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BEHAVIOUR OF C_p NEAR THE λ -POINT
IN ^4He FROM THE RENORMALIZATION GROUP EQUATIONS

1. - INTRODUCTION

We intend to expose here a treatment of the critical behaviour ⁽¹⁾ of a system of non relativistic bosons, whose interaction conserves the number of particles, making a systematical use of the renormalization group techniques ⁽²⁾. The actual physical system which should be described by such a model is the ⁴He liquid and the related critical phenomenon is the fluid-superfluid transition, where a set of very precise experimental data is available concerning the heat capacity (at constant pressure) around the transition temperature ^(3,4). For these reasons the critical behaviour actually studied will be the rate in which the heat capacity grows when the temperature decreases toward the critical value.

More precisely the technique we intend to use throughout this paper is the one of the renormalization group differential equations for the Green functions of a renormalizable field theory. The Green functions that naturally come into the problem are the finite-temperature (& time independent) Green functions of quantum statistics. By the use of the Wilson procedure ⁽²⁾ of functional integration over the higher frequency modes of the field it will be shown that, for the kind of problems we are interested in, the formalism of the finite-temperature Green function is not necessary and the theory can be developed by means of usual Green functions in an Euclidean space. We will use the device of considering a space of variable dimensions D and we recover the physical situation $D=3$ through an ϵ - expansion, where $D = 4 - \epsilon$.

The material of this paper is based on a series of lectures prepared for the Scuola di Perfezionamento in Fisica dell'Università di Trieste. Our scope is to present a rather self contained picture of the theoretical problem of the lambda point transition, in the framework of the renormalization group equation and the related approximate expansion making use of a language close to the one usually employed in particle physics. The theoretical work we describe is therefore not original, in particular there exist many excellent reviews on the calculations of the critical indices with the ϵ -expansion, which in general differ in the approach to the renormalization group and also, as it is obvious, in the practical purpose ^(5,6,7).

We have found it useful to expose again in detail the construction of the solution according to our own point of view, and keeping our attention toward the definite feature of the physical problem in which we are interested.

In particular we will, as a program, derive explicitly every relation we will need in the treatment, such as the relation between the observable thermodynamical quantities and their formal expressions as product field operators, or the relation among different critical indices. A brief discussions of the comparison of the results with the experimental data is also included.

2. - SHORT REVIEW OF THE FORMAL APPARATUS

A system of boson, with an interaction conserving the number of particles, repulsive and quadrilinear is described by a grand-canonical Hamiltonian (^{8,9}) having the following form (*):

$$H = H_0 + H_I = \int dX^D \left\{ \psi^\dagger(x) \left[-\nabla^2 - \mu \right] \psi(x) + \frac{1}{4} g \psi^{\dagger 2}(x) \psi^2(x) \right\} \quad (2.1)$$

Here $\psi^\dagger(x)$ [$\psi(x)$] is the creation [destruction] boson field operator, μ is the chemical potential, which must be determined imposing a given value for the number density (i.e., in the thermodynamical limit the total particle number N and the volume V go to infinity at fixed density N/V); g is the positive interaction strength. We have also taken for the mass $m = \frac{1}{2}$. For future convenience we work in $D = 4 - \epsilon$ spatial dimensions, the physical situation is obtained for $\epsilon = 1$. We wish to study the quantum statistical behaviour of the system; to this end the fundamental tool is the partition function

$$Z = \text{Tr} \exp [-\beta H] \quad , \quad \beta = 1/k_B T \quad . \quad (2.2)$$

In order to calculate the partition function it is convenient to develop a perturbative formalism similar to the ordinary Feynman graph expansion

(*) We will see, in the following, that the relevant results hold also in the case of more complicated interactions.

of quantum field theory. The generating functional of the Green functions of the quantum field theory is the analogue of the partition function in statistical mechanics (¹⁰). This similarity is well displayed by using the path integral technique, by means of which the partition function takes the expression

$$Z = \int d\psi d\psi^\dagger \exp \left[\frac{1}{\hbar} \int_0^{\beta\hbar} d\tau \int d\vec{x} \left\{ \psi^\dagger \left[-\frac{1}{2} \frac{\partial}{\partial \tau} + \nabla^2 + \mu \right] \psi - \frac{1}{4} g \psi^\dagger{}^2 \psi^2 \right\} \right]. \quad (2.3)$$

