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PERSPECTIVES OF SUPERLUMINAL LORENTZ TRANSFORMATIONS

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ABSTRACT

Difficulties connected with introduction of Superluminal Lorentz transformations both as maps between frames of a single manifold and as maps between two manifolds are briefly discussed. It is suggested that fibre bundles with more than one canonical projection should be used. An example is given in which the bundle manifold is a Lie group.

1. - INTRODUCTION

It happened several times in the history of physics that completion of a mathematical theory by making it more symmetrical led to new important discoveries. Thus it is not surprising that the beautiful symmetry that exists between the subluminal and Superluminal Lorentz transformations in two dimensions (one space and one time)[1] led to many efforts seeking the extension to the four-dimensional case[2].

Mathematically, one seeks an extension of the symmetry between the proper orthochronous and antichronous Lorentz transformations that corresponds to the physi-

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cal symmetry between particles and antiparticles[3]. The extension should consist of including the change of the sign of the metric $g_{ij} \rightarrow - g_{ij}$ within the group, and should correspond to a symmetry between bradyons and tachyons[4, 5]. Realization of such a symmetry as a transformation of frames, however, meets with some difficulties.

We shall review here the most substantial of the difficulties and suggest a generalization of the concept of a frame that could help to avoid them.

2. - SUBLUMINAL AND SUPERLUMINAL FRAMES

Consider a manifold M with a coordinate system x^μ, $\mu = 1, \ldots, n$, and a metric g:

$$g_{\mu\nu}(x) = g_x(\partial/\partial x^\mu, \partial/\partial x^\nu) . \tag{1}$$

Forming the bundle of linear frames on M we can define a basis of the tangent vector space by $(h_i^\mu \partial/\partial x^\mu; \ i = 1, \ldots, n)^{(*)}$, where h_i^μ are invertible $n \times n$ matrices with the inverse h_μ^i. Then the metric of the frame is given by

$$g_{ij} = g_x(h_i^\mu \partial/\partial x^\mu, \ h_j^\nu \partial/\partial x^\nu) = h_i^\mu h_j^\nu g_{\mu\nu}(x). \tag{2}$$

When a passage to another frame is made by means of the action of the general linear group

$$\bar{h}_i^\mu = h_j^\mu a_i^j; \quad a_i^j \in GL(n; \mathbb{R}), \tag{3}$$

the frame metric transforms as

$$\bar{g}_{ij} = a_i^k a_j^\ell g_{k\ell} . \tag{4}$$

A Lorentz frame is characterized by $g_{k\ell}$ being diagonal with the appropriate number of $+1$ and -1 terms on the diagonal. The Lorentz group leaves such a frame metric-invariant, while the Lorentz group extended by Superluminal transformations should contain a transformation that changes the sign:

$$- g_{ij} = a_i^k a_j^\ell g_{k\ell} . \tag{5}$$

$(*)$ The usual summation convention is used throughout.

It is well known that such a transformation exists only for a metric containing e-qual numbers of +1 and -1 terms on the diagonal[4,6], but that simple argument is usually connected in literature with many other considerations so that it may be worthwhile to show it here.

Let g_{ij} be diagonal with s elements +1 and r elements -1, $r+s=n$, and denote the corresponding matrix G. We seek a square matrix A such that

$$A^T G A = -G.$$
(6)

Writing down A in the column notation

$$A = \begin{bmatrix} \underset{\sim}{u}_1, & \ldots \ldots, & \underset{\sim}{u}_n \end{bmatrix},$$
(7)

we have

$$\begin{aligned} u_i^T G \underset{\sim}{u}_j &= 0 && \text{for } i \neq j, \\ u_i^T G \underset{\sim}{u}_i &= -1 && \text{for } i = 1, \ldots, s, \\ u_i^T G \underset{\sim}{u}_i &= +1 && \text{for } i = s+1, \ldots, n. \end{aligned}$$
(8)

If, say, $r > s$, then there exist at most s orthonormal vectors while eq. (8) requires r of such vectors. Similar argument applies to $r < s$. Matrix A of course exists for $r = s$.

