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PARASUPERSYMMETRY IN QUANTUM MECHANICS AND FIELD THEORY

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Abstract

We study para-supersymmetric quantum mechanical models containing many bosonic and parafermionic variables. The extension to an infinite number of degrees of freedom naturally leads to a simple two-dimensional field theory.

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1. INTRODUCTION

In the usual approach to Quantum Mechanics the time-evolution of the observables is described by the Heisenberg equations of motion. Using the usual canonical commutation relations for the dynamical variables (and the equivalence principle), one can generally check that these equations are compatible with the classical equations of motion. However, it is well known that the canonical quantization rules, although sufficient, are not necessary to ensure this consistency. There is in fact an infinity of quantization procedures for which the quantum equations of motion agree with the classical ones. These schemes are referred to as para-quantizations,[1] and the dynamical variables satisfying the corresponding generalized quantization rules are said to be parabosonic or parafermionic.

In the simplest situation of a single bosonic harmonic oscillator, with Hamiltonian

$$H = \frac{1}{2}(a^\dagger a + a a^\dagger) , \tag{1.1}$$

the quantum equations of motion $\dot{a} = -ia$, $\dot{a}^\dagger = ia^\dagger$, imply

$$[a, H] = a \qquad [a^\dagger, H] = -a^\dagger . \tag{1.2}$$

[The dot signifies time derivative.] The standard way of realizing these relations is to assume that the commutator between a and $a^\dagger$ is a c-number:

$$[a, a^\dagger] = 1 . \tag{1.3}$$

Nevertheless, the relations (1.2) are not incompatible with $[a, a^\dagger]$ being an operator. This is in fact the case in all parabosonic quantization schemes; in particular, in the simplest case one finds that the operators a and $a^\dagger$ obey the following trilinear parabosonic "commutation relations":

$$a^2 a^\dagger - a^\dagger a^2 = 2a \qquad a(a^\dagger)^2 - (a^\dagger)^2 a = 2a^\dagger . \tag{1.4}$$

Generalization to the case of many oscillators or to fermionic variables is straightforward.[1] The basic commutation relations of paraquantization, generalizing (1.2), are

$$[a_i, [a_j^\dagger, a_k]_\pm] = 2\,\delta_{ij} a_k , \tag{1.5a}$$
$$[a_i, [a_j^\dagger, a_k^\dagger]_\pm] = 2\,\delta_{ij} a_k^\dagger \pm 2\,\delta_{ik} a_j^\dagger , \tag{1.5b}$$
$$[a_i, [a_j, a_k]_\pm] = 0 , \tag{1.5c}$$

together with the hermitian conjugate relations, with $[A, B]_- \equiv [A, B] = AB - BA$ and $[A, B]_+ \equiv \{A, B\} = AB + BA$. The upper signs in (1.5) refer to para-Bose oscillators and the lower signs to para-Fermi oscillators.

As in the usual case, the Fock space on which the operators a_i and $a_i^\dagger$ act is built on a vacuum state $|0\rangle$,

$$a_i|0\rangle = 0 , \qquad i = 1, 2, \dots , \tag{1.6}$$

by successive applications of the creation operators $a_i^\dagger$. The requirement that the number operator $\mathcal{N}_i$ be an hermitian non-negative operator for each mode i, implies

$$a_i a_j^\dagger |0\rangle = p\,\delta_{ij}|0\rangle\ , \tag{1.7}$$

with p a non-negative integer, called the order of paraquantization. One finds that

$$\mathcal{N}_i = \frac{1}{2}[a_i^\dagger, a_i]_\pm \mp \frac{p}{2}\ . \tag{1.8}$$

One can show that for fixed p, the conditions (1.6) and (1.7) imposed on the vacuum state uniquely determine, up to unitary equivalences, an irreducible highest weight representation of the paracommutation relations (1.5). In the usual canonical case $p = 1$.

