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Abstract. Starting from finite temperature we study, in the zero temperature limit, the interquark force for the pure SU(2) gauge theory in the (3+1) dimensional continuum in function of the relative distance ℓ. We find that the instantons induce a sharp rise of the confining force, we compute the magnitude of the effect and we discuss the picture and the possible extrapolation for $\ell\to\infty$.

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A growing evidence that confinement is the only phase of the non-abelian gauge theories without Higgs fields in (3+1) dimensions is now accumulating. Quantitative results within the lattice formulation of the theory have been obtained by the beautiful numerical studies of ref.(1), (2) and (3) which show a continuous but very sharp transition between the strong and weak coupling regime. This picture appears in agreement with approximation schemes where extrapolations on the behaviour of the β-function are performed starting either from the strong coupling regime[4] or from the weak coupling one. In this last case the Princeton group[5] indicates that the instatons trigger the transition to the strong coupling regime.

Here we study further the question of the transition from weak to strong coupling by adopting a different approach, i.e. by considering the theory always in the continuum and by computing directly the force (called the string tension) between a heavy quark and an antiquark at a relative distance ℓ, rather than the β-function. The method which we follow, based on the device of first considering the system at finite temperature T and then taking $T\to 0$, allows us to put into evidence the effect of the semiclassical configurations, and in particular of the instantons, on the string tension. In this method, proposed by Polyakov[6] some years ago and discussed by us for various field theory models in ref. (7), the Euclidean time variable τ is compactified to a finite range $0 \leq \tau \leq \beta$, with periodic boundary conditions. It is then possible, by performing every other functional integration in the $A_0=0$ gauge, to formulate the theory in terms of the variable $\Omega(\vec{r})=P\exp i\int_0^\beta d\tau A_0(\tau,\vec{r})$, a usual path ordered integral of the matrix valued gauge field A_μ (due to the compactification of τ the path is closed). The gauge is completely specified with an additional requirement at $\tau=o=\beta$, which corresponds to $\tau = \infty$ in the zero temperature limit. We then remain with an effective theory in a 3-dimensional space which is only globally symmetric. We can then put the confinement test of a static quark-antiquark pair at space positions $\vec{r}_a$ and $\vec{r}_b$ in the form af an ordinary correlation function of a theory in a 3-dimensional space:

$$< \Omega^{+}_{n_b m_b}(\vec{r}_b)\ \Omega_{n_a m_a}(\vec{r}_a) > \sim \exp(-k|\vec{r}_a-\vec{r}_b|)\,. \tag{1}$$

Here Ω_{nm} indicates some suitable components of the matrix Ω. For instance the usual Wilson loop in the gauge $A_1(\bar{r},\tau=o=\beta) = 0$ would read $Tr(\Omega^{+}(\vec{r}_b)\ \Omega\ (\vec{r}_a))$ for $\vec{r}_a-\vec{r}_b$ pointing in the space direction $\vec{n}_1$; Polyakov[6] also proposed the choice $Tr\Omega(\vec{r}_b)Tr\Omega(\vec{r}_a)$. The r.h.s. of eq. (1) is what is expected if confinement holds. In the zero temperature limit $\beta \to \infty$ the quantity k is foreseen to become proportional to β, i.e. $k \sim \beta\sigma$, and σ is the string tension.

