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TOWARD SOLVABLE 4-DIMENSIONAL CONFORMAL THEORIES*

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ABSTRACT

The Symanzik representation , based on the Mellin transform, is explored to analyze the asymptotic behaviour of conformal invariant n-point functions, constructed through a generalization to arbitrary (even) space-time dimension of the two-dimensional Coulomb gas representation."

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1. The pioneering work of BPZ[1] explored recent mathematical results[2] on indecomposable representations of the Virasoro algebra to set a powerful method of building nontrivial two-dimensional conformal invariant theories. There are few examples of solvable conformal invariant theories on 2h=4 dimensions (see ref.3 for a overlook). Meanwhile, various infinite dimensional algebras, containing the 15-dimensional Minkowski (or Euclidean) conformal algebra, have been constructed inref.4 The setting there probably is not the ultimate one and further a representation theory should be developed . On the other hand one can generalize the approach in ref.5 where the (scalar) minimal theories of ref.1 for c<1 have been explicitly constructed in terms of euclidean n-point functions. The functions are SL(2,C) - invariant and correspond to the correlations of the primary fields. These fields generate the whole conformal theory i.e. an infinite set of SL(2,C)-covariant fields, closed under operator product expansions. The role of the infinite dimensional algebra - Virasoro, or, for c>1, a larger, "extended Virasoro" algebra, as, e.g. in ref.6 is to render this set into a finite number of classes, each accomodating an infinite number of fields. The Coulomb gas integral representation with a charge at infinity [5,6] is a basic ingredient of various known minimal theories. It is trivially extended to arbitrary 2h dimensions, h-integer. However, the analysis of the singular behaviour of the 4-point functions at coinciding points in ref.5 relies on the left -right factorizability, exploiting equivalent contour integral representations, and hence is intrinsically restricted to 2h=2 dimensions. We propose an alternative technique which is equally well suited for arbitrary 2h and for arbitrary n-point functions.

2. The generalization of the vertex representation of ref.5 using the gaussian action for the scalar field φ with Green function proportional to

$$S(x_1,x_2) = \frac{(-1)^{h+1}}{\Gamma(h)(4\pi)^h} \ln x_{12}^2 \mu^2 \quad , \qquad \Box^h S(x_1,x_2) = \delta(x_1-x_2) \; ,$$

is straightforward; μ is an arbitrary mass parameter in the Euclidean space-time. In the absence of a charge at infinity this model is well known, and in particular, it is the building block of the purely longitudinal model (see, e.g. ref. 7,3 describing the interaction of a massless fermion with a pure gauge electromagnetic potential.

The final result of the construction, paralleling that in ref.5 , are the conformal invariant functions defined for noncoinciding arguments by

$$\langle \phi_1(x_1) \ldots \phi_m(x_m) \rangle = \prod_{j=1}^{s-1} \int d^{2h} y_j \prod_{j=1}^{r-1} \int d^{2h} y'_j \langle\!\langle \prod_{i=1}^{m} V_{\alpha_i}(x_i) \prod_{i=1}^{s-1} V_{\alpha_-}(y_i) \prod_{i=1}^{r-1} V_{\alpha_+}(y'_i) \rangle\!\rangle \tag{1}$$

where $V_\alpha(x) = \exp i \alpha \varphi(x)$ and

$$\langle\!\langle \prod_{i=1}^{p} V_{\beta_i}(x_i) \rangle\!\rangle = \prod_{i<j} (x_{ij}^2)^{2\beta_i \beta_j} \tag{2a}$$

for

$$\sum_{i=1}^{p} \beta_i = 2\alpha_0 \quad , \qquad \alpha_0 \text{ -real (nonnegative);} \tag{2b}$$

otherwise (2a) is either zero, or not well defined. The fields ϕ_i have dimensions $d_i = 2\Delta_i$,

$$\Delta_i = \Delta(\alpha_i) = \alpha_i(\alpha_i - 2\alpha_0) = \Delta(2\alpha_0 - \alpha_i) \; ; \qquad \Delta(\alpha_\pm) = h. \tag{3}$$

Similarly the Coulomb gas representation of the c>1 extended Virasoro theories, described by r fields φ_i, i=1,...,r and vertex charges related to labels of finite dimensional representations of a rank r semisimple Lie algebra [6] can be generalized to h≥1. Our technique applies to this case as well but for simplicity we shall illustrate it on the simplest example with a single field φ.