These representations, labelled by p, can be described without reference to the vacuum state by giving a set of algebraic relations between a_i and $a_i^\dagger$ for each order. For $p = 1$, these relations are the usual bilinear commutations and anticommutation relations:

$$[a_i, a_j^\dagger]_\mp = \delta_{ij} \qquad [a_i, a_j]_\mp = 0 = [a_i^\dagger, a_j^\dagger]_\mp\ . \tag{1.9}$$

When $p = 2$, the complete set of paracommutation relations are trilinear in a_i and $a_i^\dagger$; explicitly,

$$a_i a_j^\dagger a_k \mp a_k a_j^\dagger a_i = 2\,\delta_{ij} a_k \mp 2\,\delta_{jk} a_i\ , \tag{1.10a}$$

$$a_i a_j a_k^\dagger \mp a_k^\dagger a_j a_i = 2\,\delta_{jk} a_i\ , \tag{1.10b}$$

$$a_i a_j a_k \mp a_k a_j a_i = 0\ , \tag{1.10c}$$

plus the ones that are obtained by hermitian conjugation. For higher order p, the paracommutation relations become more and more involved. In the following we shall only consider para-variables of order $p = 2$.

In systems involving both ordinary bosonic and fermionic degrees of freedom, supersymmetry transformations can arise as dynamical symmetries.[2] These transformations mix the bosonic and fermionic variables. The corresponding generators are constants of motion and form a Lie superalgebra under the standard bilinear graded product. A natural generalization consists in systems involving dynamical variables both of ordinary Bose type and of para-Fermi type. In such situations, we might expect the presence of symmetry operations transforming the bosonic variables into the parafermionic ones and vice-versa. These operations generalize the familiar supersymmetry transformations and have been called parasupersymmetric.[3] Their generators realize new algebraic structures, referred to as parasuperalgebras, of which the ordinary superalgebras are special cases. The main feature of the parasuperalgebras lies in the fact that they involve multilinear product rules for the fermionic elements; therefore, they are not Lie algebras.

Many quantum mechanical systems exhibiting parasupersymmetries have been constructed and studied.[3-9] Generalizations along these lines of the superconformal algebra have also been discussed.[10] However, the models discussed so far have only involved a single parafermionic degree of freedom. Here, we shall discuss simple examples of parasupersymmetric systems involving many para-Fermi and ordinary Bose variables. In Section 2 we consider models which describe the motion of ordinary real bosonic and real parafermionic degrees of freedom in a constant external magnetic field. The extension to field theory is carried out in Section 3 and in Section 4 a model with N complex free bosons and N complex free parafermions is analyzed.

2. REAL VARIABLES

Let us first discuss to fix the notation, a simple quantum mechanical model free of interactions and described by N ordinary bosonic degrees of freedom $x_i(t)$ and N real parafermionic variables $\psi_i(t)$, $i = 1, \ldots, N$. [t is the time variable.] Since we are imposing $\psi_i^\dagger = \psi_i$, the parafermionic commutation relations (1.5) reduce to

$$[[\psi_i, \psi_j], \psi_k] = 2\,\delta_{jk}\psi_i - 2\,\delta_{ik}\psi_j \,. \tag{2.1}$$

We shall restrict our attention to parafermions of order $p = 2$. In this case, (2.1) is equivalent to the following trilinear relation:

$$\psi_i\psi_j\psi_k + \psi_k\psi_j\psi_i = 2\,\delta_{ij}\psi_k + 2\,\delta_{jk}\psi_i \,, \tag{2.2}$$

which can also be rewritten in a more symmetric way,

$$\psi_i\Big(\{\psi_j, \psi_k\} - 4\,\delta_{jk}\Big) + \psi_j\Big(\{\psi_k, \psi_i\} - 4\,\delta_{ki}\Big) + \psi_k\Big(\{\psi_i, \psi_j\} - 4\,\delta_{ij}\Big) = 0 \,. \tag{2.3}$$

Furthermore, bosonic and parafermionic variables are taken to commute among themselves: $[x_i, \psi_j] = 0$.