In our analisys we will discuss the contribution of the semi-classical configurations to the correlation function of eq.(1) and try to understand a possible mechanism for obtaining a finite correlation lenght. In particular, an instanton at the origin gives, for the SU(2) gauge theory and in the limit $\beta \to \infty$, $\Omega_{inst}(\bar{r}) = \exp(i\Pi\vec{\sigma}\cdot\vec{r}(1/\sqrt{\rho^2 + r^2}-1/r))$ (we assume $\Omega(\vec{r}) \to 1$ for $r \to \infty$, and the instanton is here at the origin). This corresponds to a negative value of $Tr\Omega$ for $r<\rho/\sqrt{3}$ and a positive value for $r>\rho/\sqrt{3}$. Notice that at the origin and at infinity Ω_{inst} takes the value of the elements of Z_2 and this change of sign could not be observed if the quarks were in a representation, like the adjoint one, where the elements of Z_2 are mapped into the indentity. We can therefore imagine a mechanism for destroying correlations like in the Ising model, in which the instantons play the role of blobs of negative values in a sea of positive values. In order to test this picture we will consider actually the correlation function $\langle UU \rangle$ for the variable $U = \mathrm{Tr}\Omega/|\mathrm{Tr}\Omega|$ assuming that for n uncorrelated instantons U factorizes into $U_{(n)} = \prod_i^n U_i$. This is necessary in order to make use of the dilute gas description of the n instantons configurations, which is unavoidable in the lack of a better treatment; in particular the variable we use must have $|U|=1$, otherwise instance for $|U|<1$ the replacement $U_{(n)} \to \prod U_i$ could give an artificially low value for $|U_{(n)}|$ and at the end produce an artificial enhancement of the confining effect.

We have also considered, as a complementary test, the correlation function for the variable $U=e^{i\omega} = \mathrm{Tr}\,(1+\sigma_3)\Omega\,/|\mathrm{Tr}\,(1+\sigma_3)\Omega\,|$; $\omega(\vec{r})$ is an azimuthal angle of the four dimensional sphere into which the SU(2) element Ω can be mapped. In this case the instanton corresponds [7] to a vortex line for $\vec{v}(\vec{r}) = \vec{\nabla}\omega(\vec{r})$ on a circle which for the particular position and orientation above specified is described by $Z = 0$ and $\sqrt{x^2+y^2} = \rho/\sqrt{3}$, i.e. $\mathrm{rot}\,\vec{v} = 2\Pi\int ds\frac{d\vec{q}}{ds}\,\delta^3(\vec{r}-\vec{q}(s))$ where $\vec{q}(s)$ is a parametric representation of the circle. The mechanism is now like the one by the which the vortices induce a short range correlation in the XY model. We will compute for n instantons $U_n = \prod_i U_i = e^{i\Sigma\omega_i}$, where we add in the exponent the contribution of the n vortices computing each of them from $\mathrm{rot}\,\vec{v}_i = 2\Pi\int ds\frac{d\vec{q}_i}{ds}\,\delta^3(\vec{r}-\vec{q}_i)$, $\mathrm{div}\,\vec{v}_i = 0$, disregarding spin waves contributions, in the language of the X Y model, which are not typical of the instanton, like a possible other part $\vec{v}'$, $\vec{\nabla}\omega = \vec{v}+\vec{v}'$ with rot $\vec{v}' = o$ div $\vec{v}' \neq o$. The resulting contribution to, say, $\omega(\vec{r}_a)$ of one instanton centered at $\vec{r}'$ is one half of the solid angle spanned by the corresponding vortex loop as seen from $\vec{r}_a$. This is the same, modulo 2Π, as the flux of an electric field of a pointlike charge of value 2Π staying at the position $\vec{r}_a$ through the surface enclosed by the vortex loop. Therefore the instantons, here represented by the vortex loops, appear to interact with the electric field of the quark like magnetic dipoles in a magnetic field; this picture looks similar to the one described by the Princeton group in ref. (8).

Of course in both cases the instanton just represents a particularly regular and, for small coupling, important configuration of a class of topologically equivalent objects which have the same effect. For larger coupling, i.e. for larger scale, we expect the instaton picture to become inadequate and presumably fade into a picture of configurations of higher entropy. Proceeding now with the dilute gas approximation we get for both cases

$$< U^+(\vec{r}_b)\, U(\vec{r}_a) > \; = \exp\left(-2\beta \int d\rho \tilde{D}(\rho)\, d^3r'\, d\Omega_{\vec{n}}/4\Pi \; (1-U_1^+(\vec{r}_b)U_1(\vec{r}_a))\right). \quad (2)$$

