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Abstract

A new quantum group and the calculus on quantum hyperplane is described, which are associated to R-matrix describing the braiding of certain chiral Toda fields. The classical limit of these structures relates to Poisson brackets of certain chiral WZW fields.

The Yang-Baxter equation and the related algebraic structures [1-4] play an important role in two-dimensional 'integrable physics'. Of particular interest is the appearence of quantum groups in conformal field theory [5-9]. In this short talk, building on the results of papers [10,15], I am going to describe a new quantum group and the calculus on quantum hyperplane, which are associated to R-matrix describing the braiding of chiral Toda and WZW fields.

At the classical level, the Poisson brackets of certain $Sl(n)$ fields $g_L(\xi)$ have been recasted [10] in the r-matrix form

$$\{g_L(\xi_1) \overset{\otimes}{,} g_L(\xi_2)\} = -\frac{\pi}{k} [g_L(\xi_1) \otimes g_L(\xi_2)] [\text{sign}(\xi_2 - \xi_1)C + r] , \qquad (1)$$

where

$$r = -2 \sum_{i<j} \sum_{k<l} e_{ij} \wedge e_{lk} \delta_{j-i,l-k} \theta(i-k) + \sum_{\mu=1}^{n-2} h^\mu \wedge h^{\mu+1} . \qquad (2)$$

Here θ is the usual step function ($\theta(0) = \frac{1}{2}$), h_μ are the standard Cartan generators of $sl(n)$, and $C = \lambda^a \otimes \lambda_a$ is the 'Casimir-operator'.

The fields $g_L(\xi)$ are characterized by the following properties: 1) factorization of the (periodic) WZW solution $g(\tau,\sigma) = g_L(\tau + \sigma) \cdot g_R(\tau - \sigma)$, 2) left-moving chirality $-\frac{k}{2\pi} g_L'(\xi) = I(\xi)g_L(\xi)$, where $I(\xi)$ is the Kac-Moody current with level k, 3) 'primary' character $\{I_n^a, g_L(\xi)\} = -\exp(in\xi)\lambda^a g_L(\xi)$, 4) monodromy described by saying that the columns are obtained from the first one by means of successive translations by 2π. Such fields have been constructed in [10] as $g_L = PN$, where P is the path ordered exponent of the integral of $I(\xi)$ and N is a $Sl(n)$ matrix, which depends on the zero modes of $g(\tau,\sigma)$ and on the monodromy eigenvalues of P ($g_L(\xi)$ do not depend on various choices such as the start and end points of the path ordered integral, etc.)

For other results on the 'classical exchange algebra' see [11].

There is a reduction procedure [12] by which the $Sl(n)$ chiral, say left moving, Toda fields can be identified as the last row of of g_L. Since the exchange algebra (1) amounts to matrix multiplication from the right, mixing the columns of $g_L(\xi)$, the chiral Toda fields indeed satisfy the same exchange algebra as WZW fields. For similar reason (1) also yields the exchange algebra for the gauge invariant fields of the generalized Toda theories recently discovered by O'Raifeartaigh and Wipf [13]. One interesting feature of all these models is that the conformal invariance can be extended to general covariance and that one field can be interpreted as two-dimensional graviton, to which the other fields are minimally coupled.

The analogical quantum reduction is not yet fully understood, but it can be supported by the fact that (1) is the classical limit of braiding of certain quantum chiral Toda fields constructed in [14]

$$\psi_j(\xi)\psi_k(\xi') = \sum_{l,m=1}^{n} R_{jklm}\psi_l(\xi')\psi_m(\xi) , \qquad \xi > \xi', \qquad (3)$$

where

$$R = q \sum_i^n e_{ii} \otimes e_{ii} + q \sum_{i>j} q^{-2(i-j)/n} e_{ii} \otimes e_{jj} + q^{-1} \sum_{i<j} q^{2(j-i)/n} e_{ii} \otimes e_{jj}$$
$$+ (q - q^{-1}) \left(\sum_{i>j} \sum_{k=0}^{i-j-1} q^{-2k/n} e_{i,j+k} \otimes e_{j,i-k} - \sum_{i<j} \sum_{k=1}^{j-i-1} q^{2k/n} e_{i,j-k} \otimes e_{j,i+k} \right) .$$

$$(4)$$

The R-matrix (4) is an (invertible) solution of the quantum Yang-Baxter equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} .$$

$$(5)$$

To any such a solution one can associate a bi-algebra and its (formal) dual by a universal 'Leningrad' method [3]. This method, applied to the standard $Sl(n)$ R-matrix

$$R_{DJ}(q) = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + (q - q^{-1}) \sum_{i>j} e_{ij} \otimes e_{ji} .$$

