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QUANTUM GROUPS AND FUSION RULES MULTIPLICITIES

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ABSTRACT

We perform explicitly a truncation of the tensor product of two regular representations of $\mathcal{U}_q(g)$ for q a root of unity and show that it coincides with the fusion rules for integrable representations in a WZW theory based on the affine Lie algebra $g^{(1)}$. We obtain a new formula for the multiplicities which may be useful in practical calculations and which generalizes a classical formula of Weyl.

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1. Introduction.

The chiral fusion rules [1] are one of the main characteristics of any rational conformal field theory (RCFT). These theories are described by finite sets of representations of a chiral algebra for any value of its central charge. The fusion rules (FR) provide certain information about the operator product expansions (OPE) of the primary fields generating the representations, namely, they determine which primary fields appear in the OPE and with what multiplicity. Since the expansions of $\varphi_1 \varphi_2$ and $\varphi_2 \varphi_1$ are isomorphic one arrives at the notion of an abstract associative, commutative algebra with integer structure constants [2]. For a WZW theory, associated with an affine Kac - Moody algebra $g^{(1)}$ [3], the fusion rules for integrable representations of level k [4] can be loosely described as a restricted tensor product

$$T_\alpha \overset{F}{\otimes} T_\beta = \bigoplus_{\Lambda} N_{\alpha\beta}^{\Lambda} T_\Lambda \tag{1.1}$$

of finite dimensional representations T_α, T_β of g with regular weights α, β, i.e., $2(\lambda,\theta)/(\theta,\theta) \leq k$, for θ - the highest root, such that: i) only regular weights Λ survive in the decomposition, ii) $0 \leq N_{\alpha\beta}^{\Lambda} \leq m_{\alpha\beta}^{\Lambda}$, where $m_{\alpha\beta}^{\Lambda}$ is the ordinary multiplicity of the representation T_Λ in the g-invariant product $T_\alpha \otimes T_\beta$. The fusion rules arise from the requirement of integrability in a way which is analogous in nature to the procedure by which finite dimensional irreducible representations of semisimple algebras are realized as subfactors of reducible Verma modules. In [5] the integrability condition was systematically analyzed and transformed into a set of algebraic equations for the $Vir \oplus g^{(1)}$-invariant n-point functions. In particular, the equation for the 3-point function - the "depth rule" of [5], in the case $g = A_1^{(1)}$ is sufficient to describe explicitly the FR (see also [6]). In general, however, one has to analyze the restrictions on the 4-point functions, which is rather involved. An explicit formula for the multiplicities was proposed by Verlinde in [2] and proved in [7] (see also [8]). It expresses $N_{\alpha\beta}^{\Lambda}$ in terms of the modular transformation $S_{\mu\nu} : \chi_{\hat\mu}(\tau) \to \chi_{\hat\nu}(-1/\tau)$ of the affine characters [9]. Yet it would be useful for practical calculations to have alternative methods for the computation of the multiplicities. The alternative methods might also suggest a deeper understanding of the notion of FR itself. One of the motivations for the recent interest in quantum groups (or, rather, quantum universal algebras $\mathcal{U}_q(g)$) [10-13] was that they might provide a natural framework for the formulation of the rational theories FR. The connection between the representation theory of q-deformed universal algebras $\mathcal{U}_q(g)$ and the WZW theory based on the

Kac - Moody algebra $g^{(1)}$ of level k and dual Coxeter number h arises when the deformation parameter q is a root of unity , $q^p = 1$ with $p = k + h$ #. While for generic value of q the representation theory of $\mathcal{U}_q(g)$ parallels that of the classical theory with $q = 1$ [14-17], the case $q^p = 1$ is rather degenerate [18-24].

The quantum counterparts of the highest weight finite dimensional representations of g are in general reducible, indecomposable. The set of regular representations $(\Lambda + \rho, \theta^\vee) < p$, $p = k + h$, which correspond to the integrable representations of level k of $g^{(1)}$ contains only irreducible representations. However, it is not closed under comultiplication $\Delta(a)$, $a \in \mathcal{U}_q(g)$, so that the tensor product of $\mathcal{U}_q(g)$ representations for $q^p = 1$ does not resemble the fusion rules. Indeed it repeats essentially the content of the classical g-product, but in contradistinction, it is not fully reducible, since some sets of representations get effectively glued together for $q^p = 1$ into indecomposable pieces. For $g = sl(2)$ these indecomposable representations were exhaustively described in [22]. The results of [22] imply that to make contact with the RCFT fusion rules one has to consider a truncation of the $\mathcal{U}_q(g)$ product (see also [25, 26]). It has been stressed in [20, 25] (and demonstrated on the example of multiple product $T_{\Lambda_i}^{\otimes n}$ of a fundamental representation Λ_i) that the notion of a restricted $\mathcal{U}_q(g)$ product is important if one wants to interpret the results on the irreducible representations of factors of Hecke algebras [27, 28] in the spirit of the classical Weyl reciprocity property.

In the present paper we generalize the results in [22, 25, 26] and define explicitly a truncated tensor product of two arbitrary regular representations of $\mathcal{U}_q(g)$. We show that it reproduces the Verlinde formula and hence it is equivalent to the fusion rules of the WZW theory associated with $g^{(1)}$. Meanwhile we get an alternative formula for the multiplicities which admits various interpretations. The formula expresses the multiplicity $N_{\alpha\beta}^{\lambda}$ for α, β, λ-regular, as a finite sum , i.e.

$$N_{\alpha\beta}^{\lambda} = \sum_{\lambda' \in \Omega_\lambda} \epsilon_{\lambda'}\, m_{\alpha\beta}^{\lambda'} \tag{1.2}$$

where $m_{\alpha\beta}^{\lambda'}$ is the ordinary multiplicity of the highest weight λ' entering the g- Kronecker product of $\alpha \otimes \beta$, while $\epsilon_{\lambda'} = \pm 1$. The set Ω_λ is described in the text. The same formula can

#For simplicity we indicate here one parameter q, as in the case of deformations of Lie algebras of A-D-E type.

be interpreted as a kind of a generalized Weyl formula [29] (see also [30])

$$N_{\alpha\beta}^{\lambda} = \sum_{\underline{w}\in\hat{W}} \det \underline{w}\; n^{(\alpha)}_{\underline{w}(\lambda+\rho)-\beta-\rho} \tag{1.3}$$

where the ordinary Weyl group of g is replaced by the affine Weyl group $\hat{W}$ of $g^{(1)}$ # (see [9]). In (1.3) $n_{\mu}^{(\alpha)}$ is the multiplicity of the weight μ in the representation labelled by α and the sum is actually finite due to vanishing terms. Both (1.2) and (1.3) reduce the computation of the FR multiplicities to the computation of the corresponding classical quantities. A formula for the multiplicity of a regular weight in the restricted multiple tensor product of a fundamental representation of $sl(N)$, similar in spirit to our formula, has been conjectured in [22].

The paper is organized as follows. In the next section, which is to a great extent a review of [15-24], we study the reducibility properties of the representations of $\mathcal{U}_q(g)$ for q a root of unity and we determine the set Ω_λ entering (1.2). In section 3 we discuss the tensor product of two regular representations of $\mathcal{U}_q(g)$ and in the next section 4 we show how it can be explicitly truncated. The resulting reduced multiplicities given by (1.2,3) are shown to coincide with the Verlinde fusion multiplicities. In Appendix A we illustrate formulae (1.2,3) on some examples for $g = sl(3)$. In Appendix B we recall some known facts [31-33] about the implications of the group of automorphisms of the extended Dynkin diagram of $g^{(1)}$ for the FR and conjecture that for $g = sl(N)$ they imply another explicit formula for the multiplicities, useful for practical computations.

2. Structure of the (reducible) representations of $U_q(\mathcal{G})$ for q a root of unity.

Given a simple Lie algebra g the q-deformed universal enveloping algebra $\mathcal{U}_q(g)$ is defined [12], [13] as the associative algebra generated by $X_i^{\pm}, H_i$, $i = 1,2,\ldots,l = \operatorname{rank} g$, and the identity, with relations

$$\begin{aligned}
&[H_i, H_j] = 0, \quad [H_i, X_j^{\pm}] = \pm c_{ij} X_j^{\pm}, \qquad c_{ij} = 2(\alpha_i,\alpha_j)/(\alpha_j,\alpha_j),\\
&[X_i^+, X_j^-] = \delta_{ij}[H_i]_{q_i}, \qquad q_i = q^{(\alpha_i,\alpha_i)/2},\\
&\sum_{k=0}^{1-c_{ij}} (-1)^k \begin{bmatrix} 1-c_{ij} \\ k \end{bmatrix}_{q_i} (X_i^{\pm})^k X_j^{\pm} (X_i^{\pm})^{1-c_{ij}-k} = 0, \quad i \neq j,\\
&[a]_q = \frac{q^{a/2} - q^{-a/2}}{q^{1/2} - q^{-1/2}}, \qquad \begin{bmatrix} a \\ b \end{bmatrix}_q = \frac{[a]_q!}{[b]_q![a-b]_q!}.
\end{aligned} \tag{2.1}$$

#Here $w(\lambda)$ is a shorthand notation for $w(\hat{\lambda})$ projected to its g-weight, see (2.9) below.

The scalar product $(\cdot,\cdot)$ is normalized so that $(\alpha,\alpha)=2$ for the long simple roots α.
The Chevalley generators $H_i, X_i^{\pm}$ correspond to the simple roots α_i of $\mathcal{U}_q(g)$ and we shall use the standard decomposition $g = g^- \oplus \mathcal{H} \oplus g^+$, where $\mathcal{H}$ is the Cartan subalgebra of g, spanned by H_i. The comultiplication $\Delta : \mathcal{U}_q(g) \to \mathcal{U}_q(g) \otimes \mathcal{U}_q(g)$ is defined by

$$
\begin{aligned}
&\Delta(H_i) = H_i \otimes \mathbf{1} + \mathbf{1} \otimes H_i, \quad \Delta(\mathbf{1}) = \mathbf{1} \otimes \mathbf{1} \\
&\Delta(X_i^{\pm}) = X_i^{\pm} \otimes q_i^{H_i/4} + q_i^{-H_i/4} \otimes X_i^{\pm} \ .
\end{aligned}
\tag{2.2}
$$

The highest weight module V_Λ induced by $\mathcal{U}_q(g^-)$, i.e.,

$$V_\Lambda = \mathcal{U}_q(g^-) \otimes v_\Lambda, \quad X_i^+ v_\Lambda = 0, \quad H_i v_\Lambda = \Lambda(H_i) v_\Lambda, \qquad i = 1,\ldots,l, \tag{2.3}$$

is a quantum counterpart of a Verma module of g. We shall call it a $\mathcal{U}_q(g)$-Verma module.

