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OPERATOR PRODUCT ALGEBRAS IN 2D-CONFORMAL THEORIES WITH C < 1 CENTRAL CHARGE

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Operator Product Algebras in 2d Conformal Theories with $c < 1$ Central Charge

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Abstract

A generalized integral representation involving two types of charges is explored to construct correlation functions on the plane for $c = 1 - 6/\, m(m+1) < 1$ discrete unitary Virasoro series. The various local operator product algebras emerging contain (half)-integer spin fields. The examples include also a generalization for arbitrary m of the $\mathbb{Z}_2$- statistics of the Ising model order - disorder fields.

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1. Capelli, Itzykson and Zuber (CIZ) [1] classified all modular invariants corresponding to the minimal conformal theories of [2]. For any value of the central charge they describe one to three different sets of integer spin combinations of the Virasoro representations, which eventually correspond to local operator product algebras (OPAs). However, the classification in [1] does not give information about the concrete content of a given operator product (OP), which content can be extracted if the correlation functions involving the fields are available. The spinless OPAs ((A,A)-case in [1]) have been exhaustively described earlier by Dotsenko and Fateev (DF)[3] and the detailed technical information already contained there should be in principle sufficient,with some generalizations, to yield a complete analysis. Another approach based on the underlying $\hat{SU}(2) \times \hat{SU}(2)$ Kac - Moody structture of the $c < 1$ conformal models has been pursued in [4],[5] .Applying a formal correspondence, some of the CIZ OPAs have been recovered along with various subalgebras. Yet we feel that this approach aiming to translate in terms of correlation functions the rigorous algebraic result in [6] is still quite formal, so an independent direct investigation of the $c < 1$ conformal theories is justified.

Our approach is much in the spirit of that in [5], however, it is more explicit giving the expressions for the (4-point) correlation functions, not just the OP rules. We start from a (volume) integral representation generalizing that in [3] which can be interpreted as arising from an electric-magnetic Coulomb gas on the plane. The monodromy invariance of the correlations relies heavily on the monodromy invariance of their DF-counterparts. The functions obtained are further extended using essentially the monodromy factorization established in [3], thus getting a large class of correlation functions sufficient, e.g., to describe the (D,A) CIZ series. The final results do not differ from those stated in [5], however, there are certain discrepances on which we comment in the text. On the other hand the approach followed here allows the analysis, with essentially the same tools, of a wider class of theories. In particular, local algebras containing half-integer spin fields appear naturally, paralleling the (D,A) integer spin series.

Furthermore, algebras of fields with a definite statistics, generalizing for arbitrary m the $\mathbb{Z}_2$-statistics of the order-disorder Ising model fields, emerge and in principle can be investigated along the same lines.

For simplicity we consider only values of c corresponding to the unitary discrete series [7].

2. We work in the Euclidean region; the coordinates are denoted by $z = x^1 + ix^2$, $\bar{z} = x^1 - ix^2$, $(x^1, x^2) = x \in \mathbb{R}_2$. As usually the equivalence of the noncompact picture to the compact one (i.e. fields defined on the Riemann sphere) is guaranteed by the correct asymptotic behaviour at infinity of the conformally invariant Schwinger functions.

We assume that the scale dimensions $(\Delta_i, \bar{\Delta}_i)$ of the primary fields $\varphi_i(x_i)$ are given by the values selected by the Kac determinant formula [8]

$$\Delta_{rs} = \Delta(\alpha_{rs}) = \alpha_{rs}(\alpha_{rs} - 2\alpha_o) , \quad \alpha_{rs} = \frac{1-s}{2}\alpha_- + \frac{1-r}{2}\alpha_+ \tag{1}$$

$$1 \leq r < m \quad ; \quad 1 \leq s < m+1 \; ; \; \alpha_+ \cdot \alpha_- = -1 ; \alpha_+ + \alpha_- = 2\alpha_o = 1/[m(m+1)]^{\frac{1}{2}}$$

One looks in general for conformally invariant N-point correlation functions $(N \geqslant 3)$ of the type

$$\langle \varphi_1(x_1) \ldots \varphi_N(x_N) \rangle_o = \sum_{\{C\}} \gamma_{\{C\}} \int_{\{C\}} \ldots \int \prod_{i=1}^{s-1} du_i \prod_{i=1}^{\bar{s}-1} dv_i \prod_{i=1}^{r-1} du'_i \prod_{i=1}^{\bar{r}-1} dv'_i \, \cdot \tag{2a}$$

$$\cdot f(\ldots z_i, \alpha_i, \ldots u_j, \alpha_-, \ldots u'_k, \alpha_+ \ldots) \; f(\ldots \bar{z}_i, \bar{\alpha}_i, \ldots v_j, \alpha_-, \ldots v'_k, \alpha_+, \ldots) \; ;$$

$$f(w_1, \alpha_1, \ldots w_p, \alpha_p) = \prod_{i>j} (w_i - w_j)^{2\alpha_i \alpha_j} \; ;$$

$$\sum_{i=1}^{N} \alpha_i + (s-1)\alpha_- + (r-1)\alpha_+ = 2\alpha_o = \sum_{i=1}^{N} \bar{\alpha}_i + (\bar{s}-1)\alpha_- + (\bar{r}-1)\alpha_+ \tag{2b}$$

$\gamma\{C\}$ is an arbitrary constant for any set of contours $\{C\}$, which are either closed, or start and end at some of the points $z_i, \bar{z}_i$.

This integral representation can be related to an electric-magnetic Coulomb gas derived from a generalized vertex representation with a "charge $-2\alpha_0$ at infinity". Each vertex depends on an electric $a=(\alpha+\bar{\alpha})/2$ and magnetic $b=(\alpha-\bar{\alpha})/2$ charge recovering as a particular case the electric Coulomb gas representation of [3];(2b) expresses the conservation of the overall charges, including the screening charges $\alpha_\pm$. The representation allows in general for a second charge at infinity,, leading to two different values of the central charges for the two components of the energy-momentum tensor (see Appendix A).

The two-point function of a field with dimension $d=\Delta+\bar{\Delta}$ and spin $s=\Delta-\bar{\Delta}$ is given up to an arbitrary constant by

$$S(x_1,x_2) = z_{12}^{2\alpha_1\alpha_2}\,\bar{z}_{12}^{2\bar{\alpha}_1\bar{\alpha}_2} = \frac{1}{|z_{12}|^{2d}}\exp -i2\,s\arg z_{12} \tag{3}$$

where $\alpha_2=2\alpha_0-\alpha_1$, $\bar{\alpha}_2 = 2\alpha_0-\bar{\alpha}_1$; $\Delta=\Delta(\alpha_1)=\Delta(\alpha_2)$, $\bar{\Delta}=\Delta(\bar{\alpha}_1)=\Delta(\bar{\alpha}_2)$.

In particular, there are now two scalar fields of dimension $\Delta(\alpha)$ represented by the pairs $[(\alpha,\alpha),\ (2\alpha_0-\alpha,2\alpha_0-\alpha)]$ and $[(\alpha,2\alpha_0-\alpha),\ (2\alpha_0-\alpha,\alpha)]$ which need not to be identified. On the 2-point level the charge conservation condition ensures the symmetry $(\alpha,\bar{\alpha}) \leftrightarrow (2\alpha_0-\bar{\alpha},\ 2\alpha_0-\bar{\alpha})$. Assuming this symmetry (i.e., assuming that any field can be equivalently represented by $(\alpha,\bar{\alpha})$ or $(2\alpha_0-\bar{\alpha},\ 2\alpha_0-\bar{\alpha})$) the charge conservation for the general 3-point function reproduces formally the right and left fusion rules and allows any arbitrary combination consistent with them. However, we cannot expect that the general expression (2) respects this symmetry and (or) to expect that the operator content extracted from (2) is consistent with the fusion rules.

