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E. Di Salvo and G.A. Viano: SOMMERFELD POLES AND BACKWARD PEAKS IN ELASTIC SCATTERING.

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ABSTRACT: In this paper we propose a new use of the complex angular momentum representation of the scattering amplitude. In particular we show that it is not possible to connect with a pole - trajectory a sequence of compound state resonances. Furthermore the high energy backward peaks in the elastic channel are analysed, and we propose a model of the mechanism responsible for these backward enhancements. The mathematical formalism of this model, which employs the poles of the S - matrix in the complex angular momentum plane, is investigated in detail. Finally, in order to test the theory, a phenomenological analysis is performed; the results that we obtain are in agreement with the theory.

RIASSUNTO: In questo lavoro proponiamo un nuovo uso della rappresentazione dell'ampiezza di diffusione nel piano del momento angolare complesso. Dimostriamo in primo luogo che una traiettoria non può interpolare più risonanze nucleari, dovute alla formazione di nuclei composti nel senso di Bohr. Studiamo poi i picchi a grandi angoli nel canale elastico ad alta energia, e proponiamo un modello del meccanismo responsabile di questi picchi all'indietro. Analizziamo in dettaglio il formalismo matematico di questo modello, che usa i poli della matrice S nel piano del momento angolare complesso. Infine viene sviluppata una analisi fenomenologica i cui risultati sono in accordo con la teoria.

1. - INTRODUCTION

In the last fifteen years many papers have been devoted to the analytical properties of the S - matrix, which was analysed both in the complex momentum and in the complex angular momentum plane. Humblet and Rosenfeld[1] formulated a theory of nuclear reactions, based on the analytical properties of the S - matrix in the complex momentum plane. On the other hand, the complex angular momentum plane (or λ-plane) polology has been essentially used in order to introduce the so called "pole - trajectories", which should connect various bound states and resonances. However this theory is mainly employed in its relativistic extension, where one tries to get information from the trajectory on the asymptotic behaviour of the scattering amplitude in the crossed channel. For this reason it has been repeated many times that the λ-plane polology can provide an alternative and instructive description in atomic and nuclear physics (and more generally in the direct channel), but has not led here to any new physical applications (Ref. (2), pag. 99).

In the present paper we intend to criticize this doctrine and to change significantly the use, which is generally done, of the analytical properties of the S - matrix in the λ-plane. In this connection we shall prove, first of all, that the compound state resonances cannot be interpolated by a trajectory. On the other hand, in Sect. 3 we shall illustrate several phenomenological examples of elastic scattering, where the angular distribution presents a forward diffractive peak and also a strongly oscillatory enhancement in the backward direction. In this sense we can properly speak of backward peaks. The main purpose of this paper is to show that this phenomenon can be appropriately pictured by a model, based on the λ-plane polology, from which one can derive a formula that properly fits these backwards angular distributions.

Many authors[3, 4, 5] explain the backward peaks through a resonance mechanism, which, in any case, is quite different from the usual Breit-Wigner model; in fact in these backward enhancements the effect is peripheral and does not involve the formation of a compound state. From this point of view one could argue that the Breit-Wigner formalism and the λ-plane polology are complemental, in the sense that the former theory is appropriate for the compound - state resonances, while the latter is proper for the peripheral ones. In any case we want to stress that the compound state or Breit-Wigner resonances cannot be connected by an interpolation in the complex angular momentum plane; on the other hand, the backward peak does not cease to exist abruptly, but it vanishes toward higher energies. For this reason the phenomenon is correctly fitted by a continuously rising pole - trajectory (see Sect. 5).

The use of the λ-plane polology in the direct channel (with particular attention to nuclear physics) was tried long time ago[6]; then it was pursued on by many authors (see for instance refs.(3, 13)) in various directions. However we think that many points remained obscure and need a deeper theoretical analysis. Moreover, up to now, a phenomenological evidence for a trajectory has not yet been obtained. In this work we have tried to fill the gaps of the theory. The paper is organized as follows. Sect. 2 is devoted to the compound state resonances and, in particular, to the proof that these cannot be interpolated by a trajectory. In Sect. 3 we illustrate a model of the mechanism which produces the backward peaks; starting from this model we derive the Sommerfeld poles[(*)], which fit these backward enhancements. In the same Section the asymmetry in the angular distribution is discussed in detail, as well as the trajectories of the Sommerfeld poles in the complex angular momentum plane. In Section 4 we tackle the question of the Coulomb interference and analyse the classes of potentials that admit the Watson - Sommerfeld transform (an essential ingredient of the model illustrated in Sect. 3). Finally, in Sect. 5, we present a phenomenological analysis and give an example of a moving-pole trajectory.

2. - COMPOUND STATE RESONANCES.