The Lagrangian describing this simple system is given by

$$L = \frac{1}{2}\dot{x}_i^2 + \frac{i}{4}[\psi_i, \dot{\psi}_i] \,. \tag{2.4}$$

The equations of motion are simply: $\ddot{x}_i = 0$, $\dot{\psi}_i = 0$ and the dynamics is trivial. Nevertheless, L possesses symmetry transformations that interchange the bosonic and the parafermionic variables. By analogy with the standard supersymmetric case for which the parameters of the transformations are Grassmann variables, one is led to take as parameters of parasupersymmetry transformations para-Grassmann numbers θ_i, which obey the following algebra (specific to order $p = 2$)[1,3]

$$\theta_i\theta_j\theta_k + \theta_k\theta_j\theta_i = 0 \,. \tag{2.5}$$

Note that this implies $(\theta_i)^3 = 0$, which naturally generalizes the condition $(\xi_i)^2 = 0$ for the Grassmann numbers ξ_i. For arbitrary order p, one would have $(\theta_i)^{p+1} = 0$. The numbers θ_i are assumed to have non-trivial commutation relations with the variables ψ_i:

$$[[\theta_i, \psi_j], \psi_k] = 2\,\delta_{jk}\theta_i \; , \tag{2.6a}$$

$$[[\theta_i, \psi_j], \theta_k] = 0 = [[\psi_i, \psi_j], \theta_k] \; . \tag{2.6b}$$

Let us now consider the following infinitesimal transformations between the bosonic and parafermionic variables,

$$\delta_\theta x_i = \frac{i}{2}[\theta, \psi_i] \qquad\qquad \delta_\theta \psi_i = -\theta\, \dot{x}_i \; . \tag{2.7}$$

The Lagrangian (2.4) changes by a total time derivative under these transformations:

$$\delta_\theta L = \dot{\Lambda} \; , \qquad\qquad \Lambda = \frac{1}{4}[\theta, \psi_i]\dot{x}_i \; . \tag{2.8}$$

The transformations (2.7) are thus symmetries of the action. The corresponding conserved Noether charge is easily computed:

$$\begin{aligned} Q_\theta &\equiv \dot{x}_i\, \delta_\theta x_i + \frac{i}{4}[\theta, \delta_\theta \psi_i] - \Lambda \\ &= \frac{i}{2}[\theta, p_i \psi_i] \equiv \frac{i}{2}[\theta, Q] \; , \end{aligned} \tag{2.9}$$

with p_i the momenta of x_i. Using the standard canonical bosonic commutation relations, $[x_i, p_j] = i\delta_{ij}$ and the parafermionic relations (2.6), it is easy to check that Q_θ indeed generates (2.7):

$$\delta_\theta x_i = i[Q_\theta, x_i] \qquad\qquad \delta_\theta \psi_i = i[Q_\theta, \psi_i] \; . \tag{2.10}$$

The infinitesimal transformations (2.7) close onto an algebra that involve trilinear relations; using the properties of the para-Grassmann numbers, one finds:

$$\Big(\delta_{\theta_1}\delta_{\theta_2}\delta_{\theta_3} + \delta_{\theta_2}\delta_{\theta_3}\delta_{\theta_1} + \delta_{\theta_3}\delta_{\theta_1}\delta_{\theta_2}\Big) x_i = -i\frac{d}{dt}\Big(\delta_{(\theta_1\theta_2\theta_3+\theta_2\theta_3\theta_1+\theta_3\theta_1\theta_2)} x_i\Big) \; , \tag{2.11}$$

and an analogous relation for ψ_i. To better understand the structure of this algebra, it is useful to pass to the Hamiltonian formulation which make no use of the relations (2.6) between θ_i and ψ_i that might seem *ad hoc*.

The Hamiltonian coming from the Lagrangian (2.4) is just: $H = \frac{1}{2}p_i^2$. The para-Fermi charge corresponding to the generator Q_θ is obtained from (2.9): $Q = p_i \psi_i$. Being conserved, Q commutes with the Hamiltonian:

$$[H, Q] = 0 \; . \tag{2.12a}$$

Moreover, using the parafermionic relations (2.2), Q is also seen to satisfy

$$Q^3 = 2\{H, Q\} , \tag{2.12b}$$

which is essentially the Hamiltonian rewriting of (2.11). Algebras with defining relations like (2.12) have been called (real) parasuperalgebras of order $p = 2$. Their distinctive feature is the occurrence of a trilinear product rule for the fermionic elements. [Parasuperalgebras of order p would involve a $(p+1)$-linear product.]