The conservation of the charges (2b) implies that for any of the volume integrals in (1)

$$I_N(u_1,\delta_1,\dots,u_N,\delta_N) = \int d^{2h}\omega \prod_{i=1}^{N} \left[(u_i-\omega)^2\right]^{-\delta_i}, \tag{4a}$$

one has

$$\sum_{i=1}^{N} \delta_i = 2h. \tag{4b}$$

Starting as in 2h = 2 from one of the possible representations of the 4-point function for $\Delta_i = \Delta$, i = 1,2,3,4, the charges α_i are restricted to the values

$$\alpha_{rs} = \frac{1-s}{2}\alpha_- + \frac{1-r}{2}\alpha_+ \quad , \qquad \text{r,s - positive integers.} \tag{5}$$

Assuming the relation

$$\alpha_+ p + \alpha_- q = 0 \qquad , \text{p, q coprime integers} \tag{6}$$

characterizing in 2h=2 the minimal theories, one obtains that

$$4\alpha_0^2 = h\frac{(p-q)^2}{pq}, \quad \alpha_{rs} = \alpha_{r+kp\ s+kq}, \quad 2\alpha_0 - \alpha_{rs} = \alpha_{p-r\ q-s}. \tag{7}$$

Requiring that $2\alpha_0 - \alpha_{rs}$ is of the type given in (5), restricts the values r,s to $1 \le r \le p-1$, $1 \le s \le q-1$.

As in 2h=2 the integral representations (1) for the three-point function select via the charge conservation condition (2b) the possible triples $(\Delta_1,\Delta_2,\Delta_3)$ providing non-zero functions (see, e.g. ref.6a). The various representations, in which any ϕ_{Δ_α} can be represented either by V_α or by $V_{2\alpha_0-\alpha}$, recover the known conformal invariant expressions for the three-point functions up to constants given by multiple volume integrals. Assuming that these constants are finite and non-zero and using (6), one obtains, given $\Delta(\alpha_1)$, $\Delta(\alpha_2)$ with $\alpha_i = \alpha_{r_i s_i}$, $i = 1,2$, $1 \leq r_i \leq p-1$, $1 \leq s_i \leq q-1$, the selection rules on the dimension $\Delta(\alpha_3) = \Delta(\alpha_{r_3 s_3})$

$$1 \leq |r_1 - r_2| + 1 \leq r_3 \leq \min\,(r_1 + r_2 - 1,\ 2p - 1 - r_1 - r_2) \leq p - 1$$

$$1 \leq |s_1 - s_2| + 1 \leq s_3 \leq \min\,(s_1 + s_2 - 1,\ 2q - 1 - s_1 - s_2) \leq q - 1 \,. \tag{8}$$

with r_3, s_3 running by two.

The result formally does not depend on h. In 2h=2 dimensions the assumption mentioned above has been justified by the explicitly found primary fields structure constants from the analysis of the asymptotic behaviour of the 4-point functions. In h>1 the argument remains formal and, as we shall see, there appear singularities in general.

3. Our next step will be to use subsequently for any of the volume integrals (4a) in the multiple integral in (1) the Symanzik representation [8]. This representation is based on the Mellin transform and the final formula has the structure

$$I_N = r_{12}^{h-\delta_1-\delta_2}\, r_{13}^{h-\delta_1-\delta_3}\, r_{23}^{\delta_1-h} \prod_{i=4}^{N} r_{1i}^{-\delta_i} \int \cdots \int \prod_{i=4}^{N} (ds_{2i}\, ds_{3i}) \prod_{i<j=4}^{N} ds_{ij}\, K(\{s_{kp}, \delta_t\}) \tag{9}$$

$$\prod_{i=4}^{N} (h_{2i}^{s_{2i}}\, h_{3i}^{s_{3i}}) \prod_{i<j=4}^{N} h_{ij}^{s_{ij}}\, \pi^h\, (2\pi i)^{-\frac{N(N-3)}{2}} \left[\prod_{k=1}^{N} \Gamma(\delta_k)\right]^{-1} ,$$

$$h_{2i} = \frac{r_{2i}\, r_{13}}{r_{1i}\, r_{23}}, \quad h_{3i} = \frac{r_{3i}\, r_{12}}{r_{1i}\, r_{23}}, \quad h_{ij} = \frac{r_{ij}\, r_{23}}{r_{2i}\, r_{3i}}, \qquad r_{ij} = x_{ij}^2 ,$$

where the kernel $K(\{s_{ij}, \delta_k\})$ is a product of Γ-functions (see ref.8 for the explicit expression). The contour integrals in (9) go along paths parallel to the imaginary axis.