Here, U_1 means the contribution of one instanton, the factor 2 takes into account the antiinstanton, the integral over the τ position gives the factor β, the integral over $d\Omega_{\vec{n}}/4\Pi$ averages over the space orientation (a global gauge rotation is here the same as a space rotation, irrelevant for the first case but relevant for the vortex orientation), $\vec{r}'$ is the space position of the instanton, ρ is the size, $\tilde{D}(\rho)$ is the usual determinant from the functional integration over the fluctuations (in a traditional notation $D(\rho) = \tilde{D}(\rho)/\rho^5$).

Concerning the integration over ρ we make the following observations. First of all, one expects that configurations of scale much larger than $\ell=|\vec{r}_a-\vec{r}_b|$ cannot destroy the correlation between $\vec{r}_a$ and $\vec{r}_b$. More precisely let us indicate with (SL) a state of the system, where S refers to the instantons of size smaller than ℓ and L to those of size larger than ℓ. Then in general a correlation function will be $\langle G(\vec{r}_a,\vec{r}_b)\rangle = \Sigma\, G_{(SL)} P_{(SL)}$, where $P_{(SL)}$ is the probability of the state (SL). Now we can write $G_{(SL)} = G_{(S0)} \cdot G'_{(SL)}$, where (S0) is the state characterized by the instantons of small scale only; $G'_{(SL)}$ will be a slowly varying function of S and therefore $G'_{(SL)} \simeq G_{(0L)}$. Notice that $G_{(S0)}$ has a widely fluctuating phase, while $G_{(0L)}$ has always a small phase except when it is small in absolute value. In our scheme we average over the large phase fluctuations of $G_{(S0)}$ by dealing with the normalized variable U. Since also $P_{(SL)} \simeq P_{(S0)} \cdot P_{(0L)}$ we have $\Sigma\, G_{(SL)} P_{(SL)} = \Sigma_S\, G_{(S0)} P_{(S0)} \cdot \Sigma_L\, G_{(0L)} P_{(0L)}$. The dilute gas approximation, which has seen implies a factorization like $U_{(n)} \simeq \prod_1^n U_i$, can only be adequate for the first factor, i.e. the (S0) contribution, and leads to the exponentiation as in eq. (2), whereas the second factor is a slowly varying function of ℓ which can be disregarded for our purposes. We will therefore restrict the integration in eq. (2) from o to $p\ell$, where p is a number $\gtrsim 1$ on which the magnitude of our results depends rather weakly; we have actually taken $p=2$, a higher value, say $p=4$, would increase the results of a few percent (taking $p \to \infty$ would erroneously give an infinite tension). Notice that this concerns the correlation and not the expectation values like $\langle U\rangle$: here the upper limit for ρ would be the dimension of the system.

We find numerically that with a satisfactory approximation for our purposes we can parametrize for both cases, in the relevant region,

$$J(\rho,\ell) \equiv \int d^3r'\, d\Omega_{\vec{n}}/4\Pi\, (1-U_1^\dagger U_1) = \lambda\rho^2\ell\, \theta(\rho-\ell/q) + \lambda\rho^3/q\, \theta(\ell/q-\rho) \tag{3}$$

with λ ranging between 1.8 and 3.1 and q between 1.7 and 1. We have taken for both $\lambda' = 2.2$ and $q = 1.5$, which we have seen to be a good compromise. (In the second case to simplify the computation we have considered rectangular rather than circular vortex lines and checked the result with another approximation). The string tension is then

$$\sigma \equiv - 1/\beta\ell \; \ell n < U^{+}(\vec{r}_b) \; U(\vec{r}_a) > = 2/\ell \int_0^{p\ell} d\rho \; \tilde{D}(\rho) \; J(\rho, \ell) \qquad (4)$$