Let us write the usual partial - wave expansion for two colliding particles, which are supposed to be spinless, neutral and distinguishable, i. e.

$$f(E,\vartheta) = \sum_{l=0}^{\infty} (2\,l + 1)\, a_l\,(k)\, P_l(\cos\vartheta) \tag{1}$$

where $a_l(k) = (e^{2i\delta_l(k)}-1)/2\,i\,k$. The terms can be continued analytically into the complex λ-plane an the sum in (1) can be converted into an integral over a counterclockwise contour C in the right half - plane enclosing the real λ-axis, as follows[13]:

$$\begin{aligned} f(E,\vartheta) = \sum_{i=1}^{N} & \frac{g_i\,(E)}{\sin\pi\,\alpha_i(E)}\, P_{\alpha_i(E)}\,(-\cos\vartheta\,) + \\ & + \frac{1}{2i}\oint_C \frac{(2\lambda+1)\, a(\lambda,k)\, P_\lambda\,(-\cos\vartheta)}{\sin\pi\lambda}\, d\lambda \end{aligned} \tag{2}$$

where $\alpha_i(E)$ gives the location of the i - th pole of $a(\lambda, k)$ enclosed by the contour C. If, for $E=E_o > 0$, Re $\alpha_i(E_o)$ is an integer, Im $\alpha_i(E_o)$ is not too large, and $\left[d(\mathrm{Re}\;\alpha_i(E))/dE\right]_{E=E_o} > 0$, then we have a strong contribution from the pole term, i. e. $(g_i(E_o)/\sin\pi\alpha_i(E_o))\,P_{\alpha_i(E_o)}(-\cos\vartheta)$, which can correspond to a resonance. In principle the same function $\alpha_i(E)$ may originate several resonances, if it happens to come close to several integers for real positive value of E; therefore we should have fa-

(*) In this paper we prefer to speak of Sommerfeld rather than Regge poles, since we model our theory by the classical one, due to Sommerfeld, on the propagation of the radio waves around the earth.

milies of resonances described by the same function $\alpha_i(E)$, i. e. by a trajectory(14)(+).

As a first step, we prove now the following statement, which turns out to be very illuminating in the following, even if it gives a mainly negative result (see also ref. 10)): let a sequence of elastic resonances be given (i. e. a family of resonances with increasing angular momentum and energy) and suppose each resonance to have the corresponding antiresonance, i. e. the phase shift δ_l descends, after the resonance, through $\pi/2$ (mod. π). Furthermore let us assume that the resonances are spaced far enough, so that each phase shift, after crossing $\pi/2$ (mod. π), in correspondence to a resonance, descends through $\pi/2$ (mod. π) before the successive one. In this case the sequence cannot be interpolated by a trajectory $\alpha_i(E)$.

<u>Proof.</u> As is wellknown, in correspondence of a resonance at $E=E_0$, a phase shift δ_l crosses $\pi/2$ (mod. π) with the derivative $\left[d\,\delta_l(E)/dE\right]_{E=E_0} > 0$. On the contrary, in correspondence of the antiresonances the phase shift descends through $\pi/2$ (mod. π), i. e. with a negative derivative. Now we know that, in the complex angular momentum plane, when $\mathrm{Re}\,\alpha_i(E)$ crosses an integer value with the derivative $d(\mathrm{Re}\,\alpha_i(E))/dE$ positive, we have a resonance; on the other hand, in correspondence to an antiresonance, $\mathrm{Re}\,\alpha_i(E)$ is an integer, but the derivative $d(\mathrm{Re}\,\alpha_i(E))/dE$ is negative (Ref. (15, pag. 110). This implies that each trajectory, after crossing an integer corresponding to a resonance of the sequence, then turns back toward the left half - plane to produce the corresponding antiresonance; therefore the interpolation of successive resonances turns out to be impossible.

Now we illustrate the previous statement with a phenomenological example. Let us consider the resonances in ^4He - ^4He elastic scattering; at low energies we have a rotational sequence of three resonances, $0^+, 2^+, 4^+$(16). At first sight one could conjecture that these resonances lie on the same trajectory, but this is not the case. In fact these resonances are well separated each other and while δ_0 descends passing downward through $\pi/2$, δ_2 rises passing through $\pi/2$, then, when δ_2 descends, δ_4 starts to rise.

Now the effect by which the phase shift gets down again after its rising at the preceding resonance in due to a hard sphere scatterer. In the compound state elastic resonances this effect cannot be neglected, because, as is well known, the scattering amplitude is composed of two parts: a resonant amplitude and the so called potential scattering amplitude. This second part determines the elastic scattering cross section σ_{el} off resonances.

In other words σ_{el} between two resonances is expected to be the same as that of a repulsive sphere of nuclear size. Therefore, in the energy interval between two resonances, the wave function of the incident particle has a very small amplitude within the target and the minimum amplitude occurs in correspondence of the antiresonance. Recalling the previously proved statement we can conclude that it is properly the effect of the potential scattering which prevents the possibility of connecting more resonances by a trajectory $\alpha_i(E)$. In this case the resonances are generated by the going through $\pi/2$ of only one phase shift. Therefore the angular distribution of the compound state elastic resonance is symmetric and is correctly described by the Legendre polynomials. The symmetry in the angular distribution is due to the fact that, in any compound system, the lifetime of the metastable state, formed by the incoming particle and the target, is long enough to cause a complete loss of memory of the incident beam direction(4).