For generic values of the deformation parameter q the Verma modules V_Λ have the same structure as in the classical case, i.e., for $q=1$. In particular, for Λ integer, dominant, i.e.,

$$(\Lambda, \alpha_i^\vee) \in \mathbb{Z}_+, \qquad \text{for all } \alpha_i \in \Delta^o, \tag{2.4}$$

where Δ^o is the set of simple roots of g, $\alpha_i^\vee = 2\alpha_i/(\alpha_i,\alpha_i)$, the modules V_Λ are highly reducible. The submodules of V_Λ are described by their highest weights (=singular vectors) which are parametrized by the Weyl group W of g, namely, these are all Verma modules with highest weights in the set

$$\{w(\Lambda) = \underline{w}(\Lambda+\rho) - \rho, \ \underline{w} \in W\}, \quad \rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha, \tag{2.5a}$$

i.e., weights in the "shifted" Weyl orbit W_Λ of Λ. As in the classical case there is a unique maximal submodule I_Λ of V_Λ

$$I_\Lambda = \mathrm{span}\left\{ \bigcup_{i=1}^{l} V_{w_i(\Lambda)} \right\} = \mathrm{span}\left\{ \bigcup_{\underline{w} \in W} V_{w(\Lambda)} \right\}, \tag{2.5b}$$

where w_i, $i=1,\ldots,l$, are the generators of W corresponding to reflections by the simple roots α_i, i.e., $\underline{w}_i(\lambda) = \lambda - (\lambda, \alpha_i^\vee)\alpha_i$. The factor module $E_\Lambda = V_\Lambda / I_\Lambda$ can be described also as

$$E_\Lambda \cong U_q(g^-) \otimes v_\Lambda, \qquad X_i^+ v_\Lambda = 0 = (X_i^-)^{(\Lambda+\rho,\alpha_i^\vee)} v_\Lambda, \quad i = 1,\ldots,l. \tag{2.6}$$

The representation T_Λ acting in E_Λ is finite dimensional, irreducible. These representations have the same (classical) dimension d_Λ as their classical counterparts and their characters are given by the classical formulae of Weyl [29]

$$\chi_\Lambda = \sum_\mu n_\mu e_\mu = \frac{\sum_{\underline{w} \in W} \det w \quad e_{w(\Lambda)}}{\sum_{\underline{w} \in W} \det w \quad e_{w(0)}} \tag{2.7}$$

Here n_μ is the multiplicity of the weight μ in the representation T_Λ; e_μ is a formal exponent, $e_{\mu+\nu} = e_\mu e_\nu$, $e_0 = 1$; $\det w = \det \underline{w} = (-1)^{l(\underline{w})}$, where $l(\underline{w})$ is the minimal number of simple reflections necessary to write $\underline{w}$.

Let now $q^p = 1$. For the sake of simplicity we shall assume in what follows that g is a simply laced (A-D-E type) algebra. (In general p is such that $q_i^p = 1$ where q_i corresponds to a short simple root α_i. Because most formulas below are also valid in the general case we continue to use the notation $\alpha^\vee$, though $\alpha^\vee = \alpha$ in a simply laced algebra.) The Verma modules V_Λ become reducible for any Λ and their structure gets more complicated (see [23] for a detailed analysis). What is important for us is that now there appear infinitely many submodules of V_Λ and further V_Λ itself is imbedded in an infinite set of Verma modules. For Λ regular (i.e., Λ dominant, cf. (2.4), and $(\Lambda+\rho, \theta^\vee) < p$) this infinite collection of Verma modules $\mathcal{M}_\Lambda^{(p)}$ (a multiplet in the language of [23]) is parametrized by the set of weights (see also [22])

$$\{(t_\beta w)(\Lambda) = \underline{w}(\Lambda+\rho) - \rho + p\beta, \quad \underline{w} \in W, \quad \beta = \sum_{i=1}^{l} k_i \alpha_i^\vee, \quad k_i \in \mathbb{Z}\} \tag{2.8}$$

The set in (2.8) can be looked upon as an orbit of an infinite group generated by $\{\omega_\theta = t_\theta w_\theta, \quad w_i, i = 1 \dots, l\}$, where $\underline{w}_i, \underline{w}_\theta$ are the reflections corresponding to the simple roots α_i and the highest root θ respectively and $t_\beta(\Lambda) = \Lambda + p\beta$, $\beta = \sum k_i \alpha_i^\vee$, $k_i \in \mathbb{Z}$. We shall similarly use the notation $\omega_\alpha = t_\alpha w_\alpha$ for $\alpha \in \Delta$. This group has the structure of a semidirect product $T^\vee \times W$ of the Weyl group W and the translation group $T^\vee$ (isomorphic to the dual root lattice). Hence this group is isomorphic to the affine Weyl group $\hat{W}$ of the untwisted Kac-Moody algebra $g^{(1)}$ (see [9]). Let $\hat{\Lambda} = k\bar{c} + \Lambda$, $(\hat{\Lambda}, \alpha_i) = (\Lambda, \alpha_i)$, $i = 1, \dots, l$. Here $\bar{c}$ is a linear functional on the Cartan subalgebra of $g^{(1)}$, $\quad (\bar{c}, \alpha_\mu^\vee) = \delta_{\mu 0}$, $\mu = 0, 1, \dots, l$ and $\alpha_\mu, \mu = 0, 1, \dots l$, are the simple roots of $g^{(1)}$; $\quad (\Lambda, \alpha_0) = -(\Lambda, \theta)$. The action of $\hat{W}$ on the projected weights Λ is determined according to

$$(\underline{w}(\Lambda), \alpha_i) := (\underline{w}(\hat{\Lambda}), \alpha_i), \quad i = 1, 2, \ldots, l, \quad \underline{w} \in \hat{W} \tag{2.9}$$

and for $p = k + h$, h-dual Coxeter number of g, we can identify ω_θ with w_0, where $w_0(\hat{\Lambda}) = \underline{w}_0(\hat{\Lambda} + \hat{\rho}) - \hat{\rho} = \hat{\Lambda} - (\hat{\Lambda} + \hat{\rho}, \alpha_0^\vee)\alpha_0$; $\hat{\rho} = h\bar{c} + \rho$ [9]. The isomorphism of the infinite group parametrizing $\mathcal{M}_\Lambda^{(p)}$ and the affine Weyl group $\hat{W}$ reflects the fact that there is a one to one correspondence between the points (but not the directions of the embedding arrows) of the $\mathcal{U}_q(g)$ multiplet of Verma modules $\mathcal{M}_\Lambda^{(p)}$ for Λ-regular and the multiplet $\hat{\mathcal{M}}_{\hat{\Lambda}}$ of integrable, level k, $g^{(1)}$-Verma modules with $\hat{\Lambda} = k\bar{c} + \Lambda$ (see [34] for a classification of the $A_l^{(1)}$ multiplets).

Let us illustrate the general case by the example $g = A_1$. The multiplet $\mathcal{M}_\Lambda^{(p)}$ of Verma modules parametrized by the shifted $\hat{W}$-orbit of $\Lambda = r\frac{\alpha}{2}$, $0 \le r \le k = p - 2$ is represented by the graph in Fig. 1. The graph can be identified also with the weight lattice filling in all integer points. The finite dimensional representation T_Λ in $E_\Lambda = V_\Lambda / V_{w(\Lambda)}$, $w(r) = -r - 2$, is irreducible and has dimension $r + 1$. On the other hand, the module $E_{\Lambda'} = V_{\Lambda'} / V_{w(\Lambda')}$, labelled by the nearest neighbour dominant weight $\Lambda' = \omega(\Lambda)$, $r' = 2p - r - 2$, is reducible, indecomposable. We could get an irreducible representation by factoring $V_{\Lambda'}$ by the highest submodule V_Λ, generated by the singular vector $v_\Lambda = X_-^{p-r-1} v_{\Lambda'}$.

The pair of representations T_Λ, $T_{\omega(\Lambda)}$ labelled by the subset $\Omega^{(2)} = \{\mathbf{1}, \omega\}$ of $\hat{W}$, played a crucial role in a paper by two of us [26]. In a functional realization of the representations of $\mathcal{U}_q(sl(2))$ as in [26] the singular vector $X_-^{p-r-1} v_{\Lambda'}$ gives rise to an invariant operator $\underline{\mathcal{D}}$ intertwing the representations $T_{\omega(\Lambda)}$ and T_Λ. These representations are partially equivalent since $Im\underline{\mathcal{D}} \simeq \mathcal{E}_\Lambda$ ($\mathcal{E}_\Lambda$ is the representation space of T_Λ, realized in [26] as the space of polynomials of degree $\le r$) and $Ker\underline{\mathcal{D}} \ne 0$. Thus in this realization the representation T_Λ is isomorphic to a factor of $T_{\omega(\Lambda)}$. The partial equivalence of T_Λ and $T_{\omega(\Lambda)}$ implies a set of relations for the Clebsch - Gordan kernels and the $6j$ - symbols, which was used to show that all polynomial identities for general $q - 6j$-symbols are consistent for $q^p = 1$ with the truncated $\mathcal{U}_q(sl(2))$ product, equivalent to the $A_1^{(1)}$-fusion rules.

In general, as for $g = A_1$, we shall be interested in the set of finite dimensional representations $T_{\Lambda'}$ (Λ' satisfying (2.4)) of the same dimension $d_{\Lambda'}$ as their classical counterparts, which are recovered as factor modules of the Verma modules $V_{\Lambda'}$ over the analogs of the generic q (or $q = 1$) submodules $I_{\Lambda'}$. These submodules are again described using the (shifted) Weyl orbit $W_{\Lambda'}$ of Λ' (cf. (2.5b)), but now $I_{\Lambda'}$ are not in general the maximal Verma submodules of $V_{\Lambda'}$ and the representations $T_{\Lambda'}$ are reducible, indecomposable. It has been shown in [24] that the characters of the irreducible representations T_Λ, $(\Lambda+\rho,\theta^\vee)\leq p$ are given by the standard Weyl formulae (2.7). Following the same reasoning one can see that the characters of the reducible, indecomposable finite dimensional representations $T_{\Lambda'}$ of dimension $d_{\Lambda'}$, discussed above, are again given by both equalities in (2.7). Note that these characters can be used to reproduce both, the classical dimension $d_{\Lambda'}$ and the quantum dimension $D_{\Lambda'}$ of the representation $T_{\Lambda'}$ given by (see below)

$$D_{\Lambda'}=\prod_{\alpha>0}\frac{[(\Lambda'+\rho,\alpha)]_q}{[(\rho,\alpha)]_q}. \tag{2.10a}$$

All such representations $T_{\Lambda'}$ with weghts $\Lambda'=w(\Lambda)$ on the shifted $\hat{W}$- orbit of a regular Λ have the same q - dimension, up to a sign, since

$$D_{w(\Lambda)}=\det w\, D_\Lambda, \quad w\in\hat{W}, \tag{2.10b}$$

where $\det w=\det(w\big|_W)$. For regular weights the q-dimension is positive. Hence the weights for which the q-dimension vanishes, cannot lie on an orbit $\hat{W}_\Lambda$ (2.8) with Λ - regular. The dominant weights $\Lambda'=w(\Lambda)$, $w\in\hat{W}$, in the $\hat{W}$ -orbit of Λ, are all, but Λ itself, irregular, i.e. $p<(\Lambda'+\rho,\theta^\vee)$.