Thus for the realistic case of metric (+1, +1, +1, -1) such a straightforward extension of the Lorentz group is not possible. It is one of the reason why during a certain period it became very popular to consider theories based on a six-dimensional space-time with three space and three time coordinates[7,4].

It could be of some interest to note that in principle one could also consider constructions with non-square h^i. Namely, the Latin indices could run from 1 to 6 with g_{ij} being (+1, +1, +1, -1, -1, -1), while the Greek indices referring to the actual space-time manifold would still run from 1 to 4.

The mathematical system corresponding to such a construction would be a principal fibre bundle with a four-dimensional base manifold and $GL(6, \mathbb{R})$ structure, presumably reducible to the extended Lorentz group structure. However, one would need an embedding of the bundle of frames of the base manifold into such a larger bundle, and to define it one would probably encounter the usual difficulties connected with elimination of superfluous dimensions.

3. - TWO SPACE-TIME MANIFOLDS

The approach described in the preceding section was based strictly on transformations of frames in a single manifold with a fixed metric. Indeed, $g_{\mu\nu}(x)$ of eq. (2) was never assumed to change when the frame metric transformed according to eq. (4). An alternative approach is to try to describe the special Superluminal transformation as a map between two space-time manifolds that changes the sign of the metric. Such a map may be non-linear, which may seem to eliminate the difficulties described in the preceding section as well as elsewhere in the literature. However, if the map is a smooth manifold map, then the induced mapping of the tangent vector spaces is a linear map[8]. The metric is defined as a bi-linear product on the tangent vector space, and thus the metric inversion is again required to be carried out by a linear map. We have to conclude that also a non-linear but smooth point-to-point map between two manifolds transforming $g_{\mu\nu}$ into $-g_{\mu\nu}$ is ruled out. It was suggested that a type of mapping called "quasi-catastrophe" should be employed[9], but the program was never carried much beyond the original suggestion.

4. - GENERALIZED FRAMES

The usual construction of the bundle of frames that describes the space-time manifold together with its metric and its parallel translations starts with the base manifold. A fiber composed of frames is attached to each point of the base manifold and the larger bundle manifold is created. A canonical projection in the bundle manifold maps the fibers back to the points of the base manifold. The success of such a description, not only for the space-time structure but also for electromagnetic, weak, and even strong interactions where the construction does not originate from the base manifold, suggests an idea that the actual fundamental space is the bundle manifold.

The canonical projection can be then connected with our physical ability of perceiving such a generalized space. Indeed, the action of the structure group along the fibers of a fiber bundle is directly connected with physical measurements. In the case of a bundle of frames, measurements of angles and velocities correspond to the Lorentz group elements, and in the Poincaré[10] or de Sitter[11] structured bundles also measurements of displacements are added corresponding to the translation group elements. Once the philosophy of the bundle manifold as being fundamental is adopted, it is easy to imagine that some manifolds may have more than one canonical projection, each with its own structure group acting on the corresponding fibers. Each of the projections is thus associated with a class of observers using a common measuring techni

que. For example, one can have two such projections, one associated with bradyonic observers, the other with tachyonic observers. Such a structure is described in the next section.

5. - LIE GROUPS AS BUNDLES WITH MORE THAN ONE PROJECTION

When one looks for examples of fibre bundles with more than one canonical projection, one should consider structures that are geometrically more homogeneous than the bundles artificially constructed from a base manifold and a group. Such structures are Lie groups, since if H is a closed subgroup of a Lie group C and G/H is the corresponding quotient manifold. G can be considered as a principal fibre bundle with the structure group H and the base manifold $G/H^{(8)}$. Using different subgroups one can obtain different projections in the same bundle manifold. For our purpose we may try to have two projections with the Lorentz group being a common subgroup of both structure groups. This reflects the usual assumption that the Lorentz group acts equally on both subluminal and Superluminal frames. The two projections should be then distinguished by the translational part of the structure group. We choose a representation of a Minkowski space by a de Sitter structured bundle in the spirit of ref. (11). This has more desirable properties than a Poincaré structured bundle. In addition to those properties discussed in ref. (11), it is important for our purpose that the de Sitter translations have a well defined non-singular inner product naturally determined by the geometry of the group manifold. Let L_{ij} and P_i, $i, j = 1, ..., 4$, denote the ten generators of the (4, 1) de Sitter group. Then the invariant Killing product of the de Sitter translations is after the appropriate normalization