$$(6)$$

reproduces the Drinfeld-Jimbo quantum deformation $SL(n)_q$, whose dual is the quantized universal enveloping algebra $\mathcal{U}_q(sl(n))$ of $Sl(n)$. The study of the quantum group defined by (4) was initiated in [14] where it was pointed out that (6) and (4) are related by a similarity transformation in $C^n \otimes C^n$. However, except the lowest dimensional case n=2, this transformation is not of the factorized form, and therefore the quantum groups defined by (6) and (4) need not be isomorphic. In the simplest nontrivial case n=3, the Hopf algebra associated to (4), denoted $\widehat{SL}_q(3)$, and its (formal) dual $\widehat{\mathcal{U}}_q(sl(3))$, were described in [15]. It was shown, that though there is a multiplicative isomorphism between $\mathcal{U}_q(sl(3))$ and $\widehat{\mathcal{U}}_q(sl(3))$, the complete Hopf algebra structures are qualitatively different, for instance the new coproduct is not cocommutative on the Cartan subalgebra.

Another difference can be seen is the classical limit. Due to the antisymmetry and Jacobi identity for (1) the matrix (2) is a skewsymmetric solution of the modified classical Yang-Baxter equation. It can be located [15] in the space of $sl(n)$ solutions classified by Belavin and Drinfeld as far opposite to the standard Drinfeld-Jimbo r-matrix

$$r_{DJ} = - \sum_{i<j} e_{ij} \wedge e_{ji} .$$

$$(7)$$

It equips $Sl(n)$ with a new Poisson-Lie group structure.

Now, restricting to n=3, I will describe the Hopf algebra $\widehat{SL}_q(3)$ in more detail. It is generated by the unit and nine generators, arranged in a matrix $T = t_{ij}$; $i,j = 1,2,3$. The coproduct and the counit are universally given by

$$\Delta(t_{ij}) = t_{ik} \otimes t_{kj} \quad \text{and} \quad \varepsilon(t_{ij}) = \delta_{ij} .$$

$$(8)$$

The relations (commutation rules) can be written in the matrix form as

$$R (T \otimes I)(I \otimes T) = (I \otimes T)(T \otimes I) R .$$

$$(9)$$

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The appropriate deformation of the classical determinant turns out to be

$$\widehat{\det}_q T = t_{11}t_{22}t_{33} - \epsilon^2 t_{11}t_{23}t_{32} - \epsilon^2 t_{12}t_{21}t_{33} + \epsilon^6 t_{12}t_{23}t_{31} + \epsilon^6 t_{13}t_{21}t_{32} - \epsilon^8 t_{13}t_{22}t_{31} ,$$
(10)

which can be compared with the standard one

$$\det_q T = t_{11}t_{22}t_{33} - q t_{11}t_{23}t_{32} - q t_{12}t_{21}t_{33} + q^2 t_{12}t_{23}t_{31} + q^2 t_{13}t_{21}t_{32} - q^3 t_{13}t_{22}t_{31} , \quad (11)$$

Here and below we use the notation $\epsilon = q^{1/3}$. Like $\det_q$, $\widehat{\det}_q$ belongs to the center and has the multiplicative property

$$\widehat{\det}_q(T \cdot T') = \widehat{\det}_q(T)\widehat{\det}_q(T') , \tag{12}$$

for any two matrices T and T' containing two commuting copies of generators. One additional constraint $\widehat{\det}_q(T) = 1$, consistent with (9), allows one to compute the antipode S (q-deformation of the inverse) from the equation $S(T_{ij}) \cdot T_{jk} = \delta_{jk}$, as

$$S(T) = \begin{pmatrix} t_{22}t_{33} - \epsilon^{-2}t_{23}t_{32} & -\epsilon^{-2}t_{12}t_{33} + t_{13}t_{32} & \epsilon^{-6}t_{12}t_{23} - \epsilon^{-4}t_{13}t_{22} \\ -\epsilon^2 t_{21}t_{33} + \epsilon^6 t_{23}t_{31} & t_{11}t_{33} - \epsilon^4 t_{13}t_{31} & -\epsilon^{-4}t_{11}t_{23} + t_{13}t_{21} \\ \epsilon^6 t_{21}t_{32} - \epsilon^8 t_{22}t_{31} & -\epsilon^4 t_{11}t_{32} + \epsilon^6 t_{12}t_{31} & t_{11}t_{22} - \epsilon^2 t_{12}t_{21} \end{pmatrix} .$$
(13)