The functional integration ranges over the functions $\psi(\tau, \mathbf{x})$ and $\psi^\dagger(\tau, \mathbf{x})$ that are periodical in the variable τ in the interval $(0, \beta\hbar)$. The restriction to the periodical functions is dictated by the trace operation in eq. (2). In comparing this expression with the analogous one in quantum field theory it appears that the difference lies in the "time" variable. The lagrangian density for our system is in fact:

$$\mathcal{L} = \psi^\dagger \left(\frac{i}{2} \frac{\partial}{\partial t} + \nabla^2 + \mu \right) \psi - \frac{1}{4} g \psi^\dagger{}^2 \psi^2.$$


The statistical expression is obtained from the field theoretical one through a sort of Wick rotation on the time, which, moreover, is bound to vary over a finite interval.

The usual diagrammatic expansion can be easily obtained in the following way: replace \mathcal{L} by $\mathcal{L} + \psi J^\dagger + \psi^\dagger J$ where J is a complex function, thus we obtain $Z(\mu, T; J, J^\dagger)$ which is a functional of J, J^\dagger . From this functional we can derive the Green functions (including vacuum diagrams and disconnected pieces), taking the functional derivatives of $Z(J, J^\dagger)$ respect to J and J^\dagger at $J = J^\dagger = 0$

$$\begin{aligned} \langle T_\tau (\psi^\dagger(\tau_1, \mathbf{x}_1) \dots \psi^\dagger(\tau_n, \mathbf{x}_n) \psi(\sigma_1, \mathbf{y}_1) \dots \psi(\sigma_m, \mathbf{y}_m)) \rangle &= \quad (2.4) \\ &= \frac{\delta}{\delta J(\tau_1, \mathbf{x}_2)} \dots \frac{\delta}{\delta J(\tau_n, \mathbf{x}_n)} \frac{\delta}{\delta J^\dagger(\sigma_1, \mathbf{y}_2)} \dots \frac{\delta}{\delta J^\dagger(\sigma_m, \mathbf{y}_m)} Z(J, J^\dagger) \Big|_{J=J^\dagger=0} \end{aligned}$$

where $\delta/\delta J$ means functional derivative, and T_τ is the τ -ordering symbol.

Following this procedure we can obtain the Feynman rules in momentum space; they are (^{8,9,11}):

Propagator :  $\frac{1}{i\omega_n - k^2 + \mu}$ where $\omega_n = \frac{2\pi}{\beta} n$
 $n = 0, \pm 1, \pm 2, \dots$ (2.5)

Vertex :  $-\frac{g}{(2\pi)^3 \beta}$

Here, and from now on, $\hbar = k_B = 1$.

At each vertex both the three-momentum and the frequencies are conserved through the δ -factors of the type $\delta(\sum \vec{k}_i) \delta_{\sum \omega}$.

The presence of discrete frequencies is due to the periodicity conditions on the interval $0 \leq \tau \leq \beta$.

It is interesting to note that the Feynman rules lead to a topological structure of the graphs which is similar to the structure of a relativistic theory i.e. the propagation backward and forward in the "time" is allowed even if the propagator has not two poles in the frequency; this is a consequence of the fact that going to the configuration space by Fourier transforming the above propagator, the integral over the energy is replaced by a discrete sum and thus the usual ϑ -factor in the time, which selects the time direction, is avoided.

A quantity of more direct interest will be in the following the thermodynamical potential $\Omega(\mu, T)$, defined by the identity:

$$\Omega(\mu, T; J, J^\dagger) = -\frac{1}{\beta} \ln Z(\mu, T; J, J^\dagger)$$

with

$$\Omega(\mu, T; J = 0, J^\dagger = 0) \equiv \Omega(\mu, T)$$

It has the following properties:

- a) Every thermodynamical quantity ⁽¹²⁾ like the entropy, the pressure etc. can be obtained by deriving $\Omega(\mu, T)$ with respect to μ or T ; the heat capacity at constant pressure C_p (of our main interest), will be obtained as linear combination of second derivatives of $\Omega(\mu, T)$.
- b) It generates the connected Green's functions ⁽¹⁰⁾:

$$\langle T_{\tau}(\psi_1^{\dagger} \dots \psi_n^{\dagger}, \psi_1 \dots \psi_m) \rangle_{\text{connected}} = -\beta \frac{\delta}{\delta J_1} \dots \frac{\delta}{\delta J_n} \frac{\delta}{\delta J_1^{\dagger}} \dots \frac{\delta}{\delta J_n^{\dagger}} \Omega(\mu, T; J, J^{\dagger}) \Big|_{J=J^{\dagger}=0} \quad (2.6)$$