For SU(2) we have[9] $\tilde{D}(\rho) = 4/\Pi^2 \exp(-\alpha_1) \; 1/\rho^5 \; (\frac{4\Pi^2}{g^2(\rho)})^4 \; e^{-8\Pi^2/g^2(\rho)}$ with $8\Pi^2/g^2(\rho) = -22/3 \; \ell n(\rho\mu)$, i.e. the one loop formula, and μ is the Pauli-Villars scale which, in our case of no quarks, is related to the more conventional scale Λ^{MOM} by $\mu = 2.75/7.7 \; \Lambda^{MOM}$. If we consider for instance the first term at the r.h.s. of eq. (3) and change integration variables in eq. (4) to $x = \ell n(\mu\rho)$, we find a contribution to σ expressed as the integral in the interval $\ell n(\ell\mu/q), \ell n(p\ell\mu)$ of a function $F(x)$ which has a peak for $x = x_p = -4/(22/3 - 2)$ with a width $\Gamma \simeq |x_p|$, goes to zero for $x = -\infty$ and $x = o$ and then exponentially increases for $x > o$. We will then find a rise of σ when $\ell n(p\ell\mu)$ crosses x_p followed by an almost constant value (until for $\ell n(p\ell\mu) > o$ the divergent behaviour of $F(x)$ would induce a rise of σ). The results of the numerical evaluation of eq. (4) are reported in Fig. (1).

As we said we take here $p = 2$, the plateau value of σ depending very weakly on p, but of course the value of ℓ for which the string tension appears depends linearly on the poorly known parameter p^{-1}. Our result for the instanton contribution to the string tension is the plateau value $\sqrt{\sigma}_{inst} = .4\Lambda^{MOM}$ i.e. a sizable fraction of the value obtained in the Creutz computer experiment[1] $\sqrt{\sigma}_c = (1.33 \pm .21)\Lambda^{MOM}$ (perhaps we have been too strict in cutting down spurious effects as we explained before; for instance, repeating the computation for the Wilson loop $<Tr(\Omega^+(\vec{r}_b)\Omega(\vec{r}_a))>$ we get a value for $\sqrt{\sigma}_{inst}$ a factor 2 higher than the previous one). This would indicate that the instantons play a rôle in the confinement but also other more irregular configurations are important.

Of course we have a typical problem here, since the important contribution to the integral in eq. (4) comes from the previously said peak region where $\alpha_s = g^2/_{4\Pi} = 1.1$, and therefore we have to consider our result as a hopeful extrapolation into a region where α_s is not small and the one loop approximation loses sense. This is in part also a question on what we have to call the instanton contribution. For this reason we have done the exercise of modifying the behaviour of $8\Pi^2/g^2(\rho)$ inside the expression for $\tilde{D}(\rho)$ according to what is observed in the computer experiment for the behaviour of the bare coupling constant g_o with the lattice spacing, i.e. taking the one loop formula up to $\alpha_s \leq .3$, which taking into account the change of scale should correspond to $g_o \leq 2$, and after that $8\Pi^2/g^2(\rho) = 8\Pi^2/(\exp(\rho^2\sigma_c)+a)$, in order to match the strong coupling regime $\ell n(g^2(\rho)) = \rho^2\sigma_c$, σ_c being the Creutz value for the string tension and a being fixed by continuity. Now, the previously said peak for $F(x)$ becomes somehow higher and narrower and consequently the rise of σ_{inst} is somehow steeper. The whole result on the plateau is $\sqrt{\sigma}_{inst} = 0.3\ \Lambda^{MOM}$, of the same magnitude as the previous one.