The asymptotics of the n-point functions can be analyzed closing the contours in the final resulting expression and accounting for the poles of the integrand. Let us illustrate the technique on some examples. The simplest example is provided by the choice s=2, r=1 or s=1, r=2, i.e. the cases with a single volume integral in (1). Then applying the general formula in ref.8 one gets (n=4)

$$K(\{\sigma,\tau;\delta_i\}) = \Gamma(-\sigma)\Gamma(-\tau)\Gamma(-\sigma+h-\delta_2-\delta_4)\Gamma(-\tau+h-\delta_3-\delta_4)\cdot$$
$$\cdot\,\Gamma(\sigma+\tau+h-\delta_1)\Gamma(\sigma+\tau+\delta_4)\,. \tag{10}$$

Assuming $u=h_{34}<1$, $v=h_{24}<1$, each of the two integrals in (9) leads in general to two series of poles, which combine giving in the leading order for, say, $u\to 0$ $(v\to 1)$

$$I(x_1,\dots,x_4) \approx r_{23}^{-\delta_2} r_{13}^{h-\delta_1-\delta_3} r_{14}^{-\delta_4}\left(\frac{r_{34}}{r_{14}}\right)^{\delta_1+\delta_2-h}\Big\{u^{A_1(\delta_i)}\left[C_1(\delta_i)+O(u)\right]+$$
$$+\,u^{A_1(\delta_i)}\left[C_2(\delta_i)+O(u)\right] \tag{11a}$$

where

$$A_j(\delta_i) = (j-1)(h-\delta_1-\delta_2)\quad,\ j=1,2\,. \tag{11b}$$

The contributions to each of the constants C_i, i=1,2 come from two terms which sum up to the values

$$C_1 = \frac{\Gamma(h-\delta_1-\delta_2)\Gamma(h-\delta_3)\Gamma(h-\delta_4)}{\Gamma(\delta_1+\delta_2)\Gamma(\delta_3)\Gamma(\delta_4)}$$

$$C_2 = \frac{\Gamma(h-\delta_3-\delta_4)\Gamma(h-\delta_1)\Gamma(h-\delta_2)}{\Gamma(\delta_3+\delta_4)\Gamma(\delta_1)\Gamma(\delta_2)}\,. \tag{11c}$$

In this trivial example the result (11) can be reproduced directly from (2) taking in the integrand of (1) $x_1 \to x_2$, or $x_3 \to x_4$ and using the well known [9] integration formula

$$I_3(\{x_i,\delta_i\}) = \pi^h \prod_{i=1}^{3} \frac{\Gamma(h-\delta_i)}{\Gamma(\delta_i)} \prod_{i<j} r_{ij}^{h-\delta_i-\delta_j} \qquad \left(\sum_{i=1}^{3}\delta_i = 2h\right) . \tag{12}$$

The generalization to arbitrary n-point functions, $n \geq 4$, represented by one integral, is straightforward. In the leading order one gets again two terms with the same exponents (11b).

Our next example corresponds to two volume integrals in (1), i.e. either s=2=r, or r=1, s=3, or s=1, r=3. Now we have to apply twice the Symanzik's formula (9), first for N=5 and second , for N=4. The final expression reads (for n=4)

$$S(x_1,\ldots,x_4) = \text{const.} \prod_{i<j} r_{ij}^{2\alpha_i\alpha_j} \, r_{13}^{2h-\delta_1-\delta_1'-\delta_3-\delta_3'-\delta_5} \, r_{14}^{-\delta_4-\delta_4'} \, r_{23}^{-\delta_2-\delta_2'}$$