3. We shall start with an important particular case, which can be looked as the straightforward generalization of the DF-integral representation. Define the N-point function of the fields represented by $(\alpha_i,\bar{\alpha}_i)$ with $\sum_{i=1}^{N}\alpha_i - \sum_{i=1}^{N}\bar{\alpha}_i = (= 2\sum_{i=1}^{N} b_i) = 0$ by a formula like (2) with

the contour integrals replaced by a multiple 2-dimensional (volume) integral (in (2a) $v_i \to \bar{u}_i$, $v'_j \to \bar{u}'_j$) , i.e., generalize the vertex representation for the fields keeping the old scalar screening operators. Consider, for example, the 4-point function $S(x_1,x_2,x_3,x_4)$ with $\alpha_1=\alpha_3= =\alpha=\bar{\alpha}_1=\bar{\alpha}_3$, $\alpha_2=2\alpha_0-\beta$, $\bar{\alpha}_2=\gamma$, $\alpha_4=\beta$, $\bar{\alpha}_4=2\alpha_0-\gamma$, to be denoted as

$$\langle(\alpha,\alpha)\ (2\alpha_0-\beta,\gamma)\ (\alpha,\alpha)\ (\beta,\ 2\alpha_0-\gamma)\rangle \equiv \prod_{i>j} z_{ij}^{2\alpha_i\alpha_j}\, \bar{z}_{ij}^{2\bar{\alpha}_i\bar{\alpha}_j}$$

$$\int\cdots\int\prod_{k=1}^{s-1} d^2y_k \prod_{k=1}^{r-1} d^2y'_k\ F(\{x_i, y_j, y'_k\}) \ , \tag{4}$$

the integrand F is recovered from (2) with $u_j = y_j^1+iy_j^2$, $v_j=y_j^1-iy_j^2$, etc.

Recall that the 2-dimensional integration prescription for the DF-analogue of (4), recovered for $\gamma=2\alpha_0-\beta$, ensures automatically the symmetry properties of the correlation under replacement of fields as required by locality. It implies, in particular, that the fields represented by (β,β) and $(2\alpha_0-\beta, 2\alpha_0-\beta)$ can be identified.

Let us first set $\beta=\gamma=\alpha$, i.e., consider a 4-point function of scalar fields of the same dimension $\Delta(\alpha)$ represented now by $(\alpha, 2\alpha_0-\alpha)$and $(2\alpha_0-\alpha,\alpha)$ along with (α,α). The symmetry $x_1 \leftrightarrow x_3$ is ensured iff $2\Delta(\alpha)+2\alpha^2$ is an integer. This implies that either r=1 and s=(m+1)/2 (s=m+3/2), m+1=6(mod 4),or s=1 and r=m/2 (r=(m+2)/2), m=6(mod 4). Let us consider in more detail the first case, i.e., $\alpha=\alpha_{1s}$, s=(m+1)/2 for m+1=6(mod 4). Then it is easily seen using the conformal invariance of (4) that it provides a correlation function symmetric under $x_2 \leftrightarrow x_4$ as well since $S(x_1,x_2,x_3,x_4) = S(x_3,x_4,x_1,x_2) = S(x_1,x_4,x_3,x_2)$. This means that $(2\alpha_0-\alpha,\alpha)$ and $(\alpha, 2\alpha_0-\alpha)$ can be identified as representing one and the same local field. One also has $S(x_1,x_2,x_3,x_4) = S(x_2,x_1,x_4,x_3)$.

4. Our choice of α implies that $c-\bar{c} \equiv 2\alpha_-\alpha_2-2\alpha_-\bar{\alpha}_2 = 2\alpha_-\bar{\alpha}_4-2\alpha_-\alpha_4 \equiv \bar{d}-d$ is an (odd) integer (cf. (2) and also Appendix B for notation), i.e., the exponents of the integrand in (4) differ from the DF-values at most by integers. The integral in (4) can be computed in exactly the same

way as in the DF-correlation by reducing it to a (definite) linear combination of products of multiple contour integrals, obtained by a shift of the $y^{(2)}$-integration contour in the complex $y^{(2)}$- plane (see [9]). There are now only some new relative phase factors in general. The fact that $c-\bar{c} = \bar{d}-d$ is an integer implies that the monodromy properties of the integral in (4) (and hence of the whole correlation) are the same as of its DF-analogue , for which $\bar{c}=c$; i.e., (4) provides a single - valued function of all coordinate differences z_{ij}. The explicit expression can be rewritten in a form exhibiting its asymptotic behaviour in one of the channels. One obtains, up to alternating minus signs in general, the same "diagonal" linear combination as that in [3] , involving the old coefficients but with c, d, replaced by $\bar{c}$, $\bar{d}$, everywhere in the $\bar{z}$-dependent unnormalized contour integrals (see Appendix B for details and explicit examples). The OP content is easily extracted using the results of [3] . In the channel $x_1, x_2 \to x_3, x_4$ it reads (recall that $\alpha = \alpha_{1\,\frac{m+1}{2}}$, $m+1=6(\mathrm{mod}\ 4)$)

$$(\alpha,\alpha) \times (2\alpha_o-\alpha,\alpha) = \sum_{\substack{k=1\\ \text{odd}}}^{m} (2\alpha_o-\alpha_{1k}, \alpha_{1\,m+1-k}) \tag{5}$$

recovering the integer spin combinations (for r=1) of the $(A_{m-1}, D_{\frac{m+3}{2}})$ series. In the channel $x_1, x_3 \to x_2, x_4$ (4) yields in this case just the old, up to sign changes in the coefficients, spinless content of $(\alpha,\alpha) \times (\alpha,\alpha)$.*) Apparently the pairs $[(\alpha,\alpha),\ (2\alpha_o-\alpha,\ 2\alpha_o-\alpha)]$ and $[(2\alpha_o-\alpha,\alpha),\ (\alpha,\ 2\alpha_o-\alpha)]$, each containing equivalent fields, cannot be identified (see also App. B), i.e., a doubling of the scalar field

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*) We have difficulties in understanding the OP (5) and its generalizations following back the tensor product prescription as briefly sketched in [5] - in particular precisely how $(j',j') \times (j,\frac{k}{2}-j)$, $k+2 = m+1$, $j = (s-1)/2$, will give for $j=\frac{k}{4}$ anything but the content of $(j',j') \times (j,j)$. Furthermore, as far as we can see, there is no trace in [5] of the sign changes mentioned, without which (4) would reproduce, up to trivial prefactors in general, the old DF-correlations.

of dimension $\Delta = \Delta(\alpha_{1\,\frac{m+1}{2}})$ has emerged (see also [5]). Preliminary analysis of the local 3-point functions involving $(\Delta_{1k}, \Delta_{1m+1-k})$ $(\Delta_{1m+1-k}, \Delta_{1k})$ with values of k yielding an odd spin suggests that we are actually dealing with complex fields, i.e., (α,α) and $(2\alpha_0 - \alpha, \alpha)$ might be thought to represent here a scalar field and its complex conjugation. This interpretation should be finally desided upon after reconciling the sign changes mentioned above with the Osterwalder-Schrader (reflection) positivity of the correlations. It might be easier to demonstrate the positivity using the explicit expressions in the channel implying (5).