Let us add a further speculation on what one can expect on general grounds for the contribution to the correlation function from topologically non trivial configurations of various shapes and sizes. Let us consider in particular the picture of the vortex lines. It is possible to develop a second quantization formalism (11,7) in which the path integration over $\vec{q}(s)$ describing the closed vortex lines is transformed in a standard way into the functional integration over a complex field $\Phi(\vec{q})$. Consider the case in which a small constant external field $\vec{k}$ is coupled to the vortex lines, i.e. compute $\langle \exp i \int ds \frac{d\vec{q}}{ds} \vec{\eta} \rangle$ where the vector potential $\vec{\eta}$ is given by $\mathrm{rot}\,\vec{\eta} = \vec{k}$. In the second quantization formalism

$$\langle \exp(i\int ds\; \dot{\vec{q}}\cdot\vec{\eta}) \rangle = N \int D\Phi \; \exp(-\int d^3q \; (|(\vec{\partial}+i\vec{\eta})\Phi|^2 + V(\Phi)))$$

where $V(\Phi)$ weighes the field configurations and takes also care of the interactions among vortex lines. We can foresee two possibilities, namely $\langle\Phi\rangle = o$ and the condensation case $\langle\Phi\rangle \neq o$. If we take the second derivative in k at $k = o$ we find a term diverging like the volume of the system L^3 for $\langle\Phi\rangle = o$ and a term $\langle\Phi\rangle^2 \int d^3q \vec{\eta}^2(\vec{q}) \sim L^5$ in the case of condensation. Suppose that we repeat the computation for circular vortex lines in the dilute gas approximation and consider the density $\tilde{D}(\rho)$ as an unknown: we will find

$$\langle \exp(i\int \dot{\vec{q}}\cdot\vec{\eta}) \rangle = \exp(-2\beta L^3 \int d\rho \tilde{D}(\rho) \; (1-\sin(\tfrac{\Pi}{3}k\rho^2)/(\tfrac{\Pi}{3}k\rho^2)))$$

which gives for the second derivative in k a term $\sim L^3 \int_o^L d\rho \; \rho^4 \tilde{D}(\rho)$. Therefore $\langle\Phi\rangle \neq o$ corresponds $\tilde{D}(\rho) \to c_o/\rho^3$ for $\rho \to \infty$ (whereas $\langle\Phi\rangle = o$ can be seen to correspond to $\tilde{D}(\rho) \sim e^{-\rho/\rho_o}$). This also agrees with the Ising like picture described above, which gives in the dilute gas approximation $\langle U(\vec{r})\rangle \sim \exp(-\alpha \int d\rho \rho^3 D(\rho))$, vanishing like the inverse exponent of the linear dimension L in the disorder phase. This behaviour for $\mu\tilde{D}(\rho)$ gives a constant string tension, since in eq. (4) $J(\rho,\ell)=\rho^3 f(\ell/\rho)$ for dimensional considerations, and then $1/\ell \int_0^{\ell p} d\rho \tilde{D}(\rho) J(\rho,\ell) = \text{const} \int_0^p dy f(1/y)$. We are therefore led to speculate that the divergence of $\tilde{D}(\rho)$ for $\rho \to \infty$ as given by the instanton determinant is a signal of the confinement, giving a string tension which even divergently increases with the separation; this divergence is possibly tamed to an equilibrium configuration by the occurrence of a condensation regime, corresponding to a constant string tension. For instance, with the parametrization of eq. (3) we get $\sigma = 2\lambda c_o(\ln(pq) + 1)$, where $c_o = \lim \rho^3 \tilde{D}$. Just to see the order of magnitude, if we take for c_o the value corresponding to the peak value of the previously introduced $F(x)$, i.e. taking $F(x) = F(x_p)$ for $x > x_p$, we get a $\sqrt{\sigma}$ between $0.9\Lambda^{MOM}$ and $1.2\Lambda^{MOM}$, for the values of p and the formulae for $8\Pi^2/g^2(\rho)$ we have considered.

Finally, we notice that the external quarks at distance ℓ, introduced in the correlation function of eq. (1), appear to feel the medium, i.e. the instantons, over a volume of the order of ℓ^3, as can be seen eq. (3) for ρ of the order of ℓ. For instance, in the vortex picture this is the volume filled by the electric field of the quarks. Therefore the perturbation introduced by the quarks does not appear to be confined in a tube of constant transverse dimensions, in agreement with recent analysis(12,13).