It would be interesting to know if the two quantum groups $\widehat{SL}_q(3)$ and $SL_q(3)$ are equivalent in some larger framework, eg. the quasi-Hopf algebras, or if they are physically equivalent. In fact they have one importrant common property: both (4) and (6) satisfy the Hecke condition

$$R^2 = (q - \frac{1}{q})R + 1 . \tag{14}$$

There are few consequences of this property. First the branching of the multiple tensor products of the defining representation is expected to be very similar for $\mathcal{U}_q(sl(n))$ and for $\widehat{\mathcal{U}}_q(sl(n))$ because of the duality relation with the Hecke algebra. Next, just as the standard R-matrix (6), (4) can be 'Yang-Baxterized', ie. a spectral parameter can be introduced. Finally, note that (forgetting the coordinate dependence) the relations (1) can be viewed as describing (n copies of) 'Poisson-hyperplane', which is a classical version of a 'quantum plane' (cf. [3]) defined by the relations (3). It is possible to introduce the q-calculus [16] on this new quantum plane in the spirit of [17], which is generated by the coordinates x^i, one-forms dx^i $(dx^i dx^i = 0)$, and vectors ∂_i. The exterior derivative, $d = dx^i \partial_i$, satisfies $d^2 = 0$. The commutation rules read

$$x^1 x^2 = \epsilon^4 x^2 x^1, \quad x^1 x^3 = \epsilon^2 x^3 x^1 + (\epsilon^4 - \epsilon^{-2})x^2 x^2, \quad x^2 x^3 = \epsilon^4 x^3 x^2, \tag{15}$$

$$dx^1 dx^2 = -\epsilon^{-2} dx^2 dx^1, \quad dx^1 dx^3 = -\epsilon^{-4} dx^3 dx^1, \quad dx^2 dx^3 = -\epsilon^{-2} dx^3 dx^2; \tag{16}$$

$$\partial_1 \partial_2 = \epsilon^{-2} \partial_2 \partial_1, \quad \partial_1 \partial_3 = \epsilon^{-4} \partial_3 \partial_1, \quad \partial_2 \partial_3 = \epsilon^{-2} \partial_3 \partial_2. \tag{17}$$

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Moreover, there are also nontrivial relations between tensors of different type

$$x^1 dx^1 = \epsilon^6 dx^1 x^1, \quad x^1 dx^2 = \epsilon^4 dx^2 x^1 + (\epsilon^6 - 1)dx^1 x^2,$$
$$x^1 dx^3 = \epsilon^2 dx^3 x^1 + (\epsilon^6 - 1)dx^1 x^3 + (\epsilon^4 - \epsilon^{-2})dx^2 x^2,$$
$$x^2 dx^1 = \epsilon^2 dx^1 x^2, \quad x^2 dx^2 = \epsilon^6 dx^2 x^2, \quad x^2 dx^3 = \epsilon^4 dx^3 x^2 + (\epsilon^6 - 1)dx^2 x^3,$$
$$x^3 dx^1 = \epsilon^4 dx^1 x^3 + (\epsilon^8 - \epsilon^2)dx^3 x^1, \quad x^3 dx^2 = \epsilon^2 dx^2 x^3, \quad x^3 dx^3 = \epsilon^6 dx^3 x^3; \tag{18}$$

$$\partial_1 x^1 = 1 + \epsilon^6 x^1 \partial_1 + (\epsilon^6 - 1)x^2 \partial_2 + (\epsilon^6 - 1)x^3 \partial_3, \quad \partial_1 x^2 = \epsilon^2 x^2 \partial_1,$$
$$\partial_1 x^3 = \epsilon^4 x^3 \partial_1, \quad \partial_2 x^1 = \epsilon^4 x^1 \partial_2 + (\epsilon^4 - \epsilon^{-2})x^2 \partial_3,$$
$$\partial_2 x^2 = 1 + \epsilon^6 x^2 \partial_2 + (\epsilon^6 - 1)x^3 \partial_3, \quad \partial_2 x^3 = \epsilon^2 x^3 \partial_2 - (\epsilon^8 - \epsilon^2)x^2 \partial_1,$$
$$\partial_3 x^1 = \epsilon^2 x^1 \partial_3, \quad \partial_3 x^2 = \epsilon^4 x^2 \partial_3, \quad \partial_3 x^3 = 1 + \epsilon^6 x^3 \partial_3; \tag{19}$$