It is easily seen that the result in (5) does not essentially change if we apply (4) with $\alpha = \alpha_{1s}$, s-odd, $\beta = \gamma = \alpha_{r\,\frac{m+1}{2}}$, m+1=6(mod 4). The general integer spin combinations represented by $(2\alpha_0 - \alpha_{rk}, \alpha_{r\,m+1-k})$ or $(\alpha_{rk}, 2\alpha_0 - \alpha_{r\,m+1-k})$, k-odd, appear in the R.H.S. of the extension of (5) with summation running from $|s - \frac{m+1}{2}|+1$ to $\min(s+\frac{m+1}{2}-1,\ 2m+1-s-\frac{m+1}{2})$. It is clear that for all $r \geqslant 1$ the scalar fields represented by $(2\alpha_0 - \alpha_{r\,\frac{m+1}{2}}, \alpha_{r\,\frac{m+1}{2}})$ and $(\alpha_{r\,\frac{m+1}{2}}, \alpha_{r\,\frac{m+1}{2}})$ cannot be identified.

5. We can further use (4) choosing β, γ to represent the newborn fields in (5). Note that the choice $\alpha = \alpha_{1s}$, $\beta = \alpha_{rk}$, $\gamma = \alpha_{r\,m+1-k}$; m+1-even, ensures the symmetry of (4) under $x_1 \leftrightarrow x_3$, for s-odd, irrespectively of the values of k, since $2\alpha(2\alpha_0 - \beta - \gamma) = (m-2r)(1-s)/2$. Then $S(x_1,x_2,x_3,x_4)$ is symmetric (antisymmetric) under $x_2 \leftrightarrow x_4$, $x_1 \leftrightarrow x_3$ (and hence under $x_2 \leftrightarrow x_4$) if k-odd (even), m+1=6(mod 4), or k-even (odd) m+1=4(mod 4), because the phase of the factor involving x_{24} (cf. (2))

$$2(2\alpha_0 - \beta)\beta - 2\gamma(2\alpha_0 - \gamma) = (m-2r)\frac{m+1-2k}{2} = 2S_{rk} \qquad (6)$$

is an even (odd) integer. These values of k and m+1 correspond to integer (respectively half-integer) spin S_{rk} of the field represented by $(2\alpha_0 - \beta, \gamma)$ or $(\beta, 2\alpha_0 - \gamma)$.

Since the difference $\bar{c} - c = d - \bar{d} = 2\alpha_-(\gamma + \beta - 2\alpha_0) = 2r-m$ is an integer for this choice of β and γ the monodromy invariance of (4) is again inherited from the monodromy invariance of the corresponding DF-correlation with $\gamma = 2\alpha_o - \beta$. Thus (4) provides for $\alpha = \alpha_{1s}$, s-odd and $\beta = \alpha_{rk}$, $\gamma = \alpha_{r\ m+1-k}$, m+1 - even, local correlation functions involving integer or half-integer spin fields depending on the parity of k and m+1 /2. We see already at this level that half-integer spin analogs of the (D,A) CIZ series naturally appear. Actually the series for m+1=4(mod 4) starts, unlike the (A,D) case, already at m+1=4; it includes $(\Delta_{13},0)$, $(0,\Delta_{13})$; $(\Delta_{13},\Delta_{13})$ and the identity. (On the other hand the correlation provided by (4) for $\alpha = \alpha_{13}$, $\beta = \alpha_{12} = \gamma$, m+1=4, coincides with its DF-counterpart.)

The formula (4) can be slightly generalized further to include a second field represented by $(\alpha_{rk'},\ 2\alpha_o - \alpha_{r\ m+1-k'})$ or $(2\alpha_o - \alpha_{rk'}, \alpha_{r\ m+1-k'})$ like, e.g., the correlation

$$\langle (\alpha,\alpha)(2\alpha_o - \alpha_{rk}, \alpha_{r\ m+1-k})(\alpha,\alpha)(\alpha_{rk'},\ 2\alpha_o - \alpha_{r\ m+1-k'}) \rangle \qquad (7)$$

where $\alpha = \alpha_{1s'}$, $2s' = 2s + k - k'$, $k-k'$ - even. The monodromy invariance is preserved since $2\alpha_2\alpha_4 - 2\bar{\alpha}_2\bar{\alpha}_4 = \frac{m-2r}{2}(m+1-k-k')$ is still an integer.

The explicit expression for the correlation function parallels again the DF result leading to the OP content

$$(\alpha_{1s'}, \alpha_{1s'}) \times (2\alpha_o - \alpha_{rk}, \alpha_{r\ m+1-k}) = \sum (2\alpha_o - \alpha_{rt}, \alpha_{r\ m+1-t}) \qquad (8)$$

while in the channel $x_1, x_3 \to x_2, x_4$ one recovers the old scalar content

$$(2\alpha_o - \alpha_{rk}, \alpha_{r\ m+1-k}) \times (\alpha_{rk'}, 2\alpha_o - \alpha_{r\ m+1-k'}) = \sum_{odd} (2\alpha_o - \alpha_{1p}, 2\alpha_o - \alpha_{1p}) \qquad (9)$$

The limits of summation in (8,9) are determined by the combination of the usual fusion rules for the two pairs in a given channel; in (8) t and k have the same parity. These formulae cover all different cases with the parity of k and (m+1)/2 related as explained above.

6. Everything above can be repeated with $\alpha=\alpha_{1s}$ replaced by α_{r1}, ets., accordingly all restrictions on the values of m+1 carry over to the values of m. The integral in (4) in all our examples up to here is very much of the type one expects in the $\hat{SU}(2)$ - case since it involves one of the screening operators (represented by (α_+,α_+) or (α_-,α_-)) but never both. And the results about the integer spin OPAs (for r=1 up to now) are up to the subtleties mentioned above essentially the same as those stated in [5]. Let us restrict to the case r=1 everywhere. If m+1 = 6(mod 4) the integer spin algebra is exactly the subalgebra which can be extracted from the CIZ result with the scalar field of dimension $\Delta_{1\,\frac{m+1}{2}}$ doubled. The (half)-integer counterpart of (D,A) for m+1=6(mod 4) consists of

$$\left\{(\Delta_{1s}\,,\,\Delta_{1s}\,),\ \text{s-odd};\ (\Delta_{1k},\,\Delta_{1\ m+1-k}),\ \text{k-even};\ m+1=6(\text{mod }4)\right\} \qquad (10)$$

For m+1=8(mod 4), we again recover the $(A_{m-1}, D_{\frac{m+3}{2}})$ subalgebra, the scalar field of dimension $\Delta_{1\,\frac{m+1}{2}}$ represented now only by $(2\alpha_0-\alpha,\alpha)\sim(\alpha,2\alpha_0-\alpha)$, $\alpha=\alpha_{1\,\frac{m+1}{2}}$. Its products with the scalar fields α_{1s}, s-odd and the integer spin fields are exhibited in (8,9) with k', t - even.

The (half)-integer spin subalgebra for m+1=4(mod 4) consists of

$$\left\{(\Delta_{1s},\,\Delta_{1s}),\ \text{s-odd};\ (\Delta_{1k},\Delta_{1\ m+1-k}),\ \text{k-odd};\ m+1=4(\text{mod }4)\right\} \qquad (11)$$

In both (half)-integer spin subalgebras the scalar fields are represented in the correlation functions by $(\alpha_{1s},\alpha_{1s})$.

Clearly, although (4) was sufficient for the complete analysis of these subalgebras, there are certainly other local correlation functions involving, say, only nontrivial spin fields, which can be obtained solely through the general representation (2). Such correlations, if allowed, might in principle enlarge the (half)-integer algebra (10) adding nonzero integer spins as well.