c) It is possible and useful to devise a slight generalization of Ω by introducing sources coupled to bilinears or in general local polynomials in ψ and ψ^{\dagger} . In this case the functional derivatives of Ω with respect to those new sources give all the connected Green functions related to those operators. As an example let us introduce $V(x)\psi^{\dagger}(x)\psi(x)$ then

$$-\beta \frac{\delta}{\delta V(\tau, x)} \frac{\delta}{\delta V(\sigma, y)} \Omega \Big|_{J, V=0} = \langle T_{\tau}(\psi^{\dagger}(\tau, x)\psi(\tau, x)\psi^{\dagger}(\sigma, y)\psi(\sigma, y)) \rangle_{\text{connected}} \quad (2.7)$$

Actually the derivative with respect to the chemical potential yields an expression of this kind ⁽¹³⁾.

3. - CRITICAL PHENOMENA

It is known that a system of noninteracting bosons can undergo a phase transition (Bose-Einstein condensation) when a finite fraction of the total number of particles goes into the ground state. Formally this is represented by the vanishing of the chemical potential. Our aim is to study the analogous phenomenon in the case of a system of interacting bosons.

In a standard approach, the mechanism of phase transitions can be illustrated in a way similar to the phenomenon of the spontaneous symmetry breaking in quantum field theory. Let us consider in fact the effective thermodynamical potential Γ defined as a Legendre transform of $\Omega(J, J^{\dagger})$

$$\Gamma(\psi_{cl}, \psi_{cl}^{\dagger}) = \Omega(J, J^{\dagger}) - \int d\tau d^D x (\psi_{cl} J^{\dagger} + \psi_{cl}^{\dagger} J) \quad (3.1)$$

where $\psi_{cl} = \delta\Omega/\delta J^\dagger$ is the so called classical field. In eq.(3.1) J and J^\dagger are to be thought as functionals of ψ_{cl} and ψ_{cl}^\dagger . It can be shown (14) that Γ is the generating functional of the one-particle irreducible Green functions $\Gamma^{(n)}$ and clearly ψ_{cl} equals $\langle\psi\rangle$, the average of the field ψ . If $J = J^\dagger = 0$ then ψ_{cl} is a constant, when translational invariance holds. Thus, if we consider Γ as a function of a constant ψ_{cl} , we obtain, dividing it by the infinite volume of the system, a quantity called V , which can be expanded in powers of ψ_{cl}^\dagger and ψ_{cl} :

$$V(\psi_{cl}, \psi_{cl}^\dagger) = -\bar{\mu} \psi_{cl}^\dagger \psi_{cl} + \frac{1}{4} \bar{g} \psi_{cl}^{\dagger 2} \psi_{cl}^2 + \dots$$

$\bar{\mu}$ and \bar{g} are constants expressible in terms of the parameters of the theory and they approach the quantities μ and g of the hamiltonian in eq. (2.1) when the interaction tends to zero. Now the condition of thermodynamical equilibrium requires that V has a minimum at $\psi_{cl} = \langle\psi\rangle$ and this gives origin to two distinct cases: if the parameter $\bar{\mu}$ is negative the minimum occurs for $\psi_{cl} = \langle\psi\rangle = 0$, i.e. we have the symmetric phase, whereas if $\bar{\mu}$ is positive the minimum is at $\psi_{cl} = \langle\psi\rangle \neq 0$, i.e. the non symmetric phase is recovered.

In our case the fact that $\langle\psi\rangle$ is different from zero can be assumed as a definition of Bose-Einstein condensation and this is consistent also with the interpretation of $|\langle\psi\rangle|^2$ as a density of particles in the condensate state (8,9).

Therefore the phase transition corresponds to $\bar{\mu} = 0$.

The study of the critical phenomena is then related to the analysis of a massless field theory. The peculiarities of a massless theory appear in its long distance behaviour, e.g. the correlation length $\xi \approx 1/\bar{\mu}$ goes to infinity. Accordingly, we will be mainly concerned with the infrared region, which gives the most important contribution to the matrix elements of field operators at the critical point $T = T_c$ (T_c being the critical temperature, for which $\bar{\mu} = 0$).

More precisely the main task is to calculate the behaviour of the relevant thermodynamical quantities, expressible in terms of matrix elements of products of field operators, as functions of $T-T_c$. For behaviour of the kind $(T-T_c)^x$, the aim is to calculate x , the critical index or exponent. We will not discuss the determination of T_c .