The "nearest neighbours" region determined by

$$(\Lambda'+\rho,\theta^\vee)<2p, \qquad (\Lambda',\alpha_i^\vee)\in\mathbb{Z}_+, \qquad \alpha_i\in\Delta^0 \tag{2.11}$$

contains some of the weights Λ' on an orbit $\hat{W}_\Lambda$, Λ regular, and in addition all Λ' such that $(\Lambda'+\rho,\alpha^\vee)=p$ for some $\alpha\in\Delta^+$. We shall denote by Ω_Λ the intersection of $\hat{W}_\Lambda$ with the region described in (2.11).

Let us consider the case $g=A_{N-1}$. Here $\alpha^\vee=\alpha$, $\theta=\sum_1^{N-1}\alpha_i$, $p=k+N$, $\Lambda=\sum_{i=1}^{N-1}r_i\Lambda_i$, where Λ_i are the fundamental weights $(\Lambda_i,\alpha_j^\vee)=\delta_{ij}$. For $N=2$ we have $\Omega_\Lambda^{(2)}=\{\Lambda,\omega(\Lambda)\}$ and these are the pairs discussed above, characterized by the subset $\Omega^{(2)}=\{\mathbf{1},\omega\}\subset\hat{W}$. For $N=3$ the set $\Omega_\Lambda^{(3)}$ is given by

$$\begin{aligned}\Omega_\Lambda^{(3)}&=\{(r_1,r_2),(p-r_2-2,p-r_1-2),(r_2+p,p-r_1-r_2-3),(p-r_1-r_2-3,r_1+p)\}\\&=\{\mathbf{1}.\quad \omega_\theta,\quad \omega_1\omega_\theta,\quad \omega_2\omega_\theta\}(\Lambda)=\Omega^{(3)}(\Lambda)\end{aligned} \tag{2.12}$$

where $\omega_i = t_{\alpha_i} w_{\alpha_i}$, $\quad t_\alpha \in T^\vee$, $\quad w_\alpha \in W$. On Fig. 2a the set $\Omega_\Lambda^{(3)}$ is shown together with the (shifted) orbit W_Λ of the ordinary Weyl group W of Λ and with some other points in $\hat{W}_\Lambda$; the picture represents a small part of the infinite multiplet parametrized by $\mathcal{M}_\Lambda^{(p)}$ and schematically pictured in [23]. Geometrically $\Omega^{(3)}$ can be described by the simple reflection of $\Lambda + \rho$ (see Fig. 2b) with respect to the plane $(\Lambda + \rho, \theta^\vee) = p$, and by the compositions of this with a reflection with respect to the plane $(\Lambda + \rho, \alpha_1^\vee) = p$ or $(\Lambda + \rho, \alpha_2^\vee) = p$. All these planes contain points with vanishing quantum dimension according to (2.10a). On Figs. 3,4 the sets $\Omega^{(N)}$ for $N = 4, 5, 6$ are described schematically as compositions of reflections with respect to $(\Lambda + \rho, \alpha^\vee) = p$, $\alpha \in \Delta_+$, where Δ_+ is the set of positive roots. The number of elements in $\Omega^{(N)}$ grows like 2^{N-1}.

Let us summarize. The conditions (2.11) select either weights Λ' such that $(\Lambda' + \rho, \alpha^\vee) = p$ for some $\alpha \in \Delta^+$ corresponding to representations $T_{\Lambda'}$ with $D_{\Lambda'} = 0$, or subsets $\Omega_\Lambda \subset \hat{W}_\Lambda$ of the shifted $\hat{W}$-orbits of regular weights Λ. The corresponding finite dimensional representations $T_{\Lambda'}$ with the same classical dimension as their classical (or generic q) counterparts are in general reducible, indecomposable. (Exceptions are provided by the weights Λ, $\quad (\Lambda + \rho, \theta^\vee) \leq p$, for which T_Λ are irreducible.) Their characters are computed according to the classical formulae (2.7). The detailed structure of the reducible representations $T_{\Lambda'}, \Lambda' \in \Omega_\Lambda$, can be inferred from the structure of the corresponding Verma modules $V_{\Lambda'}$ (see [23]). The representations $T_{\Lambda'}$ in Ω_Λ, are partially equivalent in the sense described above. In a functional realization, generalizing that in [26], the partial equivalence relations will be carried over by some intertwining operators, which correspond to singular vectors of the Verma modules.

3. $\mathcal{U}_q(g)$-invariant tensor product for $q^p = 1$.

Let E_α, E_β be the representation spaces of the representations T_α, T_β. The tensor product representation of $\mathcal{U}_q(g)$ in $E_\alpha \otimes E_\beta$ is realized by $(T_\alpha \otimes T_\beta)(\Delta(a))$, $a \in \mathcal{U}_q(g)$ where $\Delta : \mathcal{U}_q(g) \to \mathcal{U}_q(g) \otimes \mathcal{U}_q(g)$ is the comultiplication defined in (2.2). We are interested in the content of the product $T_\alpha \otimes T_\beta$ (identical with the content of $T_\beta \otimes T_\alpha$) when α and β are regular. For $q^p = 1$ the module $E_\alpha \otimes E_\beta$ contains the same states as in the case of generic q, or $q = 1$, and in particular it splits into subspaces of states of weight μ (with respect to the Cartan subalgebra $\mathcal{H}$) with the same dimension d_μ as for $q = 1$. However for $q^p = 1$ the module $E_\alpha \otimes E_\beta$ cannot be split in general into invariant subspaces generated by highest weight states.

satisfying (2.6). Thus, for example, in the case of $\mathcal{U}_q(sl(2))$, the representations with highest weight $\Lambda = (r)$ regular and $\Lambda' = \omega(\Lambda) = (2p - r - 2)$ pair up [22] in an indecomposable representation $T_{\Omega_\Lambda^{(2)}}$ with $\Omega_\Lambda^{(2)} = \{\Lambda, \omega(\Lambda)\}$, any time both Λ and Λ' appear in the interval $[|\alpha - \beta|, \alpha + \beta]$. The irreducible module E_Λ is recovered as a factor of $E_{\Omega_\Lambda^{(2)}}$ over a submodule isomorphic to the reducible, indecomposable $E_{\omega(\Lambda)}$, realized in the previous section, so that $T_{\Omega_\Lambda^{(2)}}$ has classical dimension $d_\Lambda + d_{\omega(\Lambda)}$. This is depicted by the exact sequence

$$0 \to E_{\omega(\Lambda)} \to E_{\{\Lambda,\omega(\Lambda)\}} \to E_\Lambda \to 0 \tag{3.1}$$

According to (2.10b) the quantum dimension $D_\Lambda + D_{\omega(\Lambda)}$ of the representation $T_{\{\Lambda,\omega(\Lambda)\}}$ is zero. The pairing is due essentially to the indecomposability for $q^p = 1$ of the representation $T_{\Lambda'}$, $\Lambda' = \omega(\Lambda)$ of dimension $d_{\Lambda'} = (\Lambda' + \rho, \alpha)$ described in the previous section as a factor of Verma modules. In the concrete realization of the module $E_{\Lambda'}$, which appears in the product $E_\alpha \otimes E_\beta$, the singular vector $X_-^p v_{\Lambda'}$ vanishes identically# , while the other singular vector $X_-^{p-r-1} v_{\Lambda'}$ coincides with the would be vacuum v_Λ of E_Λ. To retain both states one has to give up the conditions $X_+ v_\Lambda = 0 = X_-^{r+1} v_\Lambda$ (cp. with (2.6)) selecting the irreducible highest weight module E_Λ; this is sufficient, since the state $X_-^{r+1} v_\Lambda$ has the same weight as $X_-^p v_{\Lambda'}$. If $\Lambda \leq \min(\alpha + \beta, 2p - 4 - \alpha - \beta)$, then $\omega_\theta(\Lambda) > \alpha + \beta$, i.e., this weight cannot appear in the product $T_\alpha \otimes T_\beta$ and only the irreducible representation T_Λ contributes to the fully reducible part of the decomposition, which furthermore can include an irregular irreducible representation with $\Lambda = \omega(\Lambda)$ (i.e., $r = p - 1$).

In general a necessary condition that a dominant weight Λ' appears in the product $\alpha \otimes \beta$, α, β - regular is provided by the inequality

$$(\Lambda' + \rho, \theta^\vee) \leq (\alpha + \beta + \rho, \theta^\vee) \leq 2p - N \ . \tag{3.2}$$

Comparing (3.2) with (2.11) we see that only some of the weights Λ' in Ω_Λ, Λ regular, can eventually appear in the product, along with the weights Λ', $(\Lambda' + \rho, \alpha^\vee) = p$, $\alpha \in \Delta_+$, yielding $D_{\Lambda'} = 0$. The structure of the indecomposable pieces, labelled by subsets of Ω_Λ, gets much more complicated. The reason is that the structure of the representations $T_{\Lambda'}$,

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#This does not hold with necessity for the corresponding vector in the isomorphic module $V_{\Lambda'}/V_{w(\Lambda')}$, discussed in Sect. 2.

is more complicated, due to abundance of singular vectors (see [23] for a detailed description in terms of the corresponding Verma modules).

Let us give an example. The set $\Omega_\Lambda^{(4)}$ for $g = A_3$ is depicted on Fig. 3. Let us introduce the notation $\mu_0, \mu_1, \mu_2, \mu_3$ for the weights sitting at the outmost vertices and $\mu'_0, \mu'_1, \mu'_2, \mu'_3$ for the weights sitting at the corners of the square. We have that $\mu'_0 = \omega_\theta \mu_0$, $\mu'_1 = \omega_1 \mu_1 = \omega_{12} \mu'_0$, etc. As in the case of $sl(2)$ we have pairing of two representations connected by an affine reflection ω_θ, or ω_i, $i = 1,2,3$, or ω_{ij}, $(ij) = (12),(23)$, i.e., representations T_{μ_0,μ'_0}, T_{μ_i,μ'_i}, $T_{\mu'_0,\mu'_1}$, $T_{\mu'_0,\mu'_3}$ of q-dimension zero. Besides this we can have an indecomposable representation (again of q-dimension zero) in the space $E_{\mu'_0,\mu'_1,\mu'_2,\mu'_3}$ described as follows

$$\begin{aligned} &0 \to E_{\mu'_2} \to E_{\mu'_1,\mu'_2,\mu'_3} \to E_{\mu'_1} \otimes E_{\mu'_3} \to 0 \\ &0 \to E_{\mu'_1,\mu'_2,\mu'_3} \to E_{\mu'_0,\mu'_1,\mu'_2,\mu'_3} \to E_{\mu'_0} \to 0 \ . \end{aligned} \tag{3.3}$$

In general for $g = sl(N)$ there are various ways of getting indecomposable representations out of $2,4,8,\ldots$, etc., representations labelled by subsets of $\Omega_\Lambda^{(N)}$.