7. Our next step is to generalize the results for r>1, trying to reach at least the generality we had for r=1 subalgebras. We quit at this point

the volume integral representation (4) and work directly with the 1-dimensional contour integrals.

The idea is to extend any of the local correlations already constructed, starting from (4), to a related correlation which differs only by a change of the type $(\alpha_{1s},\alpha_{1s})\to(\alpha_{rs},\alpha_{rs})$, or $(2\alpha_o-\alpha_{rk},\alpha_{r\,m+1-k})\to(2\alpha_o-\alpha_{r'k},\alpha_{r'm+1-k})$, etc.. Thus, e.g., one can get a scalar correlation, of fields with dimensions Δ_{rs}, $\Delta_{r'\frac{m+1}{2}}$, s-odd, m+1 - even, to be denoted $\langle\!\langle(\alpha_{rs},\alpha_{rs})(2\alpha_o-\alpha_{r'\frac{m+1}{2}},\alpha_{r'\frac{m+1}{2}})(\alpha_{rs},\alpha_{rs})(\alpha_{r'\frac{m+1}{2}},2\alpha_o-\alpha_{r'\frac{m+1}{2}})\rangle\!\rangle$, starting from its counterpart with r=1.

Leaving some details to App.B *) let us only sketch the idea, which is essentially the one used in the algorithm of [5]. The crucial point established in [3] is the factorization of the monodromy transformations of the general convolution integrals in [3]. Then the monodromy invariance of the extended expression is a consequence of the invariance of its simpler counterpart built via (4) and the invariance of the DF-correlations. The OP relations one gets just incorporate the r-dependence according to the rules in [3]; e.g., in our example

$$(\alpha_{rs},\alpha_{rs})\times(2\alpha_o-\alpha_{r'\frac{m+1}{2}},\alpha_{r'\frac{m+1}{2}})=\sum_{|r-r'|+1}\ \sum_{|s-\frac{m+1}{2}|+1}(2\alpha_o-\alpha_{jk},\alpha_{j\,m+1-k}) \tag{12}$$

with the usual, fusion rules dictated, upper bounds. We shall not write down all generalizations of (8,9) (see [5]), let us only stress that they again apply to the half-integer spin fields as well. The results for the OPAs generalize accordingly to these more general OP relations, recovering the (A,D) series along with their half-integer spin counterparts.

8. The natural question arises can (4) be of some use for the investigation of OPAs corresponding to the exceptional CIZ series. We haven't much to say on this at present, let us only point out that it is rather

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* A more detailed and exhaustive presentation will be given elsewhere.

trivial to find local 4-point correlations accommodating some fields among the set characterizing any of the (A,E) series. Indeed, there are combinations of the charges, which ensure the conservation of the total charge, without the need of introducing screening operators and hence integrals. Let us give an example for m+1=18, corresponding to the (A_{16},E_7) case. The correlation

$$\langle(\alpha_{1\,15},\alpha_{19})(2\alpha_o-\alpha_{r\,17},\alpha_{r1})(\alpha_{13},\alpha_{19})(\alpha_{r1},2_{\,o}-\alpha_{r\,17}) \qquad (13)$$

$$= \prod_{i>j} z_{ij}^{2\alpha_i\cdot\alpha_j}\; \bar{z}_{ij}^{2\bar{\alpha}_i\cdot\bar{\alpha}_j}$$

$(\alpha_1=\alpha_{1\,15}\;,\;\bar{\alpha}_1=\alpha_{19}$, etc.) is built according to (2) with only coordinate factors surviving. Correspondingly, there is only one field in any of the channels; here - $(\Delta_{1\,17},\Delta_{1\,17})$, $(\Delta_{r3},\Delta_{r9})$ or $(\Delta_{r\,15},\Delta_{r9})$, all of which belong again to the set characteristic for (A_{16},E_7). This simple mechanism provides lots of examples of local 4-point (and non-zero 3-point) correlations for any of the (A,E) cases . It can be generalized to include less trivial correlations - in our example $(\alpha_{1\,15},\alpha_{19})\to(\alpha_{1'15},\alpha_{1'9})$,etc.. We shall not go further here since the results are inconclusive as far as the full description of the OPAs is concerned.

9. Finally let us point out one more class of correlation functions which can be constructed starting from formulae like((4). The crucial point in all our examples was the observation that the monodromy invariance of the integral in (4) was ensured by the invariance of its DF-counterpart since the exponents in both cases differ by integers (the overall values being not changed). This allowed actually to carry over with small modifications the results in [3] to our case. All the various possibilities accommodated in (4) provided single valued coordinate dependent prefactors preserving this invariance. Let us now abandon this last condition, but keep the crucial property, providing the link to the DF results. Then we get at most a definite valued with respect to any of the coordinates z_{ij} expression. Consider for example

(4) with $\alpha = \alpha_{1s} = \beta = \gamma$, $s=(m+1)/2$, $m+1=4 \pmod 4$. With this choice $c-\bar{c} = \bar{d}-d$ is an integer again, while $2\Delta(\alpha)+2\alpha^2$ is a half-integer. Then (4) gives a double valued in $\dot{z}_{12}, z_{14}, \dot{z}_{23}, z_{34}$ function, which changes sign when $x_1 \leftrightarrow x_3$ (or $x_2 \leftrightarrow x_4$). The simplest example is provided by the value $m+1=4$, i.e., the Ising model, studied exhaustively in the literature [2],[10] . Thus (4) gives for any $m+1=4 \pmod 4$ a correlation having essentially the properties of the Ising model order-disorder correlation $\langle \sigma \mu \sigma \mu \rangle$, the disorder field being represented by $(2\alpha_0 - \alpha, \alpha)$ or $(\alpha, 2\alpha_0 - \alpha)$, $\alpha = \alpha_{1\,\frac{m+1}{2}}$. Formulae like (4) determine all the other mixed functions. The OP content of $(\alpha_{1\frac{m+1}{2}}, \alpha_{1\frac{m+1}{2}}) \times (2\alpha_0 - \alpha_{r\frac{m+1}{2}}; \alpha_{r\frac{m+1}{2}})$ is given again by a formula like (5) (but now $m+1=4 \pmod 4$), i.e., it yields the half-integer spin fields $(\Delta_{rk}, \Delta_{r\,m+1-k}) \oplus (\Delta_{r m+1-k}, \Delta_{rk})$, $k=1,3,\ldots,(m-1)/2$, which now enter a quasi-local OPA. It should be mentioned that such representations appear along with $(\Delta_{r\frac{m+1}{2}}, \Delta_{r\frac{m+1}{2}})$ in the partition functions, invariant under a subgroup of the modular group, consistent with $\mathbb{Z}_2$-twisted boundary conditions [11] . However, the OPAs appearing here, presumably are inevitably larger, than those extracted from the partition functions on the torus.

As in our previous examples one can further enlarge the class of functions covered by (4), getting definite valued expressions (one-, or two-valued , depending on the coordinate differences). We shall not pursue this aim here. Note that a similar series exists for $m+1=6 \pmod 4$. Consider, e.g., (4) with $\alpha = \alpha_{1s}$, s-even and $\beta = \gamma = \alpha_{r\frac{m+1}{2}}$. Repeating everything with the screening operator (α_+, α_+) in (4), one further covers all m-even values.

While this work was in progress we received [10] where the order-disorder vertex representation of [12] has been used in a somewhat different way for the construction of all the Ising model mixed n-point functions.