3. - BACKWARD PEAKS IN ELASTIC SCATTERING

Now, reasoning in a semiclassical approximation, let us suppose that the target is strongly absorptive at small impact parameters, but not at large ones. In other words, we assume that the absorption is very strong near the centre of the target and decreases towards its periphery. Of course, when the absorption is strong, we observe a diffraction pattern in the forward hemisphere with a peak at $\vartheta=0$, which is the analog of the Poisson spot in optics. However in many cases (for

(+) Many trajectories have been numerically computed for various classes of potentials (Ref. (15), chap 12, and Ref. (13)); unfortunately these numerical investigations result to be not very useful and even misleading, since small changes in the potential can produce large variations in the analytical properties of the scattering amplitude (see Sect. 4).

instance in the elastic scattering of α particles from nuclei like ^{16}O, ^{40}Ca, as well as in the elastic pion-proton collision), at angles beyond 90^o, the angular distributions do not present an exponential envelope typical for diffraction scattering. The anomaly consists of a strongly oscillatory enhancement in the backward cross section, which is reminiscent of the optical glory effect[17].

The anomaly can be explained supposing that, for high values of the impact parameter, i.e. $l \simeq kR$ (where R is the radius of the target), the incoming particle can rotate around the central region of strong absorption and then emerge in the backward direction. This phenomenon is analogous to the formation of surface waves in the scattering of light by water droplets[17]. The surface waves, which travel around the droplet, generate the optical glory effect and are responsible for the backward enhancements[18,19]. Now let us consider some qualitative consequences of this heuristic model:

a) In order to picture, in a semiclassical way, the mechanism that produces the backward peaks, one could say that the incident particle (like an optical diffracted ray in the sense of Levy and Keller[20]) describes a geodesic around the target without closing its orbit. Therefore, even if one wants to speak of "resonances"[4], these are necessarily peripheral and do not involve the formation of a compound state in the sense of Bohr. Moreover the lifetime is certainly very short and therefore the angular distribution can be quite asymmetric, since the loss of memory of the incident beam direction has not been completely realized in such a short rotation time. Consequently the angular distribution cannot be fitted by the ordinary Legendre polynomials, but one must recur to the Legendre functions, such as the $P_{\alpha}(E)(-\cos\vartheta)$, which appear in the representation (2). All these considerations make evident that the usual Breit-Wigner parameters, such as the width Γ, are not useful here; for analogous reasons we need not worry about the effect of the antiresonances. In oder words we need a new parametrization, such as that furnished by the complex angular momentum polology.

b) Furthermore there is a shell structure effect on the back angle enhancement of the elastic α-nucleus scattering. The experimental data suggest the following picture[21]: at the major shell closures, the nuclei all show backward peaks independently of their neutron excess. The enhancement disappears if we cross the closed shell and it emerges again when we approach the next closure. This isotopic dependence can be qualitatively understood at the light of the model. One observes the backward scattering enhancement if the α-rotator occupies the next higher shell (for example α - ^{12}C) or if the target is a tightly bound nucleus (for instance ^{16}O). On the other hand, with an increasing number of excess neutrons, there could be a coupling of single neutron excitation to the simple rotator state, and thus a blocking effect on the elastic width (for instance α-^{18}O)[22].

c) Of course a correct theory should answer the questions whether the backward enhancement disappears toward higher energies and in which way it ceases to exist.

Now we try to elaborate the mathematical theory of the model, starting from the representation (2). In this connection let us assume that

$$\left| S(\lambda, k) - 1 \right| \xrightarrow[|\lambda| \to \infty]{} O\left(\frac{1}{\lambda}\right), \qquad \mathrm{Re}\,\lambda \geqslant 0 \tag{3}$$

where $S(\lambda, k)$ is the analytic continuation in the complex λ-plane of $S_l(k) = e^{2i\delta_l(k)}$; we shall return with much more details on the assumpion (3) in the next Section. If the condition (3) is satisfied, then the contour C of the background integral in the representation (2) can be deformed to run along the imaginary axis and we get the Sommerfeld - Watson transform, i.e.

$$f(E,\vartheta) = \sum_{i=1}^{N} \frac{g_i(E)}{\sin \pi\alpha_i(E)} P_{\alpha_i(E)}(-\cos\vartheta) + \frac{i}{2}\int_{-1/2 - i\infty}^{-1/2 + i\infty} \frac{(2\lambda+1)\, a(\lambda,k)\, P_\lambda(-\cos\vartheta)}{\sin\pi\lambda}\, d\lambda \tag{4}$$