A few non-trivial (*i.e.* interacting) quantum mechanical models with (2.12) as symmetry algebra have been constructed. Most of them however only involve a single para-Fermi variable. The system described by the Lagrangian (2.4) contains N parafermionic degrees of freedom, but it does not have interactions. In this respect, a simple extension is obtained by introducing an external constant magnetic field. The Lagrangian is then given by

$$L = \frac{1}{2}\dot{x}_i^2 + \frac{i}{4}[\psi_i, \dot{\psi}_i] + A_i \dot{x}_i - \frac{i}{4} F_{ij}[\psi_i, \psi_j] , \tag{2.13}$$

with F_{ij} the constant field strength corresponding to the vector potential A_i; in a suitable gauge one can write $A_i = -\frac{1}{2}F_{ij}x_j$. The equations of motion are easily obtained and read:

$$\ddot{x}_i = F_{ij}\dot{x}_j \qquad \dot{\psi}_i = F_{ij}\psi_j . \tag{2.14}$$

The Lagrangian (2.13) changes by a total time derivative under the infinitesimal transformations given in (2.7), which are thus also symmetries of this extended system:

$$\delta_\theta L = \dot{\Lambda} , \qquad \Lambda = \frac{i}{2}[\theta, \psi_i](\frac{1}{2}\dot{x}_i + A_i) . \tag{2.15}$$

The corresponding conserved generator is given by

$$Q_\theta = \frac{i}{2}[\theta, \pi_i \psi_i] \equiv \frac{i}{2}[\theta, Q] , \tag{2.16}$$

where $\pi_i = p_i - A_i$ are the bosonic velocities, and $[x_i, \pi_j] = i\,\delta_{ij}$, $[\pi_i, \pi_j] = iF_{ij}$.

It is again instructive to discuss the algebra of the parafermionic generators $Q = \pi_i \psi_i$ at the Hamiltonian level. First of all, the charge Q commutes with the Hamiltonian $H = \frac{1}{2}{\pi_i}^2 + \frac{i}{4}F_{ij}[\psi_i, \psi_j]$, so that (2.12*a*) is satisfied. The trilinear relation (2.12b) however is in general modified. Since $[\pi_i, \pi_j]$ is non zero:

$$Q^3 = 2\{H, Q\} + \frac{1}{2}R , \tag{2.17}$$

where

$$R = i\Big(F_{ji}\pi_k + F_{ik}\pi_j + F_{kj}\pi_i\Big)\psi_i\psi_j\psi_k , \tag{2.18}$$

is a new symmetry generator: $[H, R] = 0$. Thus, adding to the free Lagrangian (2.4) a simple interaction term has produced a symmetry algebra which is more complicated than (2.12).

Nevertheless, for the particular case $N = 2$ one can check using the paracommutation relations (2.2) that R is zero; Eq.(2.17) then reduces to (2.12b). Actually, there are more symmetries.[5] In particular, besides $Q \equiv Q_1$ there exists a second conserved para-supersymmetric charge $Q_2 = \varepsilon_{ij}\pi_i\psi_j$, also obeying (2.12). One can moreover check that: $Q_1^2Q_2 + Q_2Q_1Q_2 + Q_2Q_1^2 = 2\{Q_2, H\}$ and $Q_2^2Q_1 + Q_2Q_1Q_2 + Q_1Q_2^2 = 2\{Q_1, H\}$. These two relations together with (2.12b) can be conveniently combined into:

$$\begin{aligned} Q_i\Big(\{Q_j,Q_k\} - 8\,\delta_{jk}H\Big) + Q_j\Big(\{Q_k, Q_i\} - 8\,\delta_{ki}H\Big) \\ + Q_k\Big(\{Q_i, Q_j\} - 8\,\delta_{ij}H\Big) = 0\ . \end{aligned} \tag{2.19}$$

Introducing the hermitian conjugate operators $Q = \frac{1}{2}(Q_1 + iQ_2)$ and $Q^\dagger = \frac{1}{2}(Q_1 - iQ_2)$, (2.19) can be rewritten as

$$Q^2Q^\dagger + QQ^\dagger Q + Q^\dagger Q^2 = 2\{H, Q\}\ , \tag{2.20a}$$

$$Q^{\dagger^2}Q + Q^\dagger QQ^\dagger + QQ^{\dagger^2} = 2\{H, Q^\dagger\}\ , \tag{2.20b}$$

$$Q^3 = Q^{\dagger^3} = 0\ , \tag{2.20c}$$

$$[H, Q] = [H, Q^\dagger] = 0\ , \tag{2.20d}$$

which is the form in which the complex parasupersymmetry algebra was first presented.[3]