$$\left(\frac{r_{34}}{r_{14}}\right)^{\delta_1+\delta_1'+\delta_2+\delta_2'+\delta_5-2h} \underset{\uparrow}{\int} \cdots \underset{\uparrow}{\int} ds_{24}\, ds_{25}\, ds_{34}\, ds_{35}\, ds_{45}\, d\bar{s}_{24}\, d\bar{s}_{34} \tag{13a}$$

$$K(\{s_{ij},\delta_k\})\, u^{2h-\delta_1-\delta_1'-\delta_2-\delta_2'-\delta_5+s_{24}+\bar{s}_{24}} \, v^{s_{24}+\bar{s}_{24}-s_{45}}$$

where the kernek $K\{s_{ij}, \delta_k\})$ is given explicitly by

$$K(\{s_{ij},\delta_k\}) = \prod_{i=4}^{5}\left[\Gamma\Big(-s_{2i}+\sum_{j>i} s_{ij}\Big)\Gamma\Big(-s_{35}+\sum_{j<i} s_{ji}\Big)\Gamma(\delta_i+s_{2i}+s_{3i})\right]$$

$$\Gamma(-s_{45})\Gamma\Big(h-\delta_1+\sum_{i=4}^{5}(s_{2i}+s_{3i})-s_{45}\Big)\Gamma\Big(\delta_1+\delta_3-h-\sum_{i=4}^{5}s_{2i}\Big)\Gamma\Big(\delta_1+\delta_2-h-\sum_{i=4}^{5}s_{3i}\Big)$$

$$\prod_{k=2}^{3}\Gamma(-\bar{s}_{k4})\Gamma(\delta_1'+\delta_3'-h+\delta_5+s_{25}+s_{45}-\bar{s}_{24})\Gamma(\delta_1'+\delta_2'+\delta_5-h+s_{35}-\bar{s}_{34}) . \tag{13b}$$

$$\cdot\Gamma(\delta_4'+\bar{s}_{24}+\bar{s}_{34}-s_{45})\Gamma(h-\delta_1'-\delta_5-\sum_{k=2}^{3}(\bar{s}_{k4}+s_{k5}))$$

$$\left[\prod_{i=1}^{5}\Gamma(\delta_i)\Gamma(\delta_1'+\delta_5'+s_{25}+s_{35})\Gamma(\delta_2'-s_{25})\Gamma(\delta_3'-s_{35}+s_{45})\Gamma(\delta_4'-s_{45})\right]^{-1}.$$

Here either

i) $\delta_i=-2\alpha_-\alpha_i\,,\quad i=1,2,3,4\,,\quad \delta_i'=\frac{\alpha_+}{\alpha_-}\delta_i\,,\quad \delta_5=2h\,,$ (14a)

or ii) $\delta_i=-2\alpha_-\alpha_i=\delta_i'\,,\qquad \delta_5=-2\alpha_-^2\,,$ (14b)

or iii) $\delta_i=-2\alpha_+\alpha_i=\delta_i'\,,\qquad \delta_5=-2\alpha_+^2\,,$ (14c)

corresponding to the three cases, covered by (13): s=r=2, or s=3, r=1, or s=1, r=3, respectively.

We will again assume that $u<1$, $v<1$ and after performing the integrals in (13) we will take $u\to 0$. Then it is convenient to start with the integrals with respect to s_{14}, s_{34}. This time the poles of the integrand depend in general on the rest of the integration variables, so one has to perfrom a sequence of integrations with the appropriate direction of closing the contours. The computation of the leading terms in the series appearing thus split into several different chains of integrations with a given choice of the poles to be accounted for . Altogether one gets 4 or 3 different terms with exponents (compare with (11a)) $A_{ij}(\{\delta_i\})$ given by

$$A_{ij}=(i-1)\left[h-\delta_1-\delta_2+(i-2)\alpha_-^2\right]+(j-1)\left[h-\delta_1'-\delta_2'+(j-2)\alpha_+^2\right]+2h\,(j-1)(i-1) \tag{15}$$

i,j=1,2, or i=1,2,3, j=1, or i=1, j=1,2,3, for (14a,b,c) respectively.