The representation (4) is composed of two parts, a sum over poles and the background integral. For large values of ϑ, the regularity of $P_\lambda(-\cos\vartheta)$ and the fact that $(\sin\pi\lambda)^{-1}$ ($\lambda = -1/2 + i\,\mathrm{Im}\,\lambda$) acts as a very powerful cutoff, suggest to neglect the background term in comparison to the sum over the poles. Furthermore, if we assume that the contribution of one pole is dominant, then we can write the following approximation for the scattering amplitude

$$f(E,\vartheta) \simeq g\,\frac{P_\lambda(-\cos\vartheta)}{\sin\pi\alpha} \tag{5}$$

where the term at the right - hand - side of formula (5) will be called Sommerfeld pole. Of course this formula can work only at backwards, since, for $\vartheta = 0$, P_α $(-\cos\vartheta)$ presents a branch - cut (the finiteness of $f(E,\vartheta)$ is guaranteed by the compensation of the two terms of representation (4)).

Next we use the following asymptotic formula, which holds for large values of Re_α and Im_α (Ref. (22) pag. 288):

$$\frac{P_\alpha(-\cos\vartheta)}{\sin\pi\alpha} \approx \left(\frac{2i}{\pi\alpha\sin\vartheta}\right)^{1/2} \exp\left[\, i\left(\alpha+\frac{1}{2}\right)\vartheta\right] \qquad (\varepsilon < \vartheta < \pi-\varepsilon) \tag{6}$$

(formula (6) does not hold in small neighbourhoods of the antipodal points $\vartheta=0$ and $\vartheta=\pi$). From formula (6) we obtain the factor $e^{-\mathrm{Im}\,\alpha\cdot\vartheta}$, which gives the breaking of the rotational symmetry in the angular distribution.

Now we want to relate this symmetry breaking factor to the rotational damping of the surface waves around the absorption volume. In this connection, following Nussenzveig[23] and Fuller[4], we write the partial wave expansion for the wave function (see also ref. (24) pag. 299):

$$\psi(k,r)=\left(\frac{2\mu k}{\pi}\right)^{1/2}(4\pi k r)^{-1}\sum_{l=0}^{\infty}(2l+1)i^{l}\,\psi_l(k,r)P_l(\cos\varphi) \tag{7}$$

where μ is the reduced mass and φ is the angle between the momentum k and r. Next $\psi_l(k,r)$ can be decomposed as follows:

$$\psi_l(k,r)=1/2\; e^{i(\pi/2)(l+1)}\left[f_l^{(-)}(k,r)-e^{-i\pi l}S_l(k)\,f_l^{(+)}(k,r)\right] \tag{8}$$

where $f_l^{(\pm)}(k,r)$ are the Jost solutions, defined by

$$\lim_{r\to\infty} f_l^{(\pm)}(k,r) = e^{\pm i k r} \tag{9}$$

Substituting formula (8) into (7), we get

$$\psi(k,r)=\left(\frac{2\mu k}{\pi}\right)^{1/2}(4\pi k r)^{-1}\cdot\sum_{l=0}^{\infty}(2l+1)\,\frac{i^{(2l+1)}}{2}\left[f_l^{(-)}(k,r)-e^{-i\pi l}\cdot\right.$$
$$\left.\cdot\, S_l(k)\,f_l^{(+)}(k,r)\right]P_l(\cos\varphi) \tag{10}$$

Recalling that $P_l(-\cos\varphi)=(-1)^l P_l(\cos\varphi)$, we obtain

$$\psi(k,r)=\left(\frac{2\mu k}{\pi}\right)^{1/2}(4\pi k r)^{-1}\,\frac{i}{2}\cdot\sum_{l=0}^{\infty}(2l+1)\left[f_l^{(-)}(k,r)-e^{-i\pi l}S_l(k)\cdot\right.$$
$$\left.\cdot\, f_l^{(+)}(k,r)\right]P_l(-\cos\varphi) \tag{11}$$

Then we perform an analytic continuation in the complex λ-plane, recalling that $f^{(\pm)}(\lambda,k,r)$ are entire functions of λ at fixed k and r[14] (at least for a very large class of potentials); so we can convert the sum (11) into an integral over a counterclockwise contour C in the right half -plane enclosing the real λ- axis. We obtain

$$\psi(K,r)=\frac{1}{2}\left(\frac{2\mu k}{\pi}\right)^{1/2}(4\pi k r)^{-1}\cdot$$
$$\cdot\left[\oint_C \frac{(2\lambda+1)\left[f^{(-)}(\lambda,k,r)-e^{-i\pi\lambda}S(\lambda,k)\,f^{(+)}(\lambda,k,r)\right]P_\lambda(\cos\varphi)}{\sin\pi\lambda}\,d\lambda+\sum_{i=1}^{N}\frac{\alpha_i(k,r)}{\sin\pi\alpha_i}P_{\alpha_i}(\cos\varphi)\right] \tag{12}$$