$$(P_i, P_j) = - g_{ij} , \qquad (9)$$

where g_{ij} is the diagonal Minkowski metric (+1, +1, +1, -1). For the generators L_{ij} and Q_i of the (3, 2) de Sitter group we have

$$(Q_i, Q_j) = g_{ij} . \qquad (10)$$

The simplest group that can accomodate both de Sitter groups as subgroups is the 14-dimensional group generated by L_{ij}, P_i and Q_i with the commutation relations

$$[L_{ij}, L_{k\ell}] = g_{jk}L_{i\ell} + g_{i\ell}L_{jk} - g_{ik}L_{j\ell} - g_{j\ell}L_{ik} ;$$

$$[L_{ij}, P_k] = g_{jk}P_i - g_{ik}P_j ; \quad [L_{ij}, Q_k] = g_{jk}Q_i - g_{ik}Q_j ; \qquad (11)$$

$$[P_i, P_j] = - L_{ij} ; \quad [Q_i, Q_j] = L_{ij} ,$$

and

$$\left[P_i, Q_j \right] = 0 . \tag{12}$$

If its group manifold G is considered as a principal fibre bundle with the structure group H, generated by L_{ij} and P_i and the base manifold G/H, there are two cross-section that describe a Minkowski space in the sense of ref. (11). Namely the four-dimensional abelian subgroups generated by $Q_i + P_i$ and $Q_i - P_i$ respectively. With a connection in G being the invariant connection where the horizontal direction is determined by Q_i we can write the horizontal lift of $\partial_i = Q_i + P_i$ as

$$Q_i = \partial_i - P_i . \tag{13}$$

Similarly for the other cross-section we have

$$Q_i = \partial_i + P_i . \tag{14}$$

The two cross-sections thus correspond to Minkowski spaces connected by space and time inversion. We can also say that the gauge trasnformation that carries one cross-section into the other represents such a discrete space-time transformation, or that the two cross-sections corresponds to particles and antiparticles.

The same group manifold viewed as a principal fibre bundle with the structure group H_2 generated by L_{ij} and Q_i and the base manifold G/H_2 also generates Minkowski spaces using the two cross-sections. However, the translations are now measured in the Q-direction instead of the P-direction, leading to the opposite metric. In this way the group manifold possesses the same kind of $Z(4) \times$ Lorentz group symmetry as described in ref. (4).

The 14-dimensional Lie group described above is not semi-simple. Geometrical studies in the group manifold may thus be hindered by the fact that the group metric is singular. It is, however, easy to extend the group by one more generator D with commutation relations

$$\left[L_{ij}, D \right] = 0 ; \quad \left[P_i, D \right] = Q_i ; \quad \left[Q_i, D \right] = P_i \tag{15}$$

and to replace the relation (12) by

$$\left[P_i, Q_j \right] = - g_{ij} D , \tag{16}$$

constructing the 15-dimensional group $SO(4, 2)$. Also this group considered as a de Sitter structured fibre bundle with the invariant connection generates a four-dimensional Minkowski space in the sense of ref. (11). Although the base manifold is now

five-dimensional, the cross-section generated by $Q_i + P_i$ and D departs from the horizontal direction (given by Q_i and D) only in four dimensions. The fifth dimension defined by D remains unobservable since D is horizontal.

In conclusion we should stress that there is no hope of deriving from the above construction any kind of direct mapping (linear or non-linear) between two Minkowski spaces with opposite metrics. In order to arrive at some physically interpretable conclusions one will have to define, at first, an event in the group manifold, and then derive the description of the event by the two kinds of observers whose observational possibilities reduce it to an event in a Minkowski space.

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