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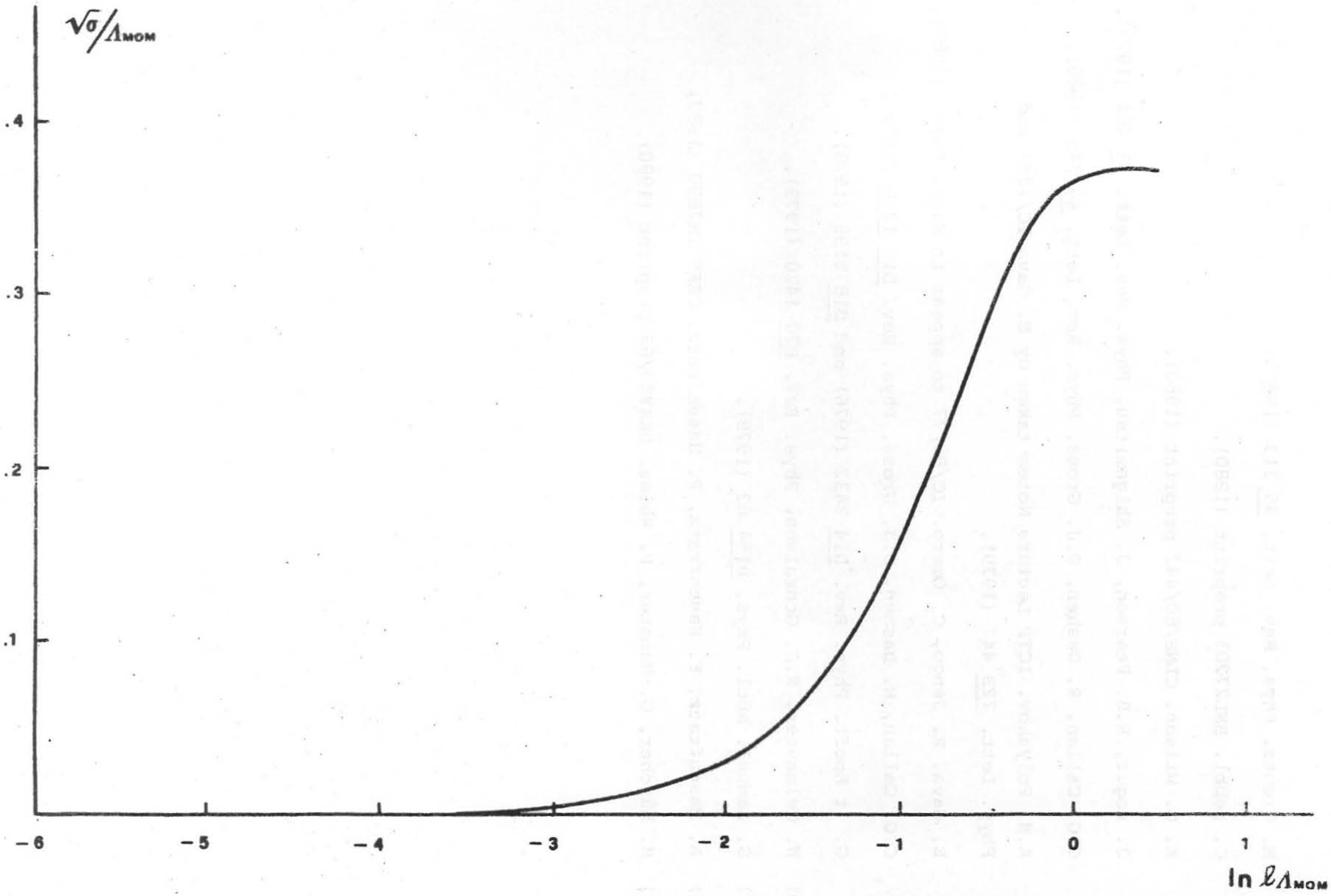

Fig. 1; The square root of the string tension as a function of the logaritm of the interquark distance.