where the sum is over the S - matrix poles lying in the domain enclosed in the contour C. This contour can be deformed to run along the imaginary axis (if we assume a suitable asymptotic behavior, in the λ-plane, for the background integrand in formula (12)), so that we can write the Watson-Sommerfeld transform for the wave function:

$$\psi(k,r) = \frac{1}{4}\left(\frac{2\mu k}{\pi}\right)(4\pi k r)^{-1} \cdot$$

$$\cdot\left[\int_{-1/2-i\infty}^{-1/2+i\infty} \frac{(2\lambda+1)\left[f^{(-)}(\lambda,k,r)-e^{-i\pi\lambda}S(\lambda,k)f^{(+)}(\lambda,k,r)\right]}{\sin\pi\lambda} P_\lambda(\cos\varphi)\, d\lambda + \sum_{i=1}^{N} \frac{c_i(k,r)}{\sin\pi\alpha_i} P_{\alpha_i}(\cos\varphi)\right] \tag{13}$$

Now it is convenient to perform the substitution $\Phi = \pi - \varphi$, from which it follows that $P_\alpha(\cos\varphi) = P_\alpha(-\cos\Phi)$. Therefore small values of Φ correspond to the upwind, while values near π correspond to the shadow or downwind region. Then, using the same arguments illustrated above, concerning the scattering amplitude, we can neglect the background integral contribution for Φ near π, i. e. in the downwind side of the interaction region relative to the incident beam (see also refs. (4, 5)). Next, according to the suggestions of ref. (4), we use the following asymptotic formula, which holds for large positive values of Reα (see ref. (25), pag. 143):

$$P_\alpha(-\cos\Phi) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\frac{3}{2})}\left(\frac{1}{2\pi\sin\Phi}\right)^{1/2} \cdot$$

$$\left\{e^{\mathrm{Im}\,\alpha(\pi-\Phi)}e^{-i\left[(\mathrm{Re}\,\alpha+\frac{1}{2})(\pi-\Phi)-\frac{\pi}{4}\right]}+e^{-\mathrm{Im}\,\alpha(\pi-\Phi)}e^{i\left[(\mathrm{Re}\,\alpha+\frac{1}{2})(\pi-\Phi)-\frac{\pi}{4}\right]}\right\}, \qquad (\varepsilon < \Phi < \pi-\varepsilon) \tag{14}$$

The equality (14), like formula (6), does not hold in small neighbourhoods of the antipodal points $\Phi = 0$ and $\Phi = \pi$. Let us observe that we are using now an approximation which is similar, but less drastic, than formula (6). In fact, for what concerns the cross - sections, we intended only to give a rough idea of the breaking of the rotational symmetry; then starting from a formula like (14), we could neglect the second term in comparison to the first one for Im α large enough (recall that the scattering angle ϑ is confined to the interval $(0, \pi)$ and formula (6) does not hold for $\vartheta = \pi$). On the other hand, we want now to analyse in detail the surface waves, which travel around the central absorption region, therefore we must retain both terms which appear in formula (14). Moreover, in order to consider the time evolution, we have to multiply the wave function by the factor e^{-iEt} ($\hbar = 1$). Therefore from the first term of (14), neglecting proportionality factors, we get:

$$\left(\frac{1}{\sin\Phi}\right)^{1/2} e^{\mathrm{Im}\,\alpha(\pi-\Phi)}\, e^{-i\left[(\mathrm{Re}\,\alpha+\frac{1}{2})(\pi-\Phi)-\frac{\pi}{4}+Et\right]} \tag{15}$$

which represents a wave traveling in the direction of increasing Φ. Analogously, from the second term of (14), we obtain

$$\left(\frac{1}{\sin\Phi}\right)^{1/2} e^{-\mathrm{Im}\,\alpha(\pi-\Phi)}\, e^{i\left[(\mathrm{Re}\,\alpha+\frac{1}{2})(\pi-\Phi)-\frac{\pi}{4}-Et\right]} \tag{16}$$

which represents a wave traveling in the direction of decreasing Φ.

Now let us suppose that we are in a region where the term (15) is strongly predominant over the term (16), so that we can approximately write the following probability distribution:

$$|\Phi|^2\, dV \approx e^{2\,\mathrm{Im}\,\alpha(\pi-\Phi)}\, d\Phi \tag{17}$$

which implies that the wave, represented by the term (15), decays for increasing values of Φ, i. e. in the direction of propagation. Next, if we force a little bit the mathematics of our approximation, assuming that it holds for any value of Φ, and suppose that we are in a region where the term (16) is predominant over the term (15), then we obtain the following probability distribution:

$$\left|\Phi\right|^2 \; dV \propto \; e^{-\mathrm{Im}\,\alpha\,(\pi - \Phi)} \, d\Phi \tag{18}$$

which implies that the wave represented by formula (16) decays for decreasing values of Φ , i. e. in the direction of propagation. Finally let us assume $\mathrm{Im}\,\alpha$ large enough to practically forbid the propagation of surface waves across the shadow; in this case the symmetry breaking in the angular distribution is very large and we observe an exponentially decreasing envelope at angles beyond 90^{o}. On the other hand, for decreasing values of $\mathrm{Im}\,\alpha$, some surface waves propagate across the shadow and emerge at backwards.