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Appendix A

Let us define the Schwinger function of N fields $V_i(x_i)$ in the region of arguments

$$x_1^{(2)} > x_2^{(2)} > \dots > x_N^{(2)} \tag{A.1}$$

by the generalized vertex representation

$$V_{[a_1,b_1]}(x_1)\dots V_{[a_N,b_N]}(x_N) = \frac{\int d\mu(\lambda,S)\,\exp J_{a,b}(\lambda,S)}{\int d\mu(\lambda,S)}, \tag{A.2a}$$

$$d\mu(\lambda,S) = D\lambda\, DS\, \exp\int d^2x\left\{-\frac{1}{2g}S_\mu(x)\,S_\mu(x) + i\varepsilon_{\mu\nu}\partial_\mu S_\nu(x)\,\lambda(x)\right\},$$

$$J_{a,b}(\lambda,S) = \int d^2x\left[J^a(x)\,\lambda(x) + \tilde{J}^b_\mu(x)\,S_\mu(x)\right],$$

$$J^a(x) = i\sum_{j=0}^{N} a_j\,\delta(x-x_j),\quad \tilde{J}^b_\mu(x) = i\sum_{j=0}^{N} b_j\int_{C_j}\delta(x-\xi)\,d\xi_\mu \tag{A.2b}$$

The contour integrals in (A.2b) go from x_j to ∞. The "electric" and "magnetic" charges a_i, b_i, $i=1,\dots N$, are real, $a_o = 2i\alpha_o$, $b_o = 2i\beta_o$, α_o, β_o - real. The expression (A.2) is understood as a limit when $x_o = R$ goes to infinity (choosing $R^{(2)} > x_1^{(2)}$). It can be given a meaning if the charges satisfy the conservation condition

$$\sum_{j=1}^{N} a_j - 2\alpha_o = 0 = \sum_{j=1}^{N} b_j - 2\beta_o \tag{A.3}$$

For $\alpha_o = 0 = \beta_o$ this vertex representation has been proposed in [12], and lattice versions have been explored earlier, see, e.g., [13]. (Actually (A.2) is a slight modification, excluding unwanted self-energy terms in [12].) The choice $b_j = 0$, $j = 0,1,\dots N$, reproduces the representation used in [3]. Similar representation emerges naturally in field theories on a torus [14]. It can be looked upon as providing a lagrangian formulation of the operator on-shell vertex constructions; e.g., $a_j = \pm b_j$ recovers chiral fields, etc.

Using a standard Gaussian integration technique (see [12] for details) one recovers from (A.2) the general "electric-magnetic" Coulomb gas representation exploited widely in statistical physics (see e.g. [15] and earlier references therein):

$$\langle V_{[a_1,b_1]}(x_1) \dots V_{[a_N,b_N]}(x_N) \rangle = \exp \frac{i}{2}\sum_{j=0}^{N} k_j a_j b_j \cdot \tag{A.4a}$$

$$\cdot \exp \frac{1}{2\pi g} \sum_{i>j}^{N} \left\{ a_i a_j \ln|z_{ij}| + g^2 b_i b_j \ln|z_{ij}| + ig(a_i b_j \arg z_{ij} + a_j b_i \arg z_{ji}) \right\}$$

$$= \prod_{i>j}^{N} \left(z_{ij}^{2\alpha_i\alpha_j} \bar{z}_{ij}^{2\bar{\alpha}_i\bar{\alpha}_j} \exp -\frac{i}{2} a_j b_i \operatorname{sign} x_{ij}^{(2)} \right) \exp \frac{i}{2} \sum_{j=0}^{N} k_j a_j b_j ,$$

$$z_{ij} = z_i - z_j \; ; \alpha_i \equiv \frac{1}{\sqrt{8\pi g}} (a_i + g\, b_i), \; \bar{\alpha}_i \equiv \frac{1}{\sqrt{8\pi g}} (a_i - g\, b_i) \tag{A.4b}$$

We rescale $gb \to b$ and set $\sqrt{8\pi g} = 1$ in what follows. We have omitted in (A.4a) an overall renormalization constant as well as a constant phase factor trivial for $\beta_0 = 0$ - it could be avoided using a less symmetric definition of the charges at infinity. In getting (A.4) one considers each $V_{[a,b]}$ as a bilocal field defined as the product of the electric and magnetic parts; the limit of coinciding arguments depends on the initial mutual position of the two points as well as on the contour C_j which might wind around them. The ambiguity created is accounted for by the integers k_j in the phase factor in (A.4a). It has been used in [12] to get the right statistics. We assume such a limiting procedure, that all $k_j = 0$ and consider (A.4) in the region (A.1) (or any other region of this type). The resulting expression in which the phase factors in (A.4) depending on sign $x_{ij}^{(2)}$ reduce to a constant, will be extended beyond (A.1). For N=2 one gets in this way up to a constant the conformal invariant function

$$\langle V_{(\alpha,\bar{\alpha})}(x_1)\, V_{(2\alpha_0-\alpha,2\alpha_0-\bar{\alpha})}(x_2)\rangle = \frac{1}{(x_{12}^2)^{\Delta+\bar{\Delta}}} \exp -2i(\Delta-\bar{\Delta})\arg(z_{12}) \qquad (A.5)$$

with $\Delta=\alpha(\alpha-2\alpha_0-2\beta_0)$, $\bar{\Delta}=\bar{\alpha}(\bar{\alpha}-2\alpha_0+2\beta_0)$; here and in what follows $(\alpha,\bar{\alpha})\equiv[a,b]$ where $\alpha,\bar{\alpha},a,b$ are related according to (A.4b).

From the Ward identities for the 3-point functions involving the modified energy-momentum tensor one obtains the values of the central charges in the model

$$c = 1 - 24(\alpha_0+\beta_0)^2\ ,\quad \bar{c} = 1 - 24(\alpha_0-\beta_0)^2 \qquad (A.6)$$

One is forced to set $\beta_0 = 0$ to ensure the invariance under space reflection.(The energy-momentum tensor components depend linearly on $\alpha_0 \pm\beta_0$.) However, (A.2) with $\alpha_0\cdot\beta_0 \neq 0$ might be useful in more general theories.

The second ingredient of the DF-algorithm apart from the charge at infinity, leading to $c<1$, is the introduction of the "screening" operators $V_{(\alpha_\pm,\alpha_\pm)}$ ($\alpha_\pm = \alpha_0 \pm\sqrt{\alpha_0^2+1}$) with scale dimensions $\Delta= 1 =\bar{\Delta}$, or of the chiral vertices with $\Delta=1$, $\bar{\Delta}= 0$ ($\Delta= 0$, $\bar{\Delta}= 1$) provided by $V_{(\alpha_\pm,0)}$, $V_{(0,\alpha_\pm)}$ ($(\alpha,0) =[\alpha,\alpha]$, etc., see above). Note that both screening conditions have more solutions here, namely $V_{(\alpha_\pm,\alpha_\mp)}$ and $V_{(\alpha_\pm,2\alpha_0)}$ $(V_{(2\alpha_0,\alpha_\pm)})$; all of these can be effectively replaced by the basic combinations $V_{(\alpha_\pm,0)}, V_{(0,\alpha_\pm)}$. Then (2) corresponds to the generalized vertex representation (A.2) ($N\geq 3$) taking arbitrary number of screening vertices $V_{(\alpha_-,0)}$, $V_{(0,\alpha_-)}$, $V_{(\alpha_+,0)}$, $V_{(0,\alpha_+)}$. The variables (u,v) replace $y^1 \pm iy^2$ when y^2 is continued in $\mathbb{C}$. It is assumed that the correlation in (2) is computed according to (A.4) - all phases are neglected (taking,say, a sum with appropriate coefficients over different regions of arguments) and the resulting expression is then integrated. Similarly one recovers the volume integral representation (4) using the screening vertices $V_{(\alpha_\pm,\alpha_\pm)}(y)$:

$$\langle \varphi_1(x_1)..\varphi_N(x_N)\rangle_o = \int..\int \prod_{j=1}^{s-1} d^2y_j \prod_{j=1}^{r-1} d^2y_j' \langle V_{(\alpha_1,\bar\alpha_1)}(x_1)..V_{(\alpha_N,\bar\alpha_N)}(x_N) \prod_{j=1}^{s-1} V_{(\alpha_-,\alpha_-)}(y_j) \prod_{j=1}^{r-1} V_{(\alpha_+,\alpha_+)}(y_j')\rangle \tag{A.7}$$

$$\left(\sum_{j=1}^{N} b_j = 0 \right)$$

Clearly this electric-magnetic Coulomb gas interpretation of the integral representations (2) or (4) is rather subtle but the original approach in [12] seems to be unapplicable directly here because of the charge $2\alpha_o$ at infinity.