$$\partial_1 dx^1 = \epsilon^{-6} dx^1 \partial_1, \quad \partial_1 dx^2 = \epsilon^{-4} dx^2 \partial_1, \quad \partial_1 dx^3 = \epsilon^{-2} dx^3 \partial_1,$$
$$\partial_2 dx^1 = \epsilon^{-2} dx^1 \partial_2 + (\epsilon^{-2} - \epsilon^{-8})dx^2 \partial_3, \quad \partial_2 dx^2 = \epsilon^{-6} dx^2 \partial_2 - (1 - \epsilon^{-6})dx^1 \partial_1,$$
$$\partial_2 dx^3 = \epsilon^{-4} dx^3 \partial_2 - (\epsilon^2 - \epsilon^{-4})dx^2 \partial_1, \quad \partial_3 dx^1 = \epsilon^{-4} dx^1 \partial_3,$$
$$\partial_3 dx^2 = \epsilon^{-2} dx^2 \partial_3, \quad \partial_3 dx^3 = \epsilon^{-6} dx^3 \partial_3 - (1 - \epsilon^{-6})dx^1 \partial_1 - (1 - \epsilon^{-6})dx^2 \partial_2. \tag{20}$$

For comparison, the calculus on the standard 3-plane is:

$$x^1 x^2 = qx^2 x^1, \quad x^1 x^3 = qx^3 x^1, \quad x^2 x^3 = qx^3 x^2, \tag{21}$$

$$dx^1 dx^2 = -q^{-1} dx^2 dx^1, \quad dx^1 dx^3 = -q^{-1} dx^3 dx^1, \quad dx^2 dx^3 = -q^{-1} dx^3 dx^2, \tag{22}$$

$$\partial_1 \partial_2 = q^{-1} \partial_2 \partial_1, \quad \partial_1 \partial_3 = q^{-1} \partial_3 \partial_1, \quad \partial_2 \partial_3 = q^{-1} \partial_3 \partial_2. \tag{23}$$

The relations between different tensors read

$$x^1 dx^1 = q^2 dx^1 x^1, \quad x^1 dx^2 = q dx^2 x^1 + (q^2 - 1)dx^1 x^2,$$
$$x^1 dx^3 = q dx^3 x^1 + (q^2 - 1)dx^1 x^3,$$
$$x^2 dx^1 = q dx^1 x^2, \quad x^2 dx^2 = q^2 dx^2 x^2, \quad x^2 dx^3 = q^2 dx^3 x^2 + (q^2 - 1)dx^2 x^3,$$
$$x^3 dx^1 = q dx^1 x^3, \quad x^3 dx^2 = q dx^2 x^3, \quad x^3 dx^3 = q^2 dx^3 x^3; \tag{24}$$

$$\partial_1 x^1 = 1 + q^2 x^1 \partial_1 + (q^2 - 1)x^2 \partial_2 + (q^2 - 1)x^3 \partial_3, \quad \partial_1 x^2 = q x^2 \partial_1,$$
$$\partial_1 x^3 = q x^3 \partial_1, \quad \partial_2 x^1 = q x^1 \partial_2,$$
$$\partial_2 x^2 = 1 + q^2 x^2 \partial_2 + (q^2 - 1)x^3 \partial_3, \quad \partial_2 x^3 = q x^3 \partial_2,$$
$$\partial_3 x^1 = q x^1 \partial_3, \quad \partial_3 x^2 = q x^2 \partial_3, \quad \partial_3 x^3 = 1 + q^2 x^3 \partial_3; \tag{25}$$

$$\partial_1 dx^1 = q^{-2} dx^1 \partial_1, \quad \partial_1 dx^2 = q^{-1} dx^2 \partial_1, \quad \partial_1 dx^3 = q^{-1} dx^3 \partial_1,$$
$$\partial_2 dx^1 = q^{-1} dx^1 \partial_2, \quad \partial_2 dx^2 = q^{-2} dx^2 \partial_2 - (1 - q^{-2})dx^1 \partial_1,$$
$$\partial_2 dx^3 = q^{-1} dx^3 \partial_2, \quad \partial_3 dx^1 = q^{-1} dx^1 \partial_3, \quad \partial_3 dx^2 = q^{-1} dx^2 \partial_3,$$
$$\partial_3 dx^3 = q^{-2} dx^3 \partial_3 - (1 - q^{-2})dx^1 \partial_1 - (1 - q^{-2})dx^2 \partial_2. \tag{26}$$

It can be checked that it is precisely the Hecke condition (14) together with the Yang-Baxter equation (5), which guarantee the consistency of these calculi. Denote by $pol(x^i)$ a generic polynomial in x^i. Then, ∂_i applied to both sides of (15, 21), and (the ∂-linear part of) $pol(x^i)$ times (17, 23) require the condition (14), while (15, 21) times $pol(x^i)$, ∂_i applied to (18, 24) and the (∂-nonlinear part of) $pol(x^i)$ times (17, 23) require the equation (5) for consistency

The extension of these algebraic structures to $n > 3$, the quantization of all other Belavin-Drinfeld classical r-matrices, and the study of their physical relevance is currently under investigation.

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