4. - REMARKS ON THE SOMMERFELD-WATSON TRANSFORM.

Firstly we recall that in many cases the colliding particles are charged and we must rewrite the expansion (1) as follows:

$$f(E,\vartheta) = f_c(\vartheta) + \frac{1}{2ik}\sum_{l=0}^{\infty}(2l+1)e^{2i\sigma_l}(S_l-1)P_l(\cos\vartheta) \tag{19}$$

where the Rutherford amplitude is given by

$$f_c(\vartheta) = -\frac{\eta}{2k\sin^2(\vartheta/2)}\exp(-i\log\sin^2\frac{\vartheta}{2} + 2i\,\sigma_0) \tag{20}$$

($\eta = \frac{ZZ'e^2}{\hbar v}$, v being the velocity of the incident particle) and

$$e^{2i\sigma_l} = \frac{\Gamma(l+1+i\eta)}{\Gamma(l+1-i\eta)} \tag{21}$$

Now, if Z and Z' are not too large and the energy of the colliding particle is high, then the Coulomb effect at backwards is small and can even be neglected (see Sect. 5). Otherwise one must perform the subtraction of the Coulomb effect; in fact the repulsive Coulomb field distorts strongly the behaviour of the pole trajectory[26]. Of course the subtraction procedure has the great disadvantage of requiring a phase - shift analysis of the experimental data (see, for example, ref. (27)).

Another more subtle problem is related to the assumption (3), concerning the asymptotic behaviour of $S(\lambda, k)$ in the complex λ-plane. The property (3) has been proved only in the case of a superposition of Yukawa potentials. On the other hand, the potentials that are generally used in nuclear physics have the Woods-Saxon shape[13]. For these potentials one can only show that

$$\left|S(\lambda)\right| \longrightarrow \begin{cases} 0, & \mathrm{Im}\,\lambda \longrightarrow +\infty \\ \infty, & \mathrm{Im}\,\lambda \longrightarrow -\infty \end{cases} \tag{22}$$

Thus, the corresponding scattering amplitude can be represented by poles and background terms like in formula (2),, but the background integral cannot be made to run along the imaginary axis and so the approximation (5) cannot be justified. However we must advance the following remarks: arbitrarily small changes in the potentials can produce arbitrarily large variations in the asymptotic behaviour of the scattering amplitude in the complex λ-plane. Let us consider, for instance, the cutoff potentials; in this case the contour of the background integral in the representation (2) cannot be made to run along the imaginary axis (i. e. $-1/2 - i\infty$, $-1/2 + i\infty$). On the other hand, in the case of Yukawian potentials, the asymptotic behaviour of $S(\lambda)$ for $|\lambda| \longrightarrow \infty$ (Re $\lambda > 0$) is such that one can use the representation (4) instead of the representation (2). Now one could argue that cutting off an exponentially decreasing potential at sufficiently large distances should produce negligibly small physical effects, and yet it drastically alters the analytic and asymptotic behaviour of the scattering amplitudes[18]. This pathology must be regarded as an unphysical aspect of the potential scattering theory. In conclusion, we think that, in view of the crudeness of the potential scattering model, not much information can be extracted out of it and the best attitude is of going to a detailed phenomenological analysis of the experimental data at our disposal.

5. - PHENOMENOLOGICAL ANALYSIS.

In this section we present the phenomenological analysis of π^+-p backward elastic scattering in the momentum range between 3.55 and 7.00 GeV/c; the data considered have been determined at CERN by Baker et al. [28] and at Saclay by Banaigs et al.[29, 30]. The choice of these experimental data has been forced by the following arguments:

a) the great accuracy and richness of the data at very backward angles, i.e. in a region which is crucial for testing our theory;
b) the smallness of the Coulomb effect; in the momentum range considered the parameter η appearing in formulas (20) and (21) results to be less than $\sim 0.78 \times 10^{-2}$.

We were not able to find equally satisfactory and accurate data in α - nuclei scattering. Furthermore in this case the Coulomb interaction is not negligible and therefore we need a precise phase shift analysis, in order to extract the quasi - nuclear phase shifts (i.e. to perform a Coulomb subtraction).

