = -\beta/\sqrt{q}$, $\bar{w} = -w/\sqrt{q}$, the following relation between the q-Hermite and the q-Laguerre polynomials is found:

$$\begin{aligned} &\sum_{m=0}^{n}\sum_{k=0}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix}_q q^{\frac{1}{2}[m(m-2n+1)+k(k+1)]}\ \frac{(-1)^k}{\Gamma_q(k+1)}\ \bar{\alpha}^k\,\bar{\beta}^{-m}\ H_{m+k}(\bar{w}; q) \\ &\qquad\qquad = \sum_{k=0}^{\infty} q^{\frac{1}{2}k(k-2n+1)}\,\bar{\beta}^{-k}\ L_k^{(n-k)}(\bar{\alpha}\bar{\beta}; q)\ H_k(\bar{w}; q) \, . \end{aligned} \tag{51}$$

From[32]

$$w^n = \sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix}_q (-1)^r q^{\frac{r(r-1)}{2}} H_{n-r}(w;q) , \tag{52}$$

and $(w + T_w)^k w^n = w^n q^{kn} H_k(q^{-n}w;q)$, one arrives at the identity:

$$\sum_{r=0}^{n} \begin{bmatrix} n \\ r \end{bmatrix} (-1)^r q^{\frac{r(r-1)}{2}} H_{n+k-r}(w;q) = w^n q^{nk} H_k(q^{-n}w;q) , \tag{53}$$

which is of help in deriving the following simpler relation between $L_k^{(\lambda)}(x;q)$ and $H_k(x;q)$. Upon setting $\tilde{\beta} = -1$ and letting $\tilde{\alpha} \to -q^{-(n+1)}\tilde{\alpha}$, $\tilde{w} \to q^n \tilde{w}$ in (51), one obtains:

$$\begin{aligned}
&(-\tilde{w})^n E_q\Big((1-q)\tilde{\alpha}\,(\tilde{w} + T_{\tilde{w}})\Big) \cdot 1 \\
&\quad \equiv (-\tilde{w})^n \sum_{k=0}^{\infty} \frac{q^{\frac{k(k-1)}{2}}}{\Gamma_q(k+1)} \tilde{\alpha}^k H_k(\tilde{w};q) \\
&\quad = \sum_{k=0}^{\infty} (-1)^k q^{\frac{1}{2}[k(k-n+1)-n(n+k+1)]} L_k^{(n-k)}(q^{-(n+1)}\tilde{\alpha};q)\, H_k(q^n\tilde{w};q) .
\end{aligned} \tag{53}$$

It should be noted that

$$\lim_{q\to 1^-} H_n(w;q) = (w+1)^n . \tag{54}$$

Substituting z for $-(\tilde{w}+1)$, it easily seen that (53) goes into (38) in the limit $q \to 1^-$.

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References

1. Drinfel'd, V.G., Quantum groups, in: *Proceedings of the International Congress of Mathematicians*, Berkeley (1986), vol. **1**, 798-820, (The American Mathematical Society, 1987)
2. Jimbo, M., A q-difference analogue of $U(g)$ and the Yang-Baxter equation, Lett. Math. Phys. **10**, 63-69 (1985); A q-analogue of $U(gl(N+1))$, Hecke algebra and the Yang-Baxter equation, *ibid.* **11**, 247-252 (1986)
3. Woronowicz, S.L., Compact matrix pseudogroups, Comm. Math. Phys. **111**, 613-665 (1987)
4. Faddeev, L.D., Reshetikhin, N.Yu. and Takhatajan, L.A., Quantization of Lie groups and Lie-algebras, in *Algebraic analysis*, vol. 1, 129 (Academic Press, New York, 1988)
5. Manin, Yu.I., *Quantum Groups and Non-Commutative Geometry*, (Centre de Recherches Mathematiques, Montréal, 1988)
6. Gasper, G. and Rahman, M., *Basic Hypergeometric Series*, (Cambridge University Press, Cambridge, 1990)
7. Exton, H., *q-Hypergeometric Functions and Applications*, (Ellis Horwood, Chichester, 1983)
8. Askey, R., Ramanujan and hypergeometric and basic hypergeometric series, Russian Math. Surveys **45**, 37-86 (1990)
9. Vilenkin, N.Ya., *Special Functions and the Theory of Group Representations*, Amer. Math. Soc. Transl. of Math. Monographs **22**, (Amer. Math. Soc., Providence, 1968)
10. Miller, W., *Symmetry Groups and Their Applications*, (Academic Press, New York, 1972)
11. Vaksman, L.L. and Soibelman, Ya.S., Algebra of functions of the quantum group $SU(2)$, Funct. Anal. Appl. **22**, 1-14 (1988)
12. Masuda, T., Mimachi, K., Nakagami, Y., Noumi, M. and Ueno, K., Representations of quantum groups and a q-analogue of orthogonal polynomials, C. R. Acad. Sci. Paris **307**, 559-564 (1988)
13. Koornwinder, T.H., Representations of the twisted $SU(2)$ quantum group and some q-hypergeometric orthogonal polynomials, Nederl. Akad. Wetensch. Proc. Ser. **A92**, 97-117 (1989)
14. Vaksman, L.L. and Soibelman, Ya.S., An algebra of bounded functions on the quantum group of the motions of the plane, and q-analogues of Bessel functions, Soviet Math. Dokl. **39**, 173-177 (1989)
15. Masuda, T., Mimachi, K., Nakagami, Y., Noumi, M., Saburi, Y. and Ueno, K., Unitary representations of the quantum group $SU_q(1,1)$: structure of the dual space of $\mathcal{U}_q(sl(2))$, Lett. Math. Phys. **19**, 187-194 (1990); Unitary representations of the quantum group $SU_q(1,1)$: II-matrix elements of unitary representations and the basic hypergeometric functions, *ibid.* **19**, 195-204 (1990)