Appendix B

1. The general expression (4) reads for $\alpha = \alpha_{1s}$

$$\langle(\alpha,\alpha)(2\alpha_o-\beta,\gamma)(\alpha,\alpha)(\beta,2\alpha_o-\gamma)\rangle = \text{const} \prod_{i>j} z_{ij}^{2\alpha_i\alpha_j}\, \bar z_{ij}^{2\bar\alpha_i\bar\alpha_j} \int\ldots\int \prod_{j=1}^{s-1} d^2u_j \tag{B.1}$$

$$\prod_{i<j}|u_{ij}|^{4\delta} \prod_j (z_1-u_j)^a(\bar z_1-\bar u_j)^{\bar a}(z_2-u_j)^c(\bar z_2-\bar u_j)^{\bar c}(z_3-u_j)^b(\bar z_3-\bar u_j)^{\bar b}(z_4-u_j)^d(\bar z_4-\bar u_j)^{\bar d}$$

where

$$a = \bar a = b = \bar b = 2\alpha_-\alpha\,,\quad c = 2\alpha_-(2\alpha_o-\beta);\quad \bar c = 2\alpha_-\gamma;\quad 2\delta = 2\alpha_-^2$$

$$a+b+c+d+2(s-2)\delta = -2 = \bar a+\bar b+\bar c+\bar d+2(s-2)\delta \tag{B.2}$$

$$(\alpha_1 = \alpha = \bar\alpha_1, \alpha_2 = 2\alpha_o-\beta, \bar\alpha_2 = \gamma, \text{etc.})$$

The nontrivial phases of the prefactor in front of the integral in (B.1) for $\beta = \alpha_{rk}$, $\gamma = \alpha_{r\ m+1-k}$ are equal to

$$2\alpha_1\alpha_2 - 2\bar\alpha_1\bar\alpha_2 = \frac{1-s}{2}(m-2r) = \frac{1-s}{2}(c-\bar c) = 2\bar\alpha_1\bar\alpha_4 - 2\alpha_1\alpha_4$$

$$2\alpha_2\alpha_4 - 2\bar\alpha_2\bar\alpha_4 = \frac{m+1-2k}{2}(c-\bar c) = 2S_{rk} \tag{B.3}$$

Denoting by $I(z,1-z)$, $z = \frac{z_{12}z_{34}}{z_{13}z_{24}}$, the appropriate limit of the integral in (B.1) for $x_4 \to \infty, x_3 \to (1,0)$, $x_1 \to 0$, we get instead of (B.1)

$$\frac{(x_{13}^2)^{-2\Delta(\alpha)}}{z_{42}^{2\Delta(\beta)}\bar{z}_{42}^{2\Delta(\gamma)}} \left[z(1-z)\right]^{2\alpha(2\alpha_o-\beta)}\left[\bar{z}(1-\bar{z})\right]^{2\alpha\gamma} I(z,1-z) \qquad (B.1')$$

The miltiple 2-dimensional integral in (B.1') is reduced to contour integrals as explained in [9], giving (for $c-\bar{c}$ - integer)

$$I(z,1-z) = \sum_{k=1}^{S} e^{i\pi(c-\bar{c})(k-1)} J_k(b,a,c;1-z,z)\ \overline{\tilde{I}_{s+1-k}(a,b,\bar{c};z,1-z)} \qquad (B.4)$$

where (for $z \notin [1,\infty)\cup(-\infty,0]$)

$$J_k(b,a,c;\underbrace{1-z,\ldots,1-z}_{s-k};\underbrace{z,\ldots,z}_{k-1}) = s_k(a,b)e^{-i\delta\varphi_k}\int_{-\infty}^{0}\prod_{j=1}^{s-k} dv_j \int_{1}^{\infty}\prod_{j=1}^{k-1} du_j\ g_{k-1,s-k}(u,v), \qquad (B.5a)$$

$$g_{m,n}(u,v) = \prod_{j=1}^{m} u_j^a(u_j-1)^b(u_j-z)^c \prod_{j=1}^{n}(-v_j)^a(1-v_j)^b(z-v_j)^c \prod_{i<j} u_{ij}^{2\delta} \prod_{i<j} v_{ij}^{2\delta} \prod_{i,j}(u_i-v_j)^{2\delta}$$

$$s_k(a,b) = \prod_{j=0}^{s-k-1} s(a+j\delta)\prod_{i=0}^{k-2} s(b+i\delta),\quad s(a)=\sin\pi a;\quad \varphi_k = \binom{s-k}{2}\pi + \binom{k-1}{2}\pi$$

$$\tilde{I}_{s+1-k}(a,b,c;\underbrace{z,\ldots z}_{s-k};\underbrace{1-z,\ldots 1-z}_{k-1}) = s_k(\delta,\delta)^{-1}\left[s(\delta)\right]^{s-1} e^{-i\delta\varphi_k}\int_{0}^{z}\prod_{j=1}^{s-k} du_j \int_{z}^{1}\prod_{i=1}^{k-1} dv_i\ \bar{h}_{s-k,k-1}(u,v) \qquad (B.5b)$$

$$\bar{h}_{m,n}(u,v) = \prod_{j=1}^{m} u_j^a(1-u_j)^b(z-u_j)^{\bar{c}} \prod_{i=1}^{n} v_i^a(1-v_i)^b(v_i-z)^{\bar{c}} \prod_{i<j} u_{ij}^{2\delta} \prod_{i<j} v_{ij}^{2\delta} \prod_{i,j}(v_i-u_j)^{2\delta}$$

and the integrals in (B.5) are contour integrals in the complex $u_j(v_i)$ plane; the contours C_j are ordered in such a way that (for z-real) the differences u_{ij}, $i<j$; $v_{i'j'}$, $i'<j'$, have nonnegative imaginary parts. The $(z,1-z)$ dependence of the integrals (B.5) is made clear if a change of variables $v_i \to 1-v_i$ is performed.

For $c = \bar{c}$ (B.4,5) reproduces the DF result in a form, in which the invariance under the change $(z,\bar{z}) \leftrightarrow (1-z, 1-\bar{z})$ is automatic, but the monodromy as well as the asymptotic properties are not explicit. Deforming the contours, (B4) can be brought to a form (analogous to the diagonal form of [3] for $\bar{c}=c$) in terms of integrals having simple monodromy properties around one of the points $0, 1, \infty$. Transforming $\tilde{I}_J(\bar{c})$ in (B.4) brings in sign changes due to $\sin\pi(x+\bar{c}) = (-1)^{\bar{c}-c} \sin\pi(x+c)$.