Of course the π^+-p interaction must be treated relativistically and the spin of the proton has to be taken into account. However the main results of the previous Sections, for what concerns the scattering amplitude, can be extended to our case. Let's consider the amplitude

$$f(s,t,u) = g(s,t,u) + i\ \vec{\sigma}\cdot\vec{n}\, h(s,t,u) \qquad (23)$$

where g(s,t,u) and h(s,t,u) are the non-spin-flip and spin-flip amplitudes respectively, s, t, and u being the Mandelstam variables. Now we can expand these scattering amplitudes into partial waves and then perform a Sommerfeld-Watson transform. At this point, following the same arguments illustrated in Sect. 3, we can neglet the background integrals at backwards. For a more detailed analysis of this point, the reader is referred to ref. (31). In conclusion, the backward differential cross section for elastic scattering of pions from an unpolarized proton target turns out to be given by (see also ref. (7)):

$$\frac{d\sigma}{du} \simeq A \left| P_{\alpha(s)}(-\cos\vartheta) \right|^2 + B \left| P^1_{\alpha(s)}(-\cos\vartheta) \right|^2 \qquad (24)$$

where P^1_α is an associated Legendre function[25].

Now we fit the experimental data by the formula (24), using $a \equiv \mathrm{Re}\,\alpha$, $b \equiv \mathrm{Im}\,\alpha$, A and B as free parameters. In our numerical analysis we are faced with a nonlinear least - squares fitting problem, which we solve by the "grid" method. More precisely, we compute $\left| P_{\alpha(s)}(-\cos\vartheta_j) \right|^2$ and $\left| P^1_{\alpha(s)}(-\cos\vartheta_j) \right|^2$ (denoting by ϑ_j the j - th angle where the differential cross section has been measured) for ordered sequences of values of a and b; these sequences form a grid in the complex angular momentum plane. Then for any value of α, where the squared moduli of the Legendre functions have been evaluated, we determine A and B through a linear least-squares fitting program. As a last step, we select those values of a, b, A, B, which minimize the χ^2.

In Figs. 1, 2, 3 we present the fits of the backward peaks at three different beam momenta, i.e. 3.55, 5.20 and 7.00 GeV/c. The results of the fits are summarized in Table I. These results are

TABLE I

p(GeV/c)	s $[(\mathrm{GeV})^2]$	a	b	A $[\mu b/(\mathrm{GeV/c})^2]$	B $[\mu b/(\mathrm{GeV/c})^2]$	χ^2
3.55	7.57	5.0	0.70	133.818	0.508	13.07
5.20	10.66	7.1	1.34	39.650	0.016	17.6
7.00	14.04	8.7	1.39	16.871	0.000	48.89

much more complete and more precise and clear than those previously obtained by one of us[11].

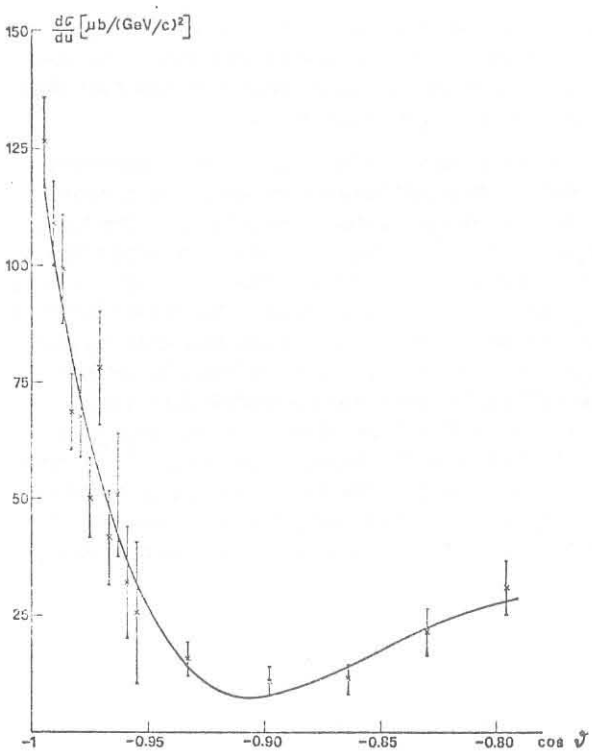

FIG. 1- π^+-p elastic scattering at s=7.57(GeV)^2 and for 0.796<-cos$\vartheta_{c.m}$ 1. The parameters of the fit are a=5.0, b=0.70 A=133.818 μb/(GeV/c)^2, B= =0.508 μb/(GeV/c)^2. The χ^2 value is 13.07 and the χ^2 test gives a probability of ~44%.

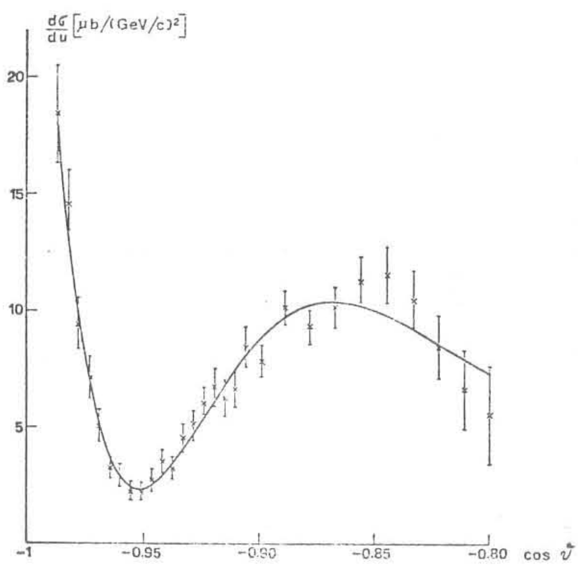

FIG. 2- π^+-p elastic scattering at s=10.66 (GeV)^2 and for 0.80< -cos$\vartheta_{c.m.}$ 1. The parameters of the fit are a=7.1, b=1.34 A=39.65 μb/(GeV/c)^2, B=0.016 μb/(GeV/c)^2. The χ^2 value is 17.6 and the χ^2 test gives a probability of ~ 85%.