The idea implicitly present in [22] is that one can define a truncated product keeping only the irreducible regular representations in $\alpha \otimes \beta$ (type II representations in the terminology of [22]) and throwing away as in the case of $g = A_1$ all indecomposable pieces along with the irreducible, but irregular T_Λ, such that $(\Lambda + \rho, \theta^\vee) = p$ (type I representations). It is expected that all indecomposable parts have zero q-dimension . However, the quantum dimension itself is not sufficient to describe which representations exactly have to be thrown away since there are in general different representations with the same q-dimension.

As we shall see, the precise structure of the indecomposable representations is fortunately irrelevant for the explicit truncation of the $\mathcal{U}_q(g)$-products. This is clear in the explicit definition of the truncated product for $g = A_1$ in [26]. There, as a first step in achieving the truncated product, the indecomposable representations $T_{\{\Lambda,\omega(\Lambda)\}}$ were effectively split into T_Λ, $T_{\omega(\Lambda)}$, using an analytic continuation procedure. Then the partial equivalence of T_Λ and $T_{\omega(\Lambda)}$ was used to show that their contribution to the invariant functions cancels down if they both violate the FR. The same idea, although realized in a different way, will be followed here.

4. Explicit truncation of the $\mathcal{U}_q(g)$-tensor product. Formulae for the multiplicities.

We shall start with the classical (or generic q) character formula for the tensor product $T_\alpha \otimes T_\beta$

$$\chi_\alpha \chi_\beta = \sum_\Lambda m_{\alpha\beta}^\Lambda \chi_\Lambda . \tag{4.1}$$

When q is a root of unity it can be still used since the characters account just for the weights of the states in the representation space $E_\alpha \otimes E_\beta$ and their multiplicities and these states are in one to one correspondence with the states for generic q, or $q = 1$. The only difference is that now some of the characters in the r.h.s. of (4.1) (in which we assume that α and β are regular) are characters of reducible, indecomposable representations $T_{\Lambda'}$ of the same dimension as their classical counterparts. As already discussed in Sect.2, these characters are defined by the classical formulae (2.7). The multiplicities in (4.1) are identical to those for $q = 1$ and are described by the standard Weyl formula (see, e.g., [30])

$$m_{\alpha\beta}^\Lambda = \sum_{\underline{w} \in W} \det w \;\, n_{w(\Lambda)-\beta}^{(\alpha)} \tag{4.2}$$

where $n_\mu^{(\alpha)}$ is the multiplicity of the weight μ in the representation with highest weight α, and $w(\lambda) = \underline{w}(\Lambda + \rho) - \rho$ as above, $\underline{w} \in W$.

In using (4.1) we have replaced any complicated indecomposable representation, labelled by a subset of weights in Ω_Λ, by the representations $\{T_{\Lambda'}\}$ described by the same subset. For example the character (given by the first formula in (2.7)) of the representation $T_{\mu_0',\mu_1',\mu_2',\mu_3'}$ described in the previous section (see (3.3)) is equal to the sum of characters $\chi_{\mu_0'} + \chi_{\mu_1'} + \chi_{\mu_2'} + \chi_{\mu_3'}$, each of which is defined according to both equalities in (2.7).

Our next step will be to reduce the product. We have to account for the partial equivalence of the representations $\{T_{\Lambda'}\}$ described by the shifted affine Weyl orbits $\hat{W}_\Lambda$. To do that we impose a constraint on the formal exponents e_Λ

$$e_\Lambda = e_{\Lambda + p\beta}, \qquad \beta = \sum_1^l k_i \alpha_i^\vee, \quad k_i \in \mathbb{Z}, \quad \text{i.e.,} \quad e_\Lambda = e_{T^\vee(\Lambda)} . \tag{4.3}$$

We shall denote the exponents satisfying (4.3) and the corresponding characters by $\overline{e}_\Lambda$ and $\overline{\chi}_\Lambda$. The constraint reads in terms of the characters $\overline{\chi}_\Lambda$

$$\overline{\chi}_{w(\Lambda)} = \det w \overline{\chi}_\Lambda , \quad w \in W . \tag{4.4}$$

The last equality says that a standard symmetry of the characters with respect to the Weyl group W is extended to $\hat{W}$, since (4.3) implies that $\overline{\chi}_\Lambda = \overline{\chi}_{T^\vee(\Lambda)}$. Clearly $\overline{\chi}_\Lambda = 0$ for all Λ, such that $D_\Lambda = 0$, since then one can find a positive root α for which $w_\alpha(\Lambda) = t_{k\alpha}\Lambda$, while $\det w_\alpha = -1$ for all $\alpha \in \Delta_+$.

The r.h.s. of (4.1) can be split into $\hat{W}$-orbits, which,as we have seen,participate at most with the finite sets $\Omega_\Lambda \subset \hat{W}_\Lambda$. Using (4.4) the contribution of any $\Omega_\Lambda \subset \hat{W}_\Lambda$, Λ-regular, can be written as

$$\sum_{w\in\Omega} m_{\alpha\beta}^{w(\Lambda)} \overline{\chi}_{w(\Lambda)} = \left(\sum_{w\in\Omega} \det w \quad m_{\alpha\beta}^{w(\Lambda)}\right) \overline{\chi}_\Lambda \ . \tag{4.5}$$

Note that the regular weight Λ, generating Ω_Λ might not be present itself, i.e. we could have $m_{\alpha\beta}^\Lambda = 0$, and also in general only a subset of Ω_Λ survives in (4.5). We have finally

$$\overline{\chi}_\alpha \overline{\chi}_\beta = \sum_{\Lambda \text{ regular}} \overline{m}_{\alpha\beta}^\Lambda \quad \overline{\chi}_\Lambda \ , \tag{4.6}$$

$$\overline{m}_{\alpha\beta}^\Lambda = \sum_{w\in\Omega} \det w \quad m_{\alpha\beta}^{w(\Lambda)} \ , \tag{4.7}$$

i.e., we have explicitly reduced $T_\alpha \otimes T_\beta$ and the respective truncated product is described according to (4.6) by

$$T_\alpha \overset{T}{\otimes} T_\beta = \bigoplus_{\Lambda \text{ regular}} \overline{m}_{\alpha\beta}^\Lambda \, T_\Lambda \tag{4.8}$$

with $\overline{m}_{\alpha\beta}^\Lambda$ given by (4.7). The formula for the reduced multiplicity (4.7) can be also rewritten as (see (4.2))

$$\overline{m}_{\alpha\beta}^\Lambda = \sum_{w\in W\Omega} \det w \quad n_{w(\Lambda)-\beta}^{(\alpha)} . \tag{4.9}$$

The characters χ_λ can be viewed as functions $\chi_\lambda(\cdot)$ on the space of weights by considering the formal exponents to be $e_{\lambda+\rho}(\cdot) = \exp(-(\lambda+\rho), \cdot\,)$. The periodicity constraint (4.3) is solved by $\overline{e}_{\lambda+\rho}(\frac{2\pi i}{p}(\lambda'+\rho)) = q^{-(\lambda+\rho,\lambda'+\rho)} (q := exp(\frac{2\pi i}{p}))$, where without lack of generality λ' is regular. Then the periodic characters in (4.4) coincide with the quotient

$$\overline{\chi}_\Lambda \left(\frac{2\pi i}{p}(\Lambda'+\rho)\right) = \frac{S_{\Lambda\Lambda'}}{S_{0\Lambda'}}, \tag{4.10}$$

where $S_{\Lambda\Lambda'}$ is the matrix performing the modular transformation (see [9]) of the affine charachers of $g^{(1)}$, i.e.,

$$ch_{\hat{\Lambda}}(-1/\tau) = \sum_{\Lambda'} S_{\Lambda\Lambda'} \ ch_{\hat{\Lambda}'}(\tau).$$

Here $\hat{\Lambda}, \hat{\Lambda}'$ are labels of integrable representations of $g^{(1)}$ of level k (recall that $p = k + h$), $(\hat{\Lambda}, \alpha_i) = (\Lambda, \alpha_i)$, $i = 1, 2, \ldots, l$. In particular we recover the q-dimension evaluating $\overline{\chi}_\Lambda(\cdot)$ at a particular value of the argument

$$\overline{\chi}_\Lambda\left(\frac{2\pi i}{p}\rho\right) = \frac{S_{\Lambda 0}}{S_{00}} = D_\Lambda\,, \tag{4.11}$$

while the classical dimension is given by

$$d_\Lambda = \chi_\Lambda(0) = \lim_{p\to\infty} \overline{\chi}_\Lambda\left(\frac{2\pi i}{p}\rho\right) = \lim_{p\to\infty} D_\Lambda. \tag{4.12}$$

Taking into account (4.10) we can rewrite (4.6) as

$$\frac{S_{\alpha\lambda'}S_{\beta\lambda'}}{S_{0\lambda'}} = \sum_\lambda \overline{m}^\lambda_{\alpha\beta} S_{\lambda\lambda'} \tag{4.13}$$

This is nothing else but the Verlinde formula for the fusion rules multiplicity $N^\Lambda_{\alpha\beta}$ since (4.13) can be inverted using the unitarity of S. Hence

$$\overline{m}^\Lambda_{\alpha\beta} = N^\Lambda_{\alpha\beta} \tag{4.14}$$

which ensures that $0 \leq \overline{m}^\Lambda_{\alpha\beta} \leq m^\Lambda_{\alpha\beta}$.

It is far from surprising that the truncation achieved by imposing the restriction (4.3) leads to the Verlinde formula (4.13). Essentiallly the same truncation was considered in [2] explicitly for $g = su(2)$ starting from the classical Weyl formula (4.1). Guided by the properties of the $\mathcal{U}_q(g)$ representations we were just able to carry over explicitly the truncation in general, getting (4.6,7).