2. Let us first apply (B.1,4) to $\alpha = \alpha_{13} = \beta = \gamma$; $m+1=6$. We obtain up to (single-valued) prefactors

$$\sim \left\{ I_1(a,a,c;z)\overline{I_1(a,a,\bar{c};z)} + I_3(c;z)\overline{I_3(\bar{c};z)} + e^{i\pi(\bar{c}-c)} 2 I_2(c,z)\overline{I_2(\bar{c},z)} \right\} \tag{B.6}$$

Now we have used the DF-notation for the s=3 independent multiple contour integrals $I_k(a,a,c;z) = I^{1s}_{1k}(a,a,c;z)$ given in general by (cf. [3])

$$I^{rs}_{lk}(a',b',c',a,b,c;z) = C_{lk}(\delta,\delta') \int_1^\infty \prod_{j=1}^{s-k} du_j \int_0^z \prod_{j=1}^{k-1} dv_j \int_1^\infty \prod_{j=1}^{r-l} du'_j \int_0^z \prod_{j=1}^{l-1} dv'_j \; g_{s-k}(u) h_{k-1}(v) \cdot$$
$$\cdot g'_{r-1}(u')\, h'_{l-1}(v') \prod_{i,j}(u_i - v_j)^{2\delta} \prod_{i,j}(u'_i - v'_j)^{2\delta'} \delta(u,v;u',v') \quad , \tag{B.7}$$

$$\delta(u,v;u',v') = \prod_{i,j}(u_j-u_i)^{-2} \prod_{i,j}(u_j-v_i)^{-2} \prod_{i,j}(v_j-u_i)^{-2} \prod_{i,j}(v_j-v_i)^{-2},$$

$$C^{-1}_{lk}(\delta,\delta') = s_k(\delta,\delta)\; s_l(\delta',\delta')\; \exp(i\delta\varphi_k + i\delta'\varphi_l) / s^{s-1}(\delta)\; s^{r-1}(\delta')$$

$$a' = 2\alpha_+\alpha_1,\; b' = 2\alpha_+\alpha_3,\; \text{etc.},\; 2\delta' = 2\alpha_+^2;\; g_m \equiv g_{(m,0)}\;, \text{etc.},\; (\text{cf.}(B.5)),$$

h (h') is defined as in (B.5b) with $a,b,\bar{c} \to a,b,c$ (a',b',c'). For generic values of the parameters the integrals in (B.7) provide a canonical basis for z=0.

Recalling that the leading terms in $I_k(a,b,c,z)$ for $z \to 0$ are given by $\sim z^{A_k}$, $A_k = (k-1)(1+a+c+(k-2)\delta)$, one sees that $I_k(a,b,\bar{c};z)$ and

$I_k(a,b,c;z)$ give contributions which differ by integer powers of z. Note that (B.6) with the prefactors added is not invariant under $x_1 \leftrightarrow x_2$, or, equivalently, under $(\alpha_1, \bar{\alpha}_1) \leftrightarrow (\alpha_2, \bar{\alpha}_2)$ Indeed the correlation function $\langle (2\alpha_0-\alpha,\alpha)(\alpha,\alpha)(\alpha,\alpha)(\alpha,2\alpha_0-\alpha)\rangle$ computed in the same way is given up to prefactors (and a sign $(-1)^{c-\bar{c}}$) by

$$\sim z^{-2\Delta(\alpha)}(\bar{z}\,|1-z|^2)^{2\alpha^2}\sum_{k=1}^{3}\gamma_k\, I_k(c,a,a;z)\,\overline{I_k(\bar{c},a,a;z)} = \tag{B.8}$$

$$= \left[z(1-z)\right]^{-2\Delta(\alpha)}\left[\bar{z}(1-\bar{z})\right]^{2\alpha^2}\sum_{k=1}^{3}\gamma_k\, I_k(a,a,c;z)\,\overline{I_k(a,a,\bar{c};z)}\ ;\gamma_2=2\gamma_1=2\gamma_3=1.$$

We have reexpressed the z-dependent integrals in (B.8) by a change of variables and used that $d=a=\bar{c}$ (cf. (B.2)). Comparing with (B.6) we see that the singlevaluedness in z_{12} still allows for relative sign changes, the monodromy transformation being effectively the "square" of reversing coordinates. It is clear that we cannot identify (α,α) and $(2\alpha_0-\alpha,\alpha)$ preserving the locality. Let us finally rewrite (B.8) in the basis which exhibits the asymptotics for $1-z\to 0$. We get

$$\sim |z|^{-2\Delta(\alpha)}\,|1-z|^{2\alpha^2}\sum_{k=1}^{3} e^{-i\pi(c-c)(k-1)}\,\gamma_k\left|I_k(a,c,a;1-z)\right|^2 \tag{B.8'}$$

The last expression differs from its DF-counterpart only by the relative minus sign in the sum.

3. This example can be generalized for $\alpha=\alpha_{1s}$, s-odd, $\beta=\gamma=\alpha_{r'\frac{m+1}{2}}$, m+1-even. The contour integrals $\{\exp(i\pi(\bar{c}-c)(k-1))\,\tilde{I}_{s+1-k}(a,b,\bar{c};z)\}$ are expressed as a linear combination of the integrals $\{\exp i\pi(\bar{c}-c)(j-1)\ I^{1s}_{1j}(a,b,\bar{c},;z)\}$ in (B.7) with the same coefficients $\tilde{\alpha}_{s+1-k\ j}(a,b,c)$ as if $\bar{c}=c$, since for $\bar{c}-c$ - odd integer (s-odd) $\tilde{\alpha}_{s+1-k\ j}(a,b,\bar{c}) = \exp(i\pi(\bar{c}-c)(k+j))\cdot \tilde{\alpha}_{s+1-k\ j}(a,b,c)$ *). Thus all off-diagonal terms will cancel as in the

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*) This property of $\tilde{\alpha}_{kj}(a,b,c)$ is much easier to get than its full explicit expression, which we actually do not need, if we exploit the fact that the two representations for the DF-correlations (in terms of a volume integral, or as a linear combination of contour integrals) should coincide up to an overall constant, which can be fixed.

DF-case, while the diagonal terms enter with the old DF-coefficients $\gamma_j(a,b,c)$, up to an overall constant; see [3]for the explicit expression. We obtain

$$I(z,1-z) = \sum_{j=1}^{s} \gamma_j(a,b,c)\, e^{i\pi(\bar{c}-c)(j-1)}\, I^{1s}_{1j}(a,b,c;z)\overline{I^{1s}_{1j}(a,b,\bar{c};z)} \qquad (B.9)$$

Using the general relation

$$I^{rs}_{1k}(a',b',c';a,b,c;z) = z^{A(a,a')}(1-z)^{A(b,b')} I^{rs}_{r+1-\ell\; s+1-k}(b',a',d';b,a,d;z), \qquad (B.10)$$

$$A(a,a') = (s-1)(a+c+1+(s-2)\delta + (r-1)(a'+c'+1+(r-2)\delta') - 2(s-1)(r-1),$$

$$d = -2-a-b-c-2(s-2)\delta+2(r-1);\quad d' = -2-a'-b'-c'-2(r-2)\delta' + 2(s-1),$$

one can rewrite (B.9) as

$$[\bar{z}(1-\bar{z})]^{(s-1)(a+\bar{c}+1+(s-2)\delta)} \sum_{j=1}^{s} \gamma_j(a,b,c)\, e^{i\pi(\bar{c}-c)(j-1)} \qquad (B.11)$$

$$\cdot\, I_{1j}(a,b,c;z)\, \overline{I_{1s+1-j}(b,a,\bar{d};z)} .$$

In our case $\bar{d} = c$ (cf.(B.2)) while the exponent of the prefactor in (B.11) is equal to $2\alpha_{1s}(2\alpha_o - 2\bar{\alpha}_2) \equiv 2\alpha_1\alpha_2 - 2\alpha_1\bar{\alpha}_2$.