As we shall see the critical exponents can be obtained in a free theory by the dimensional analysis of the relevant operators. However a simple dimensional analysis of the leading operators does not give the correct answer when the interaction is present: it is known in fact that the critical indices are non trivial real numbers whose origin, in the language of quantum field theory, must lie in some nonperturbative effects which modify the canonical dimensions of the field operators. The usual tool for the investigations of the anomalous dimensions in field theory is the renormalization group ⁽¹⁵⁾. The actual calculations are done through the technique of the ϵ -expansion, developed by Wilson, which allows the estimate of the critical indices by the computation of few Feynman graphs.

4. - THE RENORMALIZATION GROUP METHODS FOR THE INFRARED BEHAVIOUR

Since we are looking to an infrared phenomenon it is intuitive that the degrees of freedom corresponding to high frequencies should not play any rôle. Those are the fields $\psi_n = \psi(\omega_n, p)$ $\psi_n^\dagger = \psi^\dagger(\omega_n, p)$ for $n \neq 0$ and $|p| > \Lambda$, where Λ is an arbitrary quantity. This fact is actually the starting point of the renormalization group technique. Let us consider the partition function

$$Z = \int \prod_n d\psi_n d\psi_n^\dagger e^{-A(\psi_n, \psi_n^\dagger)} \quad (4.1)$$

where A is the integral of the lagrangian, and put

$$\psi_n = \begin{cases} \psi \begin{pmatrix} 1 \\ p \end{pmatrix} & \text{for } n = 0, \quad |p| < \Lambda \\ \psi \begin{pmatrix} 2 \\ n \end{pmatrix} (p) & \text{otherwise.} \end{cases}$$

We define a new action through the functional integration of $\psi_n^{(2)}$

$$e^{-A'(\psi^{(1)})} = \int \prod_n d\psi_n^{(2)} d\psi_n^{+(2)} e^{-A(\psi^{(1)} + \psi_n^{(2)})}$$

The various terms of A' are obtained perturbatively as the amputated

connected Green functions in which the external lines are of the type $\psi^{(1)}$ and the internal lines of the type $\psi^{(2)}$. It is clear that the diagrams which give rise to A' are free from infrared divergences. A' contains in general a lot of couplings (the only limit is put by the rules following from the symmetry of the original A) which are in general non local, i.e. have a non trivial dependence on the external momenta. However they can be expanded around $p = 0$ giving rise to a series of local terms, with derivative couplings of increasing order. A' is called an effective action (*). The graphs which can be computed starting from A' are obtained from those of A by considering connected subgraphs of internal lines carrying $|p| > \Lambda$, $\omega_n \neq 0$ as effective interactions among the fields $\psi^{(1)}$. The procedure can now be iterated, putting:

$$\psi \begin{pmatrix} 1 \\ p \end{pmatrix} = \begin{cases} \psi \begin{pmatrix} 1,1 \\ p \end{pmatrix} & \text{for } |p| < \Lambda \\ \psi \begin{pmatrix} 1,2 \\ p \end{pmatrix} & \text{for } |p| > \Lambda \end{cases}$$

and integrating over $\psi^{(1,2)}$. In that way one gets A'' from A' , etc. The aim is to look for the convergence of the sequence $A^{(n)} \rightarrow A^*$; A^* would be the effective action at the fixed point (if it exists).

Actually we will not follow closely this approach up to the end. Rather, we will use the more standard formalism of the renormalization group equation, as it is written for a normalizable field theory. However we can always suppose that we have performed a few steps of the afore mentioned programme. We do not lose much information since it is clear that the "fundamental interaction" that must be put in the action is not wholly known, apart from the symmetry properties and the conservation rules known to hold in the problem. Therefore in particular we can take in our effective action only the $n = 0$ component of the field.

(*) An example of similar procedure in standard quantum field theory in the introduction of the effective interaction among photons in Q.E.D. when all the energies involved are smaller than the electron mass so that the electronic degrees of freedom can be suppressed ⁽¹⁶⁾.

Formally, this amounts to solve a "classical" statistical problem, where the effective partition function takes the "classical" form (*)

$$Z = \int d\psi(p) d\psi^\dagger(p) e^{-\beta H} \quad (4.2)$$

(the ψ , ψ^\dagger are now only functions of p).