Let us now go back to our formulae (4.7), (4.9). Due to the regularity of α, β and Λ, far from all $w \in W\Omega$ do actually contribute to (4.9), since the corresponding weights Λ' do not intersect the translated weight diagram $\Gamma_\alpha + \beta$. Here $\Gamma_\alpha = \{$all weights μ in $T_\alpha\} = \{\mu = \alpha - \sum_1^l k_i\alpha_i,\ k_i \in \mathbb{Z}_+, \mu^+ \leq \alpha\}$, where μ^+ is the only dominant weight among the weights $\{\underline{w}(\mu),\quad \underline{w} \in W\}$. Because of the same reasons we can safely add to (4.9) the Weyl orbits of all other dominant weights, i.e., we can extend the set $W\Omega$ to the full affine Weyl group $\hat{W}$. In this way we arrive at formula (1.3) of the introduction, while (1.2) corresponds to (4.7).

Let us compare these formulae with their classical counterparts (4.1), (4.2). While in the classical formula (4.1) the sum runs over dominant weights, now in (4.6) only regular weights are taken into account. This is illustrated for $g = sl(3)$ on Fig.5; the dominant region P_+ described by the sector bounded by the fundamental axis is reduced by the axis $(\Lambda+\rho,\theta) = p$ to the shaded triangle region $P_+^{(p)}(\ni \Lambda+\rho)$. On the other hand the Weyl group W in formula (4.2) for the classical multiplicity $m_{\alpha\beta}^{\Lambda}$ is replaced by the affine Weyl group $\hat{W}$ in formula (1.3) for the reduced multiplicity $\overline{m}_{\alpha\beta}^{\Lambda}$.

The equivalent formula (4.7) seems to be more useful for practical computations. One only needs the classical multiplicities $m_{\alpha\beta}^{\Lambda}$ of the highest weights Λ in the g- product $T_\alpha \otimes T_\beta$, which can be computed by some of the existing algorithms. We illustrate in Appendix A our methods of computing the fusion rules multiplicities on several examples of $g = sl(3)$.

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Note added in proof.

In a recent preprint [36] it is conjectured that the fusion multiplicities $N_{\alpha\beta}^{\Lambda}$ can be obtained by a formula generalizing (4.2) by adding just the reflection (of $\Lambda+\rho$) with respect to the hyperplane $(\Lambda+\rho,\theta^\vee) = p$, i.e. the reflection in $\hat{W}$ corresponding to the affine simple root α_0. It does indeed give the correct result for the simple examples of all level $k = 1$ representations of $g^{(1)}$, checked by the author. However, in view of our results, the conjecture is wrong in general and the simple method of [36] cannot recover $N_{\alpha\beta}^{\Lambda}$; the sum in the generalization (4.9) of (4.2) runs over $W\Omega$, not just $W\omega_\theta$. A counter example is provided, e.g., for $sl(3)$ by the product $(6,2)\otimes(6,2)$ for $k = 8$, illustrated on Fig.6 (see Appendix A). In particular, the method of [36] would give a negative multiplicity for the weight $\Lambda = (8,0)$.

Appendix A. Examples.

In this Appendix we show how formulae (4.7), (4.9) can be used in practical calculations.

To apply (4.7) we shall exploit one of the existing algorithms for the computation of the representations in the decomposition of the ordinary g-product $\alpha\otimes\beta$ and their multiplicities $m_{\alpha\beta}^{\Lambda}$. This is the Weyl determinant method (see [30]) which we breafly recall.

Let $g = sl(N)$. We shall use the notation $(r_1,\ldots,r_{N-1})$ or $[m_1,\ldots,m_N]$ with $r_i =$

$m_i - m_{i+1} \in \mathbb{Z}_+$ for the weight $\Lambda = \sum_{i=1}^{N-1} r_i \Lambda_i$.

Consider the formal determinant

$$F_\Lambda := \begin{vmatrix} F_{m_1} & F_{m_1+1} & \dots & F_{m_1+N-1} \\ F_{m_2-1} & F_{m_2} & \dots & F_{m_2+N-2} \\ & & \ddots & \\ F_{m_N-N+1} & F_{m_N-N+2} & \dots & F_{m_N} \end{vmatrix} \tag{A.1}$$

where $F_m = 0$ if $m < 0$, $F_0 = 1$ and the operators F_m for $m > 0$ act on the weight $[m'_1, \dots, m'_N]$ as follows

$$F_m [m'_1, \dots, m'_N] = \sum_{\{\varepsilon_1, \dots, \varepsilon_N\}} [m'_1 + \varepsilon_1, \dots, m'_N + \varepsilon_N] , \tag{A.2}$$

where $\sum_{i=1}^{N} \varepsilon_i = m$, $0 \le \varepsilon_1 \le m$, $0 \le \varepsilon_{i+1} \le r'_i$, $i = 1, \dots, N-1$. The multiplicities $m_{\alpha\beta}^{\Lambda}$ in $\alpha \otimes \beta$ are computed according to the rule

$$F_\alpha \, \beta = \sum_{\Lambda} m_{\alpha\beta}^{\Lambda} \, \Lambda . \tag{A.3}$$

Let us illustrate the method on the example $\alpha = (2,1) = [3,1,0]$ and $\beta = (2,2) = [4,2,0]$. We have

$$\begin{aligned} F_\alpha \, \beta &= (F_1 F_3 - F_4) [420] \\ &= F_1 ([720] + [630] + [621] + [540] + [531] + [522] + [441] + [432]) \\ &\quad - ([820] + [730] + [721] + [640] + [631] + [622] + [541] + [532] + [442]) \\ &= [730] + [721] + [640] + 2\,[631] + [622] + [550] + 2\,[541] + [442] + 2\,[532] + [433], \end{aligned}$$

i.e., we get

$$(2,1) \otimes (2,2) = (4,3) + (5,1) + (2,4) + 2\,(3,2) + (4,0) + (0,5) + (0,2) + 2\,(1,3) + 2\,(2,1) + (1,0). \tag{A.4}$$

Since the sign $\det w$ in front of $m_{\alpha\beta}^{w(\Lambda)}$ in (4.7) coincides with the sign of the q- dimension $D_{w(\Lambda)}$, given $m_{\alpha\beta}^{\Lambda}$ it is useful first to compute the q-dimensions D_Λ for any Λ with $m_{\alpha\beta}^{\Lambda} \neq 0$. The degeneracy of the q-dimensions, if it occurs, is removed by splitting the set of all Λ' with the same , up to a sign, q-dimension $D_{\Lambda'}$ into (shifted) $\hat{W}$-orbits, i.e., into sets $\Omega_\Lambda \subset \hat{W}_\Lambda$. It remains to apply (4.7).

Let us now choose $p(= k + N) = 7$ (recall that $[a]_q = [p-a]_q = -[a+p]_q$). We have

$$
\begin{array}{ccccccccccc}
\Lambda & (4,3) & (5,1) & (2,4) & (3,2) & (0,5) & (1,3) & (4,0) & (2,1) & (0,2) & (1,0) \\
m^{\Lambda}_{\alpha\beta} & 1 & 1 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 1 \\
D_{\Lambda} & -[2]_q[3]_q & -[1]_q & -[3]_q & 0 & 0 & [3]_q & [1]_q & [2]_q[3]_q & \frac{[3]_q[3]_q}{[2]_q} & [3]_q
\end{array} \tag{A.5}
$$

The degeneracy of the q-dimensions is illustrated in (A.5) by the representations $(1,0)$, $(1,3)$ and $(2,4)$, since $D_{(1,0)} = D_{(1,3)} = -D_{(2,4)}$ for $p = 7$. Using (2.12) we see that $\{(1,3),(2,4)\} \subset \Omega^{(3)}_{(1,3)}$, while $(1,0) \in \Omega^{(3)}_{(1,0)}$. Hence according to (4.7),

$$\overline{m}^{(1,0)}_{\alpha\beta} = m^{(1,0)}_{\alpha\beta} = 1, \quad \overline{m}^{(1,3)}_{\alpha\beta} = m^{(1,3)}_{\alpha\beta} - m^{(2,4)}_{\alpha\beta} = 2 - 1 = 1.$$

Similarly

$$\overline{m}^{(2,1)}_{\alpha\beta} = m^{(2,1)}_{\alpha\beta} - m^{(4,3)}_{\alpha\beta} = 2 - 1 = 1, \text{ etc.}$$

Finally we obtain that for $p = 7$

$$(2,1) \overset{F}{\otimes} (2,2) = (1,3) + (2,1) + (0,2) + (1,0). \tag{A.6}$$

In the same way starting from the g-product (A.4) one can compute the FR multiplicities for any $p \geq 7$. For $p \geq 10$ the result coincides with the classical formula (A.4).

The analytic method based on the Weyl determinant method, which is used in this example, can be readily implemented into a computer code. Next we shall illustrate the "geometric" method for the computation of the FR multiplicities which is provided by formula (4.9). This formula generalizes the classical Weyl formula (4.2). The meaning of (4.2) is the following. We first compute the multiplicities of all the weights in the representation T_α to obtain the weight diagram Γ_α. In the case $g = sl(3)$ it consists of hexagons (or triangles - degenerate hexagons) of weights of the same multiplicity. In particular the biggest hexagon contains the ordinary (not shifted) orbit $W_\Lambda = \{\underline{w}(\Lambda)\}$ of the weight Λ. The weights lying on the outmost hexagon have multiplicity one and moving inward the multiplicities increase with one untill the hexagons degenerate into triangles; further inward they remain constant. Having constructed the weight diagram of α we translate it by β to $\Gamma_\alpha + \beta$. Only weights $\Lambda = \mu + \beta, \quad \mu \in \Gamma_\alpha$, with Λ - integer dominant enter the r.h.s. of (4.1). Their multiplicity

$m_{\alpha\beta}^{\Lambda}$ is computed following the ordinary Weyl orbit of $\Lambda+\rho$, i.e. $W_{\Lambda+\rho}=\{\underline{w}(\Lambda+\rho)\}$ and adding the multiplicity of the weights $\underline{w}(\Lambda+\rho)\in W_{\Lambda+\rho}\cap(\Gamma_\alpha+\beta+\rho)$ with a relative sign determined by $\det\underline{w}$. The formula (4.9) for the reduced multiplicity $\overline{m}_{\alpha\beta}^{\Lambda}$ says that now we have to take into account along with $W_{\Lambda+\rho}$ also the ordinary Weyl orbits of the points $\Lambda'\in\Omega_\Lambda$. Thus for $g=sl(3)$ we have to reflect $\Lambda+\rho$ with respect to the plane $(\Lambda+\rho,\theta)=p$ and then take into account the ordinary Weyl orbit of the point $\omega_\theta(\Lambda)+\rho$, next we have to reflect $\omega_\theta(\Lambda)+\rho$ with respect to the planes $(\Lambda+\rho,\alpha_i)=p,\quad i=1,2$, and again to follow the ordinary Weyl orbits of the resulting points $\omega_i\omega_\theta(\Lambda)+\rho$.