The case $\alpha = \alpha_{1s}$, s-odd , $\beta = \alpha_{r'k}$, $\gamma = \alpha_{r'm+1-k}$, m+1-even, is treated in exactly the same way. The final expression, picking up the prefactors reads

$$\sim \frac{[z(1-z)]^{2\alpha(2\alpha_o-\beta)}[\bar{z}(1-\bar{z})]^{2\alpha(2\alpha_o-\gamma)}}{(x^2_{13})^{2\Delta(\alpha)} z_{42}^{2\Delta(\beta)} \bar{z}_{42}^{2\Delta(\gamma)}} \sum_{j=1} \gamma_j(a,b,c) e^{i\pi(\bar{c}-c)j}\, I_{1j}(a,b,c;z)\overline{I_{1s+1-j}(b,a,\bar{d};z)}$$

$$(B.12)$$

The integral (B.9), with the prefactors added, can be used as a definition instead of (B.1), exploiting the property $\alpha_{jp}(a,b,\bar{c}) = \exp i(\bar{c}-c)\pi(j+p+s-1)\ \alpha_{jp}(a,b,c)$ of the coefficients relating $\{I_j(a,b,\bar{c};z)\}$ to $\{I_j(b,a,\bar{c};1-z)\}$.

4. The expression (B.12) is our starting point for an extension beyond (B.1).

Let $\alpha=\alpha_{rs}$, $\beta=\alpha_{r'k}$, $\gamma=\alpha_{r'\ m+1-k}$. Define

$$\langle\langle(\alpha,\alpha)(2\alpha_o-\beta,\gamma)(\alpha,\alpha)(\beta,2\alpha_o-\gamma)\rangle\rangle = f(\alpha,\beta,\gamma;\{z_{ij}\})\cdot \tag{B.13}$$

$$\cdot\sum_{j=1}^{s}\sum_{l=1}^{r}\gamma_j(a,a,c)\gamma_l(a',a',c')e^{i\pi(c-\bar{c})j} I^{rs}_{lj}(a',a',c';a,a,c;z)\overline{I^{rs}_{ls+1-j}(a',a',\bar{d}';a,a,\bar{d};z)}$$

where

$$a=2\alpha_-\alpha_{rs}=2\alpha_-\alpha_{1s}+1-r=b,$$

$$c=2\alpha_-(2\alpha_o-\alpha_{r'k}),\quad \bar{c}=2\alpha_-\alpha_{r'm+1-k}$$

$$\bar{d}=2\alpha_-(2\alpha_o-\alpha_{r'm+1-k})=-2-a-b-\bar{c}-(s-2)\delta+2(1-r) \tag{B.14}$$

$$a'=2\alpha_+\alpha_{rs}=b',\quad c'=2\alpha_+(2\alpha_o-\alpha_{r'k}),\quad \bar{c}'=2\alpha_+\alpha_{r'm+1-k}$$

$$\bar{d}'=-2-a'-b'-\bar{c}'-(r-2)\delta'+2(1-s)=2\alpha_+(2\alpha_o-\alpha_{r'm+1-k})=c'+m+1-2k$$

and the prefactor f is the same as that in (B.12) with the new value of α. It is clear from (B.14) that a and b differ from their old values by the integer r-1. Note that $\gamma_j(a+p,b+p,c)=\gamma_j(a,b,c+p)=\gamma_j(a,b,c)$ for p-integer (see[3]). Let r-1 be even. Hence the monodromy coefficients depending on the parameters a,b,c do not change. Then the monodromy invariance of (B.13) is a consequence of the factorization of the monodromy transformations, the invariance of (B.12) and the invariance of the DF-correlations for s=1, since $\bar{d}'=c'+$ even integer (cf.(B.14)). Using (B.10) again (B.13) can be rewritten up to prefactors as

$$\sum_{j,l}\gamma_j(a,a,c)\gamma_l(a,a,c)e^{i\pi(c-\bar{c})j} I^{rs}_{lj}(a',a',c';a,a,c;z)\overline{I^{rs}_{r+1-l\ j}(a',a',\bar{c}';a,a,\bar{c};z)} \tag{B.15}$$

For the scalar function (i.e. $\beta=\gamma=\alpha_{r',\frac{m+1}{2}}$) $\bar{d}'=c'$, $\bar{d}=c$, in (B.13).

Choosing r-odd is sufficient since for r-even, m-r is odd, but actually the argument generalizes for r-even as well.

Let us illustrate (B.15) by an example for m+1 = 6: $\alpha=\alpha_{23}=\beta=\gamma$.
Up to prefactors one has

$$\sim\left[z(1-z)\right]^{-2\Delta(\alpha)}\left[\bar{z}(1-\bar{z})\right]^{2\alpha^2}\Big\{-\gamma' I_{11}(c,z)\overline{I_{21}(\bar{c},z)}-\gamma' I_{13}(c,z)\overline{I_{23}(\bar{c},z)} \tag{B.16}$$

$$-I_{21}(c,z)\overline{I_{11}(\bar{c},z)}-I_{23}(c,z)\overline{I_{13}(\bar{c},z)}+2\left[I_{22}(c,z)\overline{I_{12}(\bar{c},z)}+\gamma' I_{12}(c,z)\overline{I_{22}(\bar{c},z)}\right]\Big\}$$

where $\gamma'=s(a'+b'+c')/s(c')$ and the exact dependence of the integrals on the parameters is as in (B.15); $\bar{c}=d$, $\bar{c}'=d'$ .It further can be rewritten in the form (B.13).

References

1. A.Capelli, C.Itzykson, J.B.Zuber, Nucl. Phys. B 280 , 445 (1987).
2. A.A.Belavin, A.M.Polyakov, A.M.Zamolodchikov, J.Stat. Phys. 34, 763 (1984); Nucl.Phys. B 241 , 333 (1984).
3. Vl.S.Dotsenko, V.A.Fateev, Nucl. Phys. B 240 [FS 12] , 312 (1984); ibid. B 251 [FS 13] , 691 (1985); Phys.Lett. 154B, 291 (1985).
4. P.Christe,R.Flume, Nucl.Phys. B 282, 466 (1987); Phys.Lett. 188B, 219 (1987).
5. P.Christe, Ph.D. thesis Bonn -IR-86-32.
6. P.Goddard, A.Kent, D.Olive, Comm.Math.Phys. 103 , 105 (1986).
7. D.Friedan, Z.Qiu, S.H.Shenker, Phys.Rev.Lett. 52, 157 (1984).
8. V.G.Kac, Lecture Notes in Physics 94, 441 (1979).
9. Vl.S.Dotsenko, Lectures on Conformal Field Theory, RIMS-559 (1986).
10. K.Rehren, B.Schroer, preprint Berlin,July (1987), B.Schroer, preprint CERN-TH.4566/87.
11. J.B.Zuber, Phys.Lett. 176B, 127 (1986).
12. E.C.Marino, J.A.Swieca, Nucl.Phys. B 170[FS 1], 175 (1980).
13. A.M.Polyakov, Lectures at ICTP, Trieste, November 1977 (IC/78/4).
14. P.di Francesco, H.Saleur, J.B.Zuber, Saclay preprint S.Ph-T/87-31.
15. B.Nienhuis, J.Stat.Phys. 34, 731 (1984).