Although R is surely non zero as an operator for $N > 2$, this does not exclude the possibility that it could well vanish in a given parafermionic Fock space. We explained in the Introduction how the representation of the para-commutation relations (2.1) is characterized by the choice of the vacuum state. Since we are dealing with $p = 2$ parafermions, it is natural to require (compare with (1.7)):

$$\psi_i\psi_j|0\rangle = 2\,\delta_{ij}|0\rangle\ . \tag{2.21}$$

The fermion Fock space built on $|0\rangle$ is $(N+1)$-dimensional, and its basis $\{|0\rangle, |i\rangle\}$ is defined by

$$\begin{aligned} \psi_j|0\rangle &= -i\sqrt{2}|j\rangle \\ \psi_j|k\rangle &= i\sqrt{2}\delta_{jk}|0\rangle\ . \end{aligned} \tag{2.22}$$

In this space, the ψ_i's are represented by $(N+1)\times(N+1)$ matrices, explicitly given by:

$$(\psi_j)_{\mu\nu} = i\sqrt{2}\Big(\delta_{j\nu}\delta_{\mu 0} - \delta_{j\mu}\delta_{\nu 0}\Big)\ , \qquad \mu,\nu = 0,\ldots,N\ , \quad k = 1,\ldots,N\ . \tag{2.23}$$

One can easily check that the parafermionic relations (2.2) are indeed satisfied by the matrices (2.23). These matrices have an additional property:

$$\psi_i\psi_j\psi_k = 0 \qquad\qquad \text{for } i \neq j \neq k\ . \tag{2.24}$$

This guarantees that the charge R is zero in this representation. There is a nice interpretation of (2.23) in terms of group representations. It is well known that one can construct explicit realizations of the classical Lie algebras and superalgebras using linears and bilinears in ordinary Bose and Fermi oscillators. This property extends to para-Bose and para-Fermi oscillators.[1,11] In particular, with the N para-Fermi operators ψ_i one can obtain realizations of the algebra of $SO(N+1)$. In fact, using (2.1) one can easily check that

$$J_{ij} = \frac{1}{2i}[\psi_i, \psi_j] \tag{2.25a}$$

$$J_{i0} = \frac{1}{\sqrt{2}}\psi_i\ , \tag{2.25b}$$

are generators of $SO(N+1)$:

$$[J_{\mu\nu}, J_{\rho\sigma}] = i\Big(\delta_{\mu\rho}\, J_{\nu\sigma} + \delta_{\nu\sigma}\, J_{\mu\rho} - \delta_{\mu\sigma}\, J_{\nu\rho} - \delta_{\nu\rho}\, J_{\mu\sigma}\Big)\ . \tag{2.26}$$

The matrix realization given in (2.23) precisely corresponds to the fundamental representation of this algebra.

3. FIELD THEORY

The simple quantum mechanical models discussed in the previous Section can be easily extended to continuum systems describing an infinite number of bosonic and parafermionic degrees of freedom.[12] We shall concentrate our attention to the field theory analog of the Lagrangian (2.13) in two space-time dimensions. The system involves a bosonic field $\chi(t,x)$ and a parafermionic field $\psi(t,x)$ satisfying the self-dual equations of motion:

$$\dot{\chi} = \chi' \qquad \dot{\psi} = \psi'\ . \tag{3.1}$$

[The dot means time differentiation, while the prime represents differentiation with respect to the space variable x.] The theory is governed by the Lagrangian:

$$\begin{aligned} L = \frac{1}{4}\int dx\, dy\, \chi(t,x)\, \mathrm{sign}(x-y)\, \dot{\chi}(t,y) - \frac{1}{2}\int dx\, \chi^2(t,x) \\ + \frac{i}{4}\int dx \Big[\psi(t,x), \big(\dot{\psi}(t,x) - \psi'(t,x)\big)\Big]\ , \end{aligned} \tag{3.2}$$

from which (3.1) arise as Euler-Lagrange equations. The extension of the paracommutation relations to field variables is straigthforward; instead of (2.1) and (2.2) one respectively has (at fixed time):

$$\Big[[\psi(x), \psi(y)], \psi(z)\Big] = 2\,\delta(y-z)\psi(x) - 2\delta(x-z)\psi(y)\ , \tag{3.3}$$

$$\psi(x)\psi(y)\psi(z)+\psi(z)\psi(y)\psi(x)=2\,\delta(x-y)\psi(z)+2\,\delta(y-z)\psi(x)\ . \tag{3.4}$$