On Fig.5 we illustrate the geometric method on our previous example, $\alpha=(2,1)$, $\beta=(2,2)$, $p=7$. In the translated weight diagram $\Gamma_\alpha+\beta+\rho$ depicted on Fig.5 we select the weights $\Lambda+\rho$ with Λ- regular ($\rho=(1,1)$). These are $\Lambda+\rho=(2,1)$, $(1,3)$, $(5,1)$, $(2,4)$, $(3,2)$, appearing with multiplicities $1,1,1,2,2$ respectively. In the ordinary g-product these weights survive with the same multiplicities (compare with (A.4)) since the reflections $\underline{w}(\Lambda+\rho),\underline{w}\in W,\underline{w}\neq\mathbf{1}$, create points beyond the diagram $\Gamma_\alpha+\beta+\rho$. Now to compute the reduced multiplicity of, say. $\Lambda=(2,1)$, we have to reflect $\Lambda+\rho=(3,2)$ with respect to the plane $(\Lambda+\rho,\theta)=p$ getting $\Lambda'+\rho=(5,4)$. Hence $\overline{m}_{\alpha\beta}^{(2,1)}=2-1=1$. Similarly one computes $\overline{m}_{\alpha\beta}^{\Lambda}$ for $\Lambda+\rho=(2,4)$, $(5,1)$, while the reflections of $\Lambda+\rho=(2,1)$, $(1,3)$ appear beyond $\Gamma_\alpha+\beta+\rho$. The weights $\Lambda+\rho=(1,6)$, $(4,3)$ lie on the plane $(\Lambda+\rho,\theta)=p$, i.e., they are invariant under the reflection with respect to this plane and hence they do not contribute to the reduced product.

In this example only one or two elements of the sets $\Omega_\Lambda^{(3)}$ do appear in the diagram $\Gamma_\alpha+\beta$. The situation is more complicated in the next example, illustrated on Fig.6. Here $\alpha=\beta=(6,2)$ and $p=11$. In this example along with $\Lambda+\rho$ and $\omega_\theta(\Lambda)+\rho$ also $\omega_1\omega_\theta(\Lambda)+\rho$ can appear in $\Gamma_\alpha+\beta+\rho$. This is illustrated by $\Lambda+\rho=(9,1)$ which is connected with dashed lines with $\Lambda'+\rho=(10,2)$, $(12,1)$, all in $\Omega_{(8,0)}^{(3)}+\rho$ (see (2.12)). The ordinary Weyl orbits of these three points each intersect $\Gamma_{\alpha+\beta+\rho}$ at two points (connected by thick lines on Fig.6). According to (4.9) the resulting reduced multiplicity is

$$\overline{m}_{\alpha\beta}^{(8,0)}=(3-2)-(3-1)+(2-1)=0.$$

Appendix B. Automorphisms of the extended Dynkin diagrams of $A^{(1)}$ and WZW fusion rules.

The group Γ of proper automorphisms of the extended Dynkin diagrams of the untwisted affine algebras $g^{(1)}$ are listed in [31] (see also [32]). They induce an action on the projection Λ of $\hat{\Lambda} = k\bar{c} + \Lambda$ according to $(\sigma(\hat{\Lambda}), \alpha_i) = (\hat{\Lambda}, \sigma(\alpha_i))$, $\sigma \in \Gamma$.

Analyzing the system of equations in [5], Fuchs and Gepner [32] have shown that the fusion products $\overset{F}{\otimes}$ satisfy

$$\Lambda \overset{F}{\otimes} \sigma(\mathbf{1}) = \Lambda', \quad \sigma \in \Gamma$$

and their results further imply $\Lambda' = \sigma(\Lambda)$, i.e.,

$$\Lambda \overset{F}{\otimes} \sigma(\mathbf{1}) = \sigma(\Lambda) \tag{B.1}$$

The associativity of the product $\overset{F}{\otimes}$ and (B.1) leads to the relation

$$\Lambda_1 \overset{F}{\otimes} \sigma(\Lambda_2) = \sigma(\Lambda_1) \overset{F}{\otimes} \Lambda_2 = \sigma(\Lambda_1 \overset{F}{\otimes} \Lambda_2), \tag{B.2}$$

a property, stated in [33]. Here $\sigma(\oplus_i \Lambda_i) = \sum_i \sigma(\Lambda_i)$. It has been stressed in [35] that (B.2) is a necessary condition for any truncated product to coincide with the fusion rules. In particular, without referring to the equivalence with the Verlinde formula, but using (1.3) one can show that the multiplicity $\overline{m}^{\Lambda}_{\Lambda_1\Lambda_2}$ satisfies (B.2), i.e., $\overline{m}^{\sigma(\Lambda)}_{\Lambda_1\sigma(\Lambda_2)} = \overline{m}^{\Lambda}_{\Lambda_1\Lambda_2}$. Indeed, $\sigma\hat{W}\sigma^{-1} = \hat{W}$ and one has to use that for any $\sigma \in \Gamma$, $\sigma(\lambda) = \sigma(0) + \underline{w}(\lambda)$ for some $\underline{w} \in W$ and furthermore that $n^{(\alpha)}_{\underline{w}(\mu)} = n^{(\alpha)}_{\mu}$.

The symmetry property (B.2) suggests that one can define another truncation of the g-tensor product according to

$$\Lambda_1 \tilde{\otimes} \Lambda_2 = \bigcap_{\sigma_1,\sigma_2 \in \Gamma} \sigma_1^{-1}\sigma_2^{-1}\left(\sigma_1(\Lambda_1) \otimes \sigma_2(\Lambda_2)\right) . \tag{B.3}$$

It follows that

$$N^{\Lambda}_{\Lambda_1\Lambda_2} \leq \tilde{N}^{\Lambda}_{\Lambda_1\Lambda_2} := \min_{\sigma_1,\sigma_2 \in \Gamma} m^{\sigma_1\sigma_2(\Lambda)}_{\sigma_1(\Lambda_1)\sigma_2(\Lambda_2)} \tag{B.4}$$

where $m^{\Lambda}_{\Lambda_1\Lambda_2}$ is the ordinary multiplicity of Λ in the ordinary g-tensor product $\Lambda_1 \otimes \Lambda_2$.

We make the conjecture that the equality in (B.4) is actually reached if $g = A_{N-1}$, i.e., $N^{\Lambda}_{\Lambda_1\Lambda_2} = \tilde{N}^{\Lambda}_{\Lambda_1\Lambda_2}$. Let us consider this case in more detail. The group Γ is isomorphic to $\mathbb{Z}_N$ generated by

$$\sigma(\alpha_0, \alpha_1, \ldots \alpha_{N-1}) = (\alpha_{N-1}, \alpha_0, \alpha_1, \ldots \alpha_{N-2})$$

where $\alpha_\mu, \mu = 0, 1, ..., N-1$, are the simple roots of $g^{(1)}$. In terms of the projection Λ, $(\Lambda, \alpha_i) = (\hat{\Lambda}, \alpha_i), i = 1, 2, ..., N-1$, one has

$$\sigma(r_1, ..., r_{N-1}) = (k - \sum_{i=1}^{N-1} r_i, r_1, ..., r_{N-2}). \tag{B.5}$$

The transformation (B.5) maps regular weights Λ into regular weights again. Formulae (B.3), (B.4) state that to get all $\tilde{N}_{\Lambda_1\Lambda_2}^{\Lambda}$ one needs to decompose all ordinary products $\sigma^k(\Lambda_1)\otimes \sigma^l(\Lambda_2)$, using some of the existing algorithms, convert the result back by $(\sigma^{k+l})^{-1}$ (so that the N-ality is preserved) and retain the sum of representations common to all products with the smallest multiplicity present. Obviously all irregular weights appearing should be dropped without applying $(\sigma_{k+l})^{-1}$ since it sends them to nondominant weights. In all examples which we have checked the equality $N_{\Lambda_1\Lambda_2}^{\Lambda} = \tilde{N}_{\Lambda_1\Lambda_2}^{\Lambda}$ was reached averaging just along the diagonal $\sigma_1 = \sigma_2$ in (B.3,4).

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Figure captions.

1. The set $\mathcal{M}_{\Lambda}^{(p)}$ for $\mathcal{U}_q(sl(2))$, $q^p = 1$. The arrows on the line point to the imbedded modules, while the dashed arrows correspond to the action of the elements of $\hat{W}$, generated by $w = w_{\alpha}$, $\omega = \omega_{\theta}$.

2a. Part of $\mathcal{M}_{\Lambda}^{(p)}$ for $\mathcal{U}_q(sl(3))$, $q^p = 1$.

2b. The points $\Lambda' + \rho$, $\Lambda' \in \Omega_{\Lambda}^{(3)}$ obtained by composition of reflections with respect to the planes $(\Lambda + \rho, \alpha) = p$, $\alpha > 0$.

3. The set $\Omega_{\Lambda}^{(4)}$.

4a,b. The sets $\Omega_{\Lambda}^{(N)}$ for $N = 5, 6$.