Accounting for the asymptotic behaviour of the prefactors in (1) , the results in (11b) and (15) are in agreement with the fusion rules (8). We expect that in the general case the explicit expressions for the exponents A_{ij} will be, as in (11b), (15) of the same type for arbitrary h, repeating the structure for 2h=2. This expectation is supported by the observation that the general multiple contour integral representing the 4-point function is the same for every h, with the same structure of the integrand. With the choice (5-7) the value of any δ_i in 2h-dimensions is h times the value of δ_i for 2h=2. Comparing the explicit expressions (10), (13b) (and this holds true in the general case) we see that the whole difference is the scaled by h dependence of the arguments of the Γ–functions $\Gamma(ah+ \sum_i (\pm) s_i)$ (s_i are integration variables). Thus we can further expect that in the generic cases the contributions to any of the constants C_{ij} (and the constants themselves) will maintain this "scaling" structure. Thus the knowledge of the 2h=2 primary fields structure constants eventually provide the values of their analogues for arbitrary h. One has to be cautious, however, about the possibly arising singularities. For instance , in our simplest example with explicitly found constants (11c) the case $\delta_1 = \delta_3 = \delta_4 = -2\alpha\_\alpha_{12}$, $\delta_2 = 2\alpha_0 - \alpha_{12}$ for $p = q - 1 = 3$ (i.e. the analogue of the Ising model $<\sigma\,\sigma\,\sigma\,\sigma>$ correlation) the constant C_2 in front of the contribution of the dimension $d=2\Delta(\alpha_{13}) = h$ explodes, due to a pole of the Γ- function, for every h-even. This is in agreement with the fact that the vertex representations (1) for the 3-point function $<\phi_{\Delta(\alpha_{12})}\,\phi_{\Delta(\alpha_{12})}\,\phi_{\Delta(\alpha_{13})}>$ provided by $\int <<V_{\alpha_{12}} V_{2\alpha_0-\alpha_{12}} V_{\alpha_{13}} V_{\alpha_-}>>$ and $<<V_{\alpha_{12}} V_{\alpha_{12}} V_{2\alpha_0-\alpha_{13}}>>$ differ by an infinite constant , as can be easily checked using

(14). Although there remains still a possibility that this infinite contribution of the primary field of dimension $\Delta(\alpha_{13})$ can be cancelled down by such a contribution from a descendent field in the class of the identity, this effectively will change the fusion rules (8). The operator subalgebra provided by the classes $[\mathbb{1}]$ and $[\phi_{\Delta(\alpha_{21})}]$ is well defined for 2h=4 *).

Presumably such singularities appear for every n-they show up at correlations represented by higher (depending on n) number of volume integrals. We expect that this problem will not arise for the $h \geq 1$ generalization of the $c>1$ models in ref.6, choosing appropriate r, increasing with h.

4. The next to leading singularities of the 4-point functions can be reorganized comparing with the general conformal covariant operator product expansion [10,11] to extract the contribution of descendent conformal covariant fields with dimensions differing by integers from the dimensions of the primary fields. These fields are the analogues of the quasi primary fields for 2h=2, which in a suitable basis span, together with their derivatives, the Virasoro (factor) representations generated by the primary fields. Further partial information about the correlations including such fields can be obtained from the knowledge of the general n-point primary fields functions [11]. However, without the missing algebraic background, there is no way to distinguish the different descendent fields at the same level of dimension (and spin).

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*)The n-point functions and the resulting OPEs are conformal invariant , as far as integer dimensions d_i bigger than h appear, strictly speaking only after going to the Minkowski space (see ref.3). E.g. the 2-point function for $d \rightarrow h$ given by (2a) provides an euclidean conformal invariant distribution only if normalized by a factor (d-h), yielding in the limit $d \rightarrow h$ a δ-function. On the other hand the 2-point Wightman distribution corresponding to (2a) is invariant under the Minkowski space conformal group .

The concrete models discussed here are still rather unrealistic. Apart from the problem of singularities in the fusion rules mentioned above, the most unpleasant feature is that they do not provide for $h>1$ unitary theories. Recall that the OS-positivity condition for the 2-point scalar function requires $d = 2\Delta \geq h-1$ [11]. One possible way to overcome this drawback is to use the vertex representation only as a subsidiary step, constructing composite fields, e.g. of the type $:\psi \exp i\alpha\varphi:$, where ψ is a free field of dimension $h-1$.

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