On the other hand, standard quantization of the bosonic part of the Lagrangian (3.2) produces the commutator:[13]

$$[\chi(x),\chi(y)]=i\delta'(x-y)\ . \tag{3.5}$$

The field $\chi(x)$ is thus the continuum analog of the variable π_i of the previous Section and the distribution $\delta'(x-y)$ appearing in (3.5) can be interpreted as a functional constant $U(1)$ field strength: $\mathcal{F}(x,y)=\delta'(x-y)$. Furthermore, as before bosonic and parafermionic fields are taken to commute: $[\chi,\psi]=0$.

It is easy to check that the following infinitesimal transformations,

$$\delta_\theta\chi(x)=\frac{i}{2}\big([\theta,\psi(x)]\big)'\qquad\qquad \delta_\theta\psi(x)=-\theta\,\chi(x)\ , \tag{3.6}$$

are symmetries of the Lagrangian (3.2). The corresponding conserved charge $Q_\theta=\frac{i}{2}\int dx\,[\theta,\psi(x)]\chi(x)$ generates (3.6). As in the finite-dimensional case, it is convenient to deal with the Hamiltonian $H=\frac{1}{2}\int dx\,\chi^2(x)+\frac{i}{4}\int dx\,dy\,\mathcal{F}(x,y)[\psi(x),\psi(y)]$ and the parafermionic charge $Q=\int dx\,\chi(x)\psi(x)$. The generator Q commutes with H and moreover:

$$Q^3=2\,\{H,Q\}+\frac{1}{2}R\ , \tag{3.7}$$

where

$$R=\int dx\,dy\,\chi(x)\Big[\psi'(y)\psi(y)\psi(x)+\psi(y)\psi(x)\psi'(y)+\psi(x)\psi'(y)\psi(y)\Big]\ . \tag{3.8}$$

Clearly, R must also be conserved and in fact, $[H,R]=0$. Note that the Hamiltonian and the symmetry operators Q and R are not well defined since they involve multiplications of fields at the same point. An infinite subtraction is sufficient to well define them. One can check that the algebra (3.7) remains unaltered when renormalized operators are used.

Since in general the charge R is non zero, the symmetry algebra of the system is complicated. A choice of the vacuum state analogous to (2.21) would reduce it to (2.12) since in the corresponding Fock space R is represented by the null operator. However, this choice makes the theory rather trivial, at least for what concerns the parafermionic spectrum. This is more easily seen by going to momentum space and writing $\psi(x)=\int dk\,e^{-ikx}\,b(k)$, with $b^\dagger(k)=b(-k)$. In terms of the Fourier components $b(k)$ the relations (3.3) and (3.4) become:

$$\Big[[b(p),b(k)],b(q)\Big]=2\,\delta(k+q)b(p)-2\,\delta(p+q)b(k)\ , \tag{3.9}$$

$$b(p)b(k)b(q)+b(q)b(k)b(p)=2\,\delta(p+k)b(q)+2\,\delta(k+q)b(p)\ . \tag{3.10}$$

The vacuum state for which $R=0$ satisfies:

$$b(p)b(k)|0\rangle=2\,\delta(p+k)|0\rangle\ , \tag{3.11}$$

the analog of (2.21). However, this implies that the Hilbert space of the system contains only single-particle parafermionic states.

It is interesting to note that the Lagrangian (3.2) actually possesses many more symmetries besides the transformations (3.6). It is in fact invariant under local conformal transformations, given by:[14]

$$\delta_f\chi = (f\chi)' \qquad \delta_f\psi = (f\psi)' - \frac{1}{2}f'\psi\ , \tag{3.12}$$

where $f(t+x)$ is an arbitrary function of its argument. The corresponding generator is given by:

$$Q_f = \frac{1}{2}\int dx\, f(x)\chi^2(x) + \frac{i}{4}\int dx\, f(x)\Big[\psi(x), \psi'(x)\Big]\ . \tag{3.13}$$

Moreover, (3.2) is also invariant under the local version of the transformations (3.6), when the parameter θ is made to depend arbitrarily on the variable $t+x$. The corresponding conserved generator is again given by $Q_\theta = \int[\theta,\psi]\chi$. Together, these local infinitesimal transformations would seem to provide a parasuperconformal generalization of the superconformal transformations. A model for a parasupersymmetric extension of the infinite conformal algebra in two dimensions was provided in Ref.[10]. The algebra that Q_f and Q_θ realize is however more involved.