5. The weight diagrams Γ_{α} and $\Gamma_{\alpha} + \beta + \rho$ for $\alpha = (2,1)$, $\beta = (2,2)$.

6. The weight diagram $\Gamma_{\alpha} + \beta + \rho$ for $\alpha = \beta = (6,2)$.

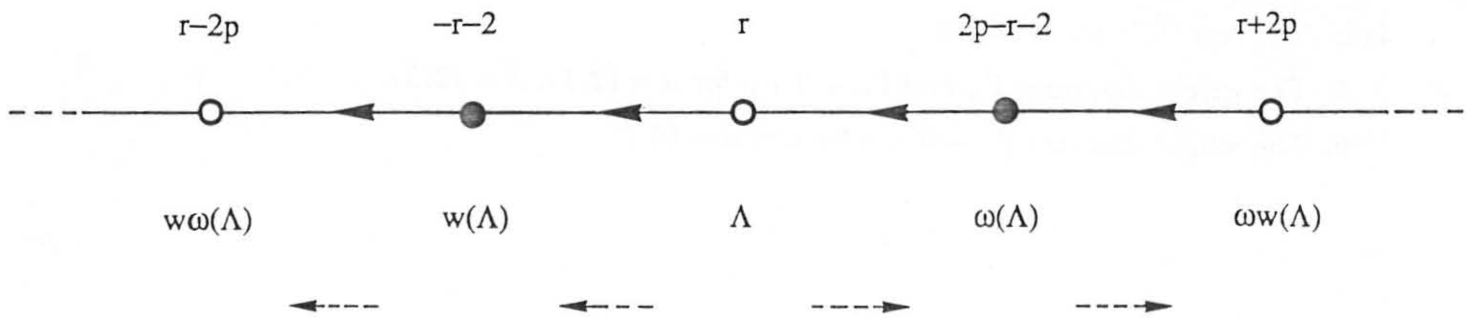
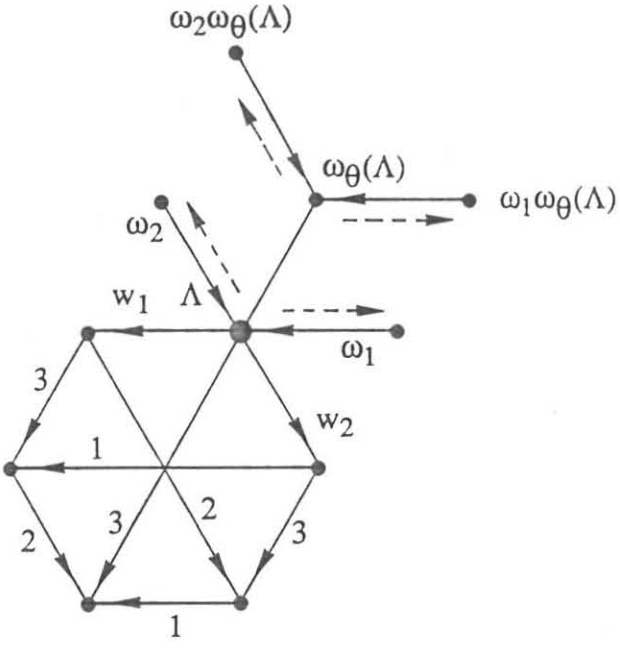
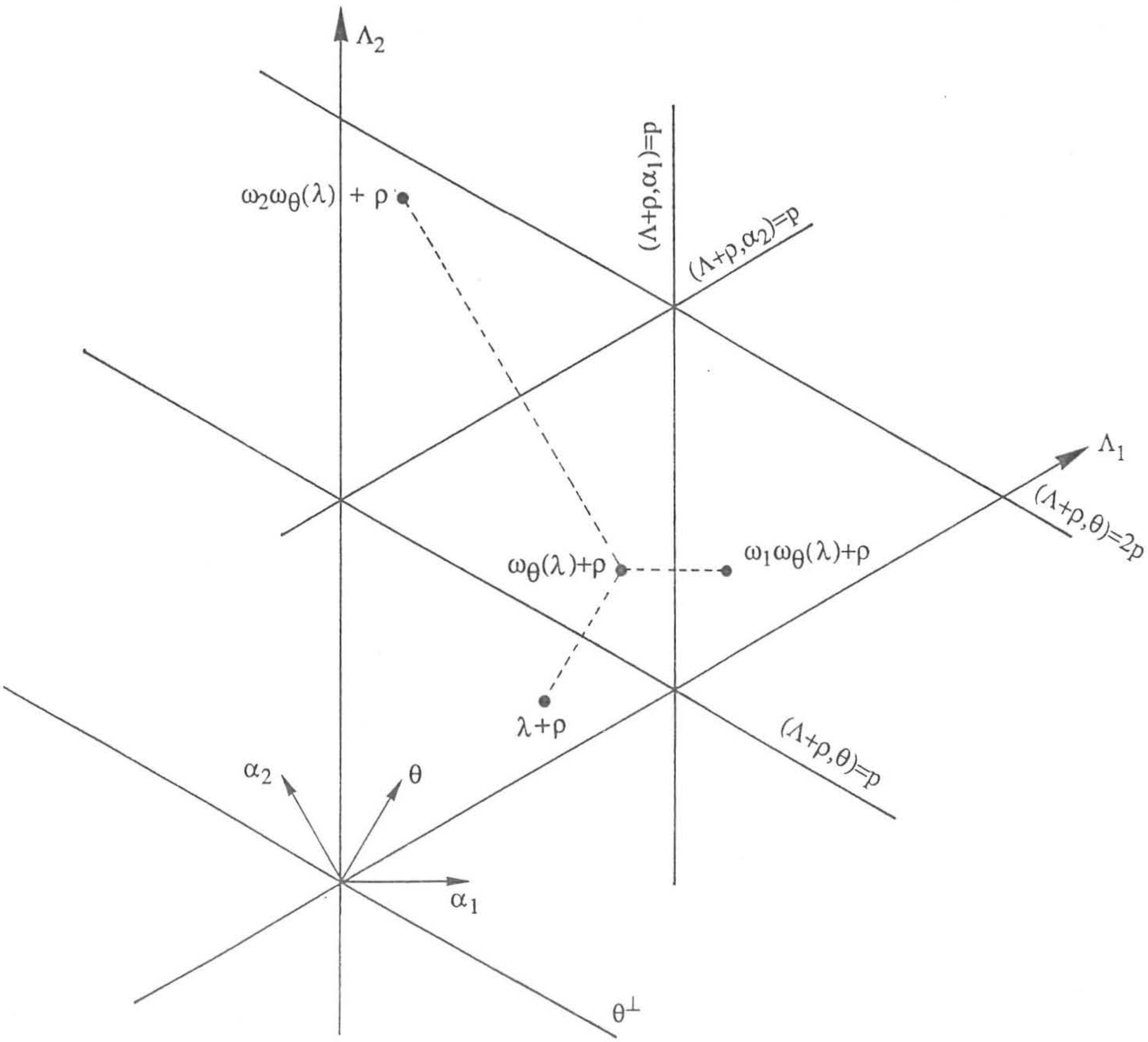


Fig. 1



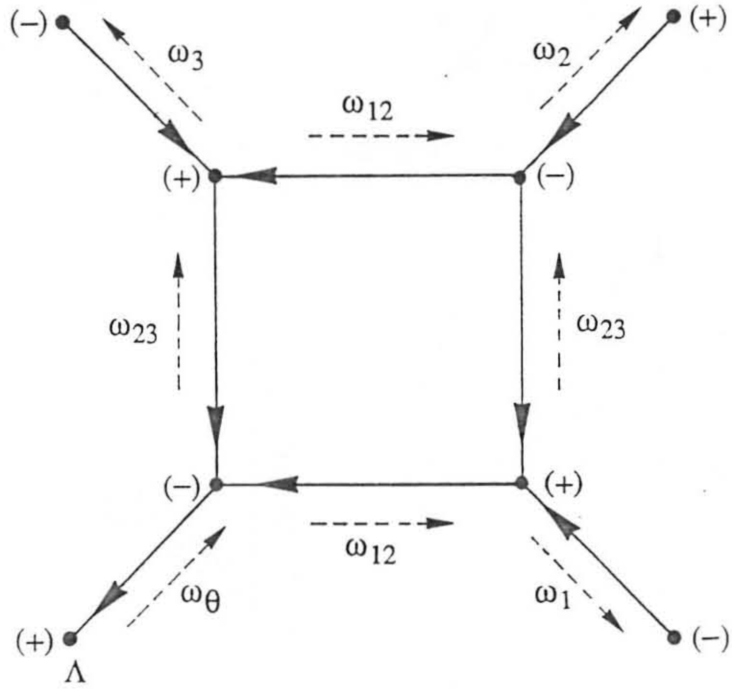
$\omega_2\omega_\theta(\Lambda)$
$\omega_\theta(\Lambda)$
$\omega_1\omega_\theta(\Lambda)$
ω_2
w_1
Λ
ω_1
w_2
3
1
2
3
2
3
1

Fig. 2a



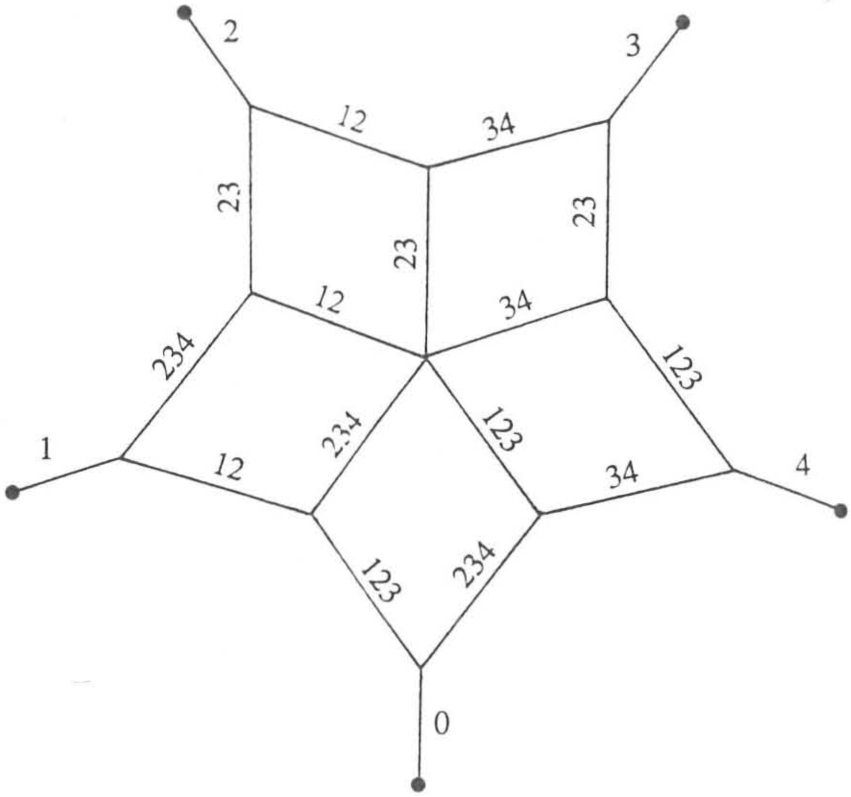
Λ_2
$(\Lambda+\rho,\alpha_1)=p$
$\omega_2\omega_\theta(\lambda)+\rho$
$(\Lambda+\rho,\alpha_2)=p$
Λ_1
$(\Lambda+\rho,\theta)=2p$
$\omega_\theta(\lambda)+\rho$
$\omega_1\omega_\theta(\lambda)+\rho$
$\lambda+\rho$
$(\Lambda+\rho,\theta)=p$
α_2
θ
α_1
$\theta^\perp$