16. Kirillov, A.N. and Reshetikhin, N.Yu., Representations of the algebra $U_q(sl(2))$, q-orthogonal polynomials and invariants of links, in *Infinite Dimensional Lie Algebras and Groups*, Kac, V.G. ed., 285-339, (World Scientific, Singapore, 1989)

17. Noumi, M. and Mimachi, K., Quantum 2-spheres and big q-Jacobi polynomials, Commun. Math. Phys. **128**, 521-531 (1990); Big q-Jacobi polynomials, q-Hahn polynomials, and a family of quantum 3-spheres, Lett. Math. Phys. **19**, 299-305 (1990)

18. Biedenharn, L.C., The quantum group $SU(2)_q$ and a q-analogue of the boson operators, J. Phys. **A22**, L873-L878 (1989)

19. Macfarlane, A.J., On q-analogues of the quantum harmonic oscillator and the quantum group $SU(2)_q$, J. Phys. **A22**, 4581-4588 (1989)

20. Sen, C.-P. and Fu, H.-C., The q-deformed boson realization of the quantum group $SU(n)_q$ and its representations, J. Phys. **A22**, L983-L986 (1989)

21. Floreanini, R. and Vinet, L., q-Analogues of the parabose and parafermi oscillators and representations of quantum algebras, J. Phys. **A23**, L1019-L1023 (1990)

22. Hayashi, T., Q-analogue of Clifford and Weyl algebras–Spinor and oscillator representations of quantum envelopping algebras, Comm. Math. Phys. **127**, 129-144 (1990)

23. Chaichan, M. and Kulish, P., Quantum Lie superalgebras and q-oscillators, Phys. Lett. **B234**, 72-80 (1990)

24. Floreanini, R., Spiridonov, V.P. and Vinet, L., Bosonic realization of the quantum superalgebra $osp_q(1,2n)$, Phys. Lett. **B242**, 383-386 (1990)

25. Floreanini, R., Spiridonov, V.P. and Vinet, L., q-Oscillator realizations of the quantum superalgebras $sl_q(m,n)$ and $osp_q(m,2n)$, Commun. Math. Phys., to appear

26. D'Hoker, E., Floreanini, R. and Vinet, L., q-oscillator realizations of the metaplectic representation of quantum $osp_q(3,2)$, J. Math. Phys., to appear

27. E. Abe, *Hopf Algebras*, (Cambridge University Press, Cambridge, 1980)

28. Yan, H., q-Deformed oscillator algebra as a quantum group, Academia Sinica-preprint, AS-ITP-90-33, (1990)

29. Agarwal, A.K., Kalnins, E.G. and Miller, W., Canonical equations and symmetry techniques for q-series, SIAM J. Math. Anal. **18**, 1519-1538 (1987)

30. Moak, D.S., The q-analogue of the Laguerre polynomials, J. Math. Anal. Appl. **81**, 20-47 (1981)

31. *Higher Transcendental Functions*, Erdélyi, A., ed., vol. 2, (McGraw-Hill, New York, 1953)

32. Carlitz, L., Some polynomials related to theta functions, Annali di Matematica Pura ed Applicata (4) **41**, 359-373 (1955)

33. Carlitz, L., Some polynomials related to theta functions, Duke Math. J. **24**, 521-527 (1957)

34. Carlitz, L., Note on orthogonal polynomials related to theta functions, Publicationes Mathematicae **5**, 222-228 (1958)