4. COMPLEX VARIABLES

We have so far examined the parasupersymmetries of simple models involving real bosonic and parafermionic degrees of freedom. We shall now study in this respect, another system which is naturally described in terms of complex variables. Our quantum mechanical model involves N ordinary bosonic oscillators and N parafermionic oscillators of order $p=2$.

The Bose annihilation and creation operators a_i and $a_i^\dagger$, $i=1,\dots,N$, obey the canonical commutation relations

$$[a_i, a_j^\dagger] = \delta_{ij} \qquad [a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0\ , \tag{4.1}$$

while the Fermi operators ψ_i and $\psi_i^\dagger$, $i=1,\dots,N$, satisfy

$$\begin{aligned} &\psi_i\psi_j^\dagger\psi_k + \psi_k\psi_j^\dagger\psi_i = 2\,\delta_{ij}\,\psi_k + 2\,\delta_{jk}\,\psi_i \\ &\psi_i\psi_j\psi_k^\dagger + \psi_k^\dagger\psi_j\psi_i = 2\,\delta_{jk}\,\psi_i \\ &\psi_i\psi_j\psi_k + \psi_k\psi_j\psi_i = 0\ . \end{aligned} \tag{4.2}$$

We further assume that the bosonic and parafermionic variables commute among themselves: $[a_i, \psi_j] = [a_i^\dagger, \psi_j] = 0$. The Hamiltonian describing the system is simply

$$H = \frac{1}{2}\{a_i^\dagger, a_i\} + \frac{1}{2}[\psi_i^\dagger, \psi_i]\ . \tag{4.3}$$

The parafermionic charges

$$Q = a_i^\dagger \psi_i \qquad Q^\dagger = a_i \psi_i^\dagger , \tag{4.4}$$

are conserved: $[H, Q] = [H, Q^\dagger] = 0$. They mix bosonic and parafermionic variables and therefore generate parasupersymmetry transformations. In one dimension ($N = 1$), Q and $Q^\dagger$, together with H, realize the relations (2.20). In general, for $N > 1$, the situation is more complicated. In addition to

$$Q^3 = 0 \qquad (Q^\dagger)^3 = 0 , \tag{4.5}$$

one also gets,

$$Q^2 Q^\dagger + Q Q^\dagger Q + Q^\dagger Q^2 = 2\{H, Q\} + 2(N-1) R , \tag{4.6b}$$
$$Q(Q^\dagger)^2 + Q^\dagger Q Q^\dagger + (Q^\dagger)^2 Q = 2\{H, Q^\dagger\} + 2(N-1) R^\dagger , \tag{4.6b}$$

with $R = a_i^\dagger \Psi_i$ and Ψ_i the following trilinear in the ψ's:

$$\Psi_i = \frac{1}{2(N-1)} \left(\psi_j^\dagger \psi_i \psi_j - \psi_i \psi_j^\dagger \psi_j + \psi_i \psi_j \psi_j^\dagger \right) . \tag{4.7}$$

The operator R commutes with the Hamiltonian and a new symmetry generator is thus obtained.