Fig. 2b



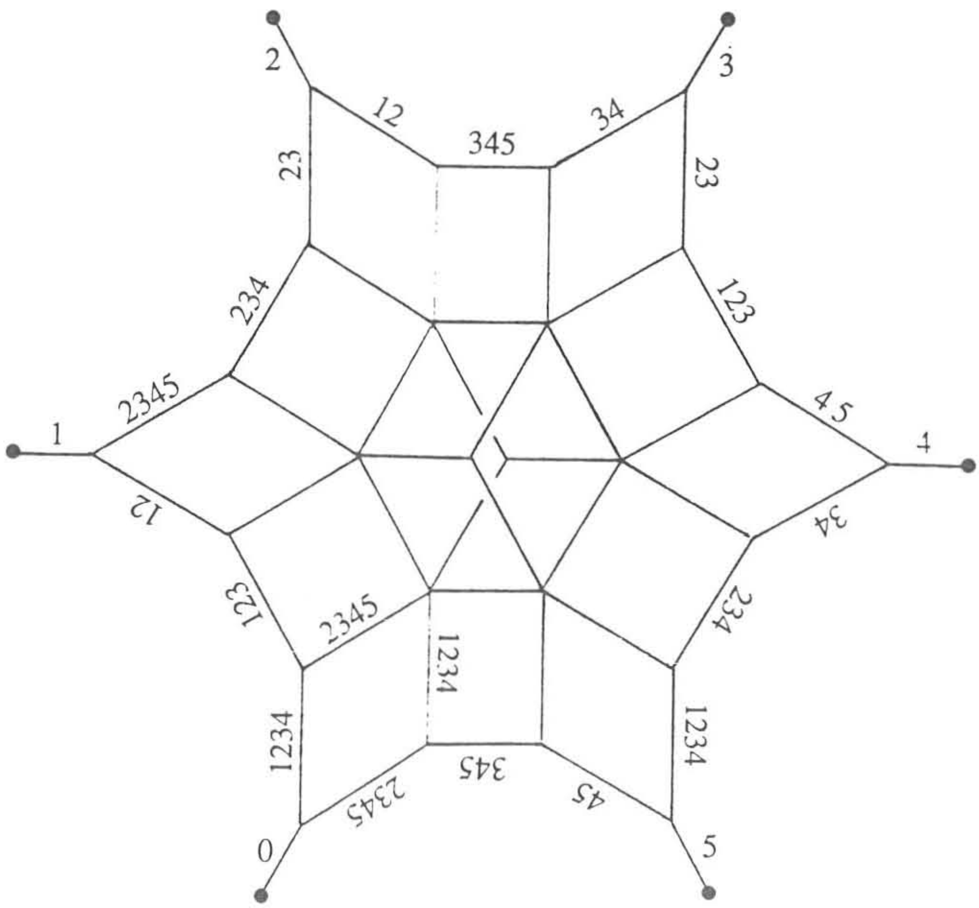
(−)
ω_3
ω_2
(+)
ω_{12}
(+)
(−)
ω_{23}
ω_{23}
(−)
(+)
ω_{12}
ω_θ
ω_1
(+)
Λ
(−)

Fig. 3



2
3
12
34
23
23
23
12
34
234
123
1
234
123
12
34
4
123
234
0

Fig. 4a



2
3
12
345
34
23
23
234
123
2345
45
1
4
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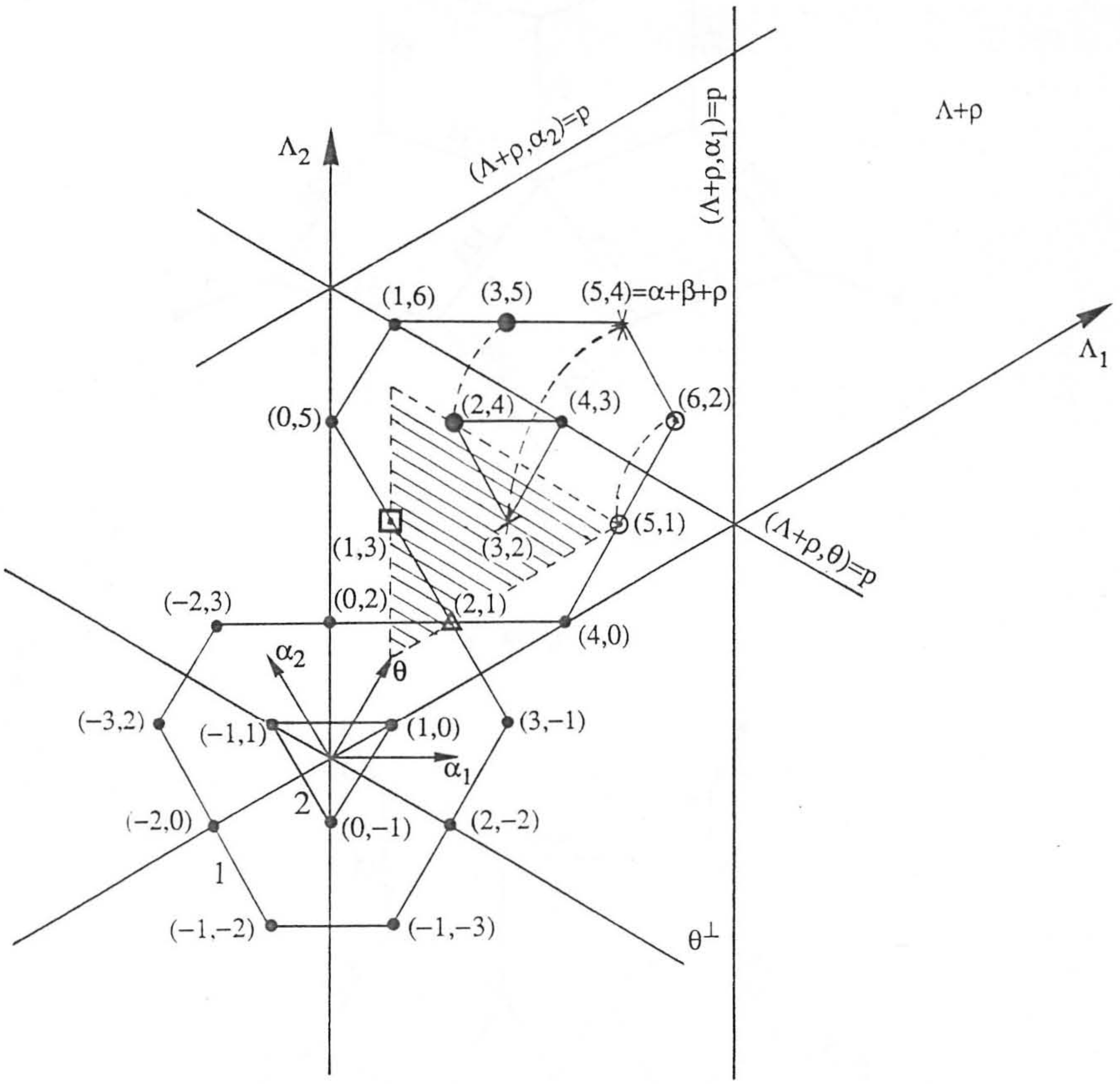
34
123
2345
234
1234
1234
1234
2345
345
45
0
5

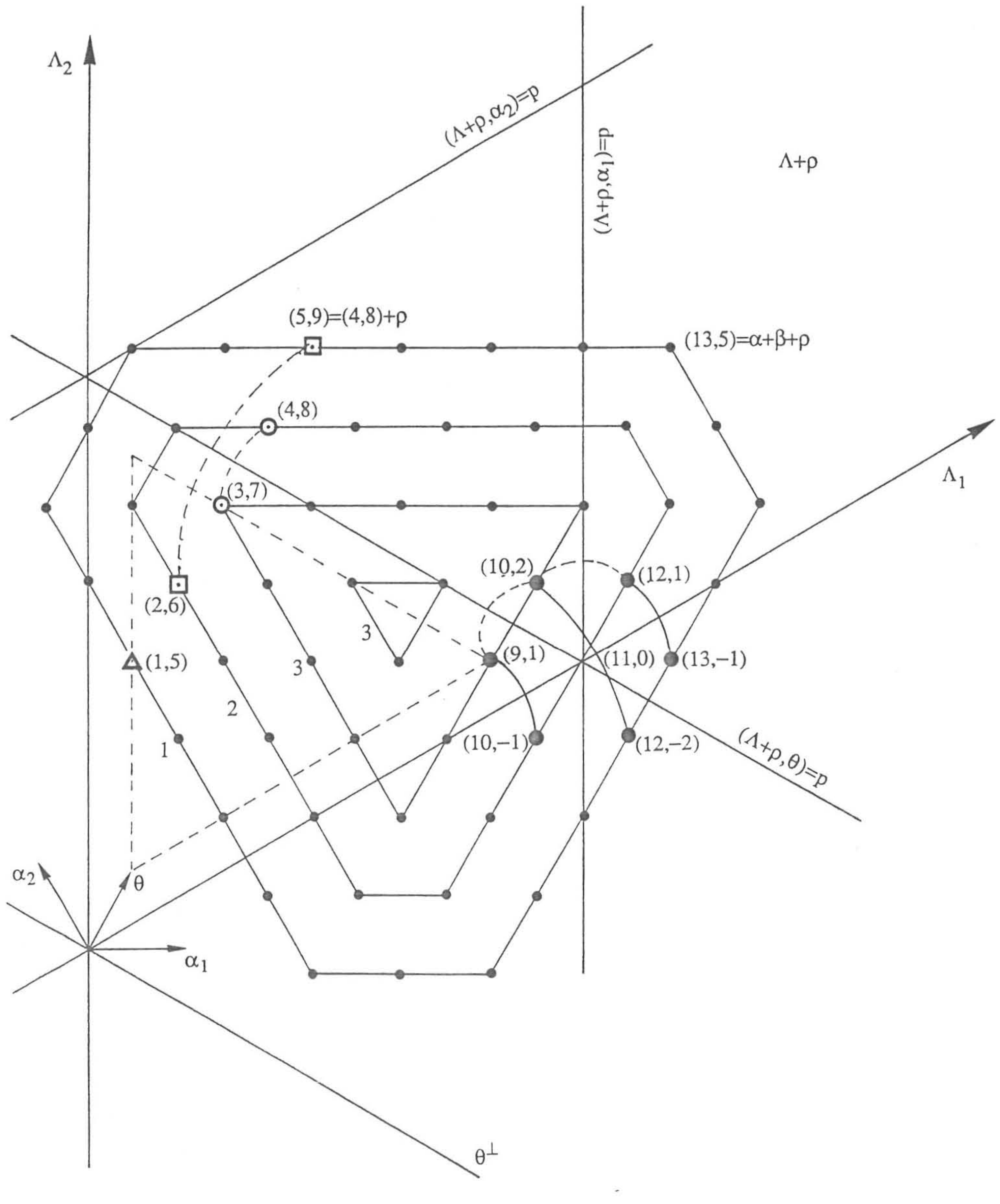
Fig. 4b



$$(2,1) \overset{F}{\otimes} (2,2) = (2,1) \oplus (1,3) \oplus (1,0) \oplus (0,2)\ ;\ p = 7$$

$\times$ ● △ □

Fig. 5



$$(6,2) \overset{F}{\otimes} (6,2) = (2,6) \oplus (1,5) \oplus (0,4) \; ; \; p = 11$$

Fig. 6