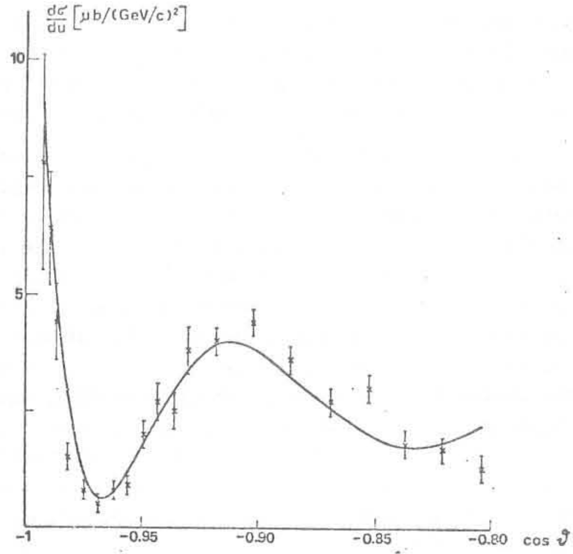

FIG. 3- π^+-p elastic scattering at s=14.04(GeV)^2 and for 0.804 -cos $\vartheta_{c.m}$ 1. The parameters of the fit are a=8.7, b=1.39 A=16.871 μb/(GeV/c)^2, B=0.000 μb/(GeV/c)^2. The χ^2 value is 48.89.

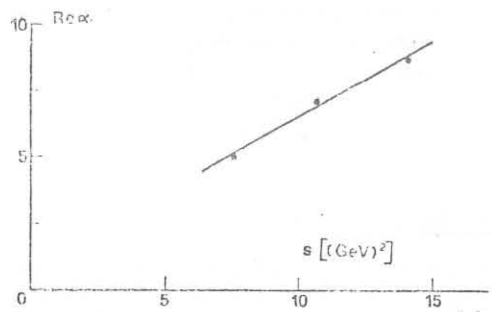

FIG. 4- The value of Re$\alpha \equiv$ a, obtained in the analysis of the experimental data shown in Figs. 1, 2, 3, are fitted by straight line. The correlation coefficient is r=0.9946.

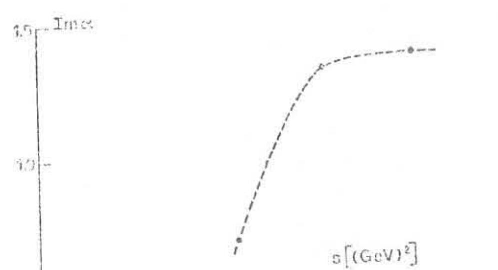

FIG. 5- The value of Im $\alpha \equiv$ b; they are connected by the dashed line.

Next we fit by a straight line the values obtained for Re $\alpha \simeq$ a (see Fig. 4). Fig. 5 shows the value obtained for Im $\alpha \equiv$ b;these points might be connected, for instance, by the dashed line drawn in the same Figure. The values of b cannot be fitted by a straight line; this is consistent with the fact that the height of the backward peaks versus s cannot be fitted by a single exponential.

The graphs of Fig. 1, 2, 3 indicate that, even if we are in a relativistic region, the backward enhancements can be fitted by the Sommerfeld poles, which were introduced in Sect. 3 in a non-relativistic framework. At this point let us remark that our model is considerably simpler than the theory based on the exchange of baryonic Regge trjectories(32), which is usually adopted to explain the π^+-p backward peaks. Furthermore the graphs of Figs. 4 and 5 show that a(s) and b(s) are increasing functions of s and α(s) does not turn back towards the left half-plane. Moreover the fact that b(s) is an increasing function of s implies the lowering of the backward peaks towards larger values of s, as the experiments confirm. In other words our theory gives a relationship between the asymmetry in the angular distribution, which is described through the Legendre functions $P_\alpha(\cos\vartheta)$, and the asymptotic vanishing of the backward peaks. For this reason the mathematics of our model seems to be much more appropriate than the one used by Bryant and Jarmie(17), who work with Bessel functions. From these considerations it follows that the backward enhancements which appear in the low energy nuclear physics can be explained by the same model; however, in this case, the clearness of the phenomenon is obscured by the strong Coulomb field, whose effect can be hardly subtracted in a satisfactory way.

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