In order to understand better the role of the charge R, let us set $N = 2$, and construct the explicit representation of the parafermionic algebra (4.2) in the Fock space built on the vacuum state $|0\rangle$, specified by (see (1.6) and (1.7))

$$\psi_i |0\rangle = 0 \qquad \psi_i \psi_j^\dagger |0\rangle = 2\,\delta_{ij}\, |0\rangle . \tag{4.8}$$

The fermionic Fock space is 10-dimensional and in addition to $|0\rangle$ the normalized basis states are:

$$\begin{aligned}
|1\rangle &= \frac{1}{\sqrt{2}} \psi_1^\dagger |0\rangle & |2\rangle &= \frac{1}{\sqrt{2}} \psi_2^\dagger |0\rangle & |3\rangle &= \frac{1}{2} (\psi_1^\dagger)^2 |0\rangle \\
|4\rangle &= \frac{1}{2} (\psi_2^\dagger)^2 |0\rangle & |5\rangle &= \psi_2^\dagger \psi_1^\dagger |0\rangle & |6\rangle &= \psi_1^\dagger \psi_2^\dagger |0\rangle \\
|7\rangle &= \frac{1}{2\sqrt{2}} \psi_2^\dagger (\psi_1^\dagger)^2 |0\rangle & |8\rangle &= \frac{1}{2\sqrt{2}} \psi_1^\dagger (\psi_2^\dagger)^2 |0\rangle & |9\rangle &= \frac{1}{4} (\psi_2^\dagger)^2 (\psi_1^\dagger)^2 |0\rangle .
\end{aligned} \tag{4.9}$$

The operators ψ_i and $\psi_i^\dagger$ are now represented by 10×10 matrices and one can straightforwardly evaluate the combinations Ψ_i. Using this explicit realization, one then discovers that Ψ_i and $\Psi_i^\dagger$ satisfies the parafermionic commutation relations (4.2); in other words, the Ψ's form a second inequivalent realization of (4.2).

One can now determine the algebra satisfied by the charge R. Since the Ψ's obey (4.2), the computation goes as it went for Q, and explicitly one gets,

$$R^3 = 0 \qquad (R^\dagger)^3 = 0 , \tag{4.10}$$

$$R^2 R^\dagger + RR^\dagger R + R^\dagger R^2 = 2\{H,R\} + 2(N-1)\,S \tag{4.11a}$$
$$R(R^\dagger)^2 + R^\dagger RR^\dagger + (R^\dagger)^2 R = 2\{H,R^\dagger\} + 2(N-1)\,S^\dagger\ . \tag{4.11b}$$

The additional generator S is again of the form $S = a_i^\dagger \mathit{\Psi}_i$, where $\mathit{\Psi}_i$ are the trilinears in the Ψ's that are obtained upon effecting $\Psi_i \to \mathit{\Psi}_i$ and $\psi_i \to \Psi_i$ in (4.7). Remarkably, the charge S is something that we already know. In fact, using the matrix representation, one discovers that

$$\mathit{\Psi}_i = \psi_i \qquad\qquad \mathit{\Psi}_i^\dagger = \psi_i^\dagger\ ; \tag{4.12}$$

in other words, $S \equiv Q$. For $N = 2$, Q, R and H therefore constitute a closed set of generators.

Unfortunately, this picture works only for $N = 2$. Consider the operators Ψ_i given in (4.7), with N arbitrary. On the standard vacuum defined in (4.8), one easily sees that:

$$\Psi_i\,|0\rangle = 0 \qquad\qquad \Psi_i\Psi_j^\dagger\,|0\rangle = 2\,\delta_{ij}\,|0\rangle\ . \tag{4.13}$$

However, one also finds that, typically

$$\Psi_i^\dagger\Psi_j^\dagger\Psi_k + \Psi_k\Psi_j^\dagger\Psi_i^\dagger \neq 2\delta_{jk}\,\Psi_i^\dagger\ . \tag{4.14}$$

In fact, acting with $\Psi_i^\dagger\Psi_j^\dagger\Psi_k + \Psi_k\Psi_j^\dagger\Psi_i^\dagger$ on $|0\rangle$, one gets

$$\left(\Psi_i^\dagger\Psi_j^\dagger\Psi_k + \Psi_k\Psi_j^\dagger\Psi_i^\dagger\right)\,|0\rangle = \frac{2}{(N-1)^2}\left[\left(1+(N-2)^2\right)\delta_{jk}\,\Psi_i^\dagger + 2(N-2)\delta_{ik}\,\Psi_j^\dagger\right]\,|0\rangle\ . \tag{4.15}$$

The right-hand side of this formula is equal to $2\,\delta_{jk}\Psi_i^\dagger$ only when $N = 2$ and the Ψ's do not therefore satisfy (4.2) in general. It follows that as N increases the set of generators of the invariance parasuperalgebra gets enlarged by more and more elements.

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