Fortunately, of the full generality of the possible interactions only few ones are actually relevant in the infrared limit, namely only the couplings which have non negative energy dimensions (²). This result can be shown perturbatively if the dynamical effects give small corrections to the expectations from the normal dimensional arguments, and this in turn must be true if the whole approach makes sense (and it is verified perturbatively for small ϵ in the ϵ -expansion we will treat). Therefore we will assume it to hold. In our case we are interested in dimensions $D = 4-\epsilon$ (ϵ ranging from 0 to the physical case $\epsilon = 1$ and the relevant terms in the hamiltonian turn out to be those written in eq.(2.1).

Of course, following this procedure we are led to an effective action in which the original parameters μ, g are replaced by new parameters which depend on the former ones and T . This dependence is free of infrared singularities and thus, from the point of view of the study of the critical behaviour of any thermodynamical quantity the effective theory is perfectly equivalent to the original one (**).

In the following, up to chapter 8, we will not take explicitly into account the factor $\beta = \frac{1}{T}$ in front of Hamiltonian of eq.(4.2). Formally, we will write the formula for the case $\beta = 1$.

(*) A naive argument is that in the infrared limit every quantity with the dimension of an energy is rescaled to infinity. Therefore the effective temperature $k_B T \rightarrow \infty$, which gives the classical limit.

(**) Notice that we have to treat the constant which appears in front of the $|\nabla\psi|^2$ term in the action as an independent parameter. This parameter is strictly related to the normalization of the field operator; see next section.

5. - THE RENORMALIZATION GROUP EQUATION FOR THE RENORMALIZED GREEN FUNCTIONS

As said before, our task can be performed starting from the following bare effective Lagrangian (*)

$$\mathcal{L} = \psi_0^\dagger (\vec{\nabla}^2 + \mu_0) \psi_0 - \frac{1}{4} g_0 \psi_0^{\dagger 2} \psi_0^2$$

in which μ_0 , g_0 , ψ_0 are respectively the bare chemical potential, coupling constant and field. As it is well known for dimensions $D = 4 - \epsilon$ ($0 \leq \epsilon \leq 1$) the quantum field theory developed from this \mathcal{L} needs (and allows) the procedure of renormalization, involving the removal of ultraviolet divergences from the Feynman integrals. Since the worst situation from the point of view of the divergences occurs at $D = 4$, we set up a renormalization procedure for this case, automatically covering the $\epsilon > 0$ case.

The superficial degree of divergence of the relevant Feynman graphs, i.e. the one-particle irreducible Green functions $\Gamma^{(n)}$, can be easily estimated by power counting: it equal, for $D = 4$, the canonical dimension of the $\Gamma^{(n)}$; in general, for $D = 4 - \epsilon$ we have:

$$\dim \left[\Gamma^{(n)} \right] = 4 - \epsilon - n(2-\epsilon)/2$$

where n is the number of external lines. Thus in 4 dimensions only $\Gamma^{(2)}$ and $\Gamma^{(4)}$ are superficially divergent, more precisely $\Gamma^{(2)}$ diverges quadratically, and $\Gamma^{(4)}$ logarithmically. The bare Green functions are defined through a cut-off prescription and then the divergent, i.e. cut-off dependent terms are isolated and made harmless in the procedure of renormalization of mass, coupling constant and field.

This programme is usually carried out expressing the above \mathcal{L} in terms of renormalized finite quantities μ_R and g_R and of cut-off dependent counterterms which realize the subtractions of divergences from the Green functions:

(*) This effective Lagrangian is the negative of the Hamiltonian appearing in eq.(4.2).

$$\mathcal{L} = \psi^\dagger (\nabla^2 + \mu_R) \psi - \frac{g_R}{4} \psi^\dagger \psi^2 + (Z_3 - 1) \psi^\dagger (\nabla^2 + \mu_R) \psi + \\ - (Z_1 - 1) \frac{g_R}{4} \psi^\dagger \psi^2 + Z_3 \delta\mu \psi^\dagger \psi .$$

From this it follows that there is a multiplicative transformation from g_0, ψ_0 to g_R , of the form

$$\psi_0 = Z_3^{1/2} \psi \tag{5.1}$$

$$g_0 = Z_1 Z_3^{-2} g_R$$

and that:

$$\mu_0 = \mu_R + \delta\mu$$

where $\delta\mu$ is the mass counterterm. For the Green functions:

$$\Gamma_{\text{bare}}^{(n)}(p; g_0, \mu_0) = Z_3^{-n/2} \Gamma^{(n)}(p; g_R, \mu_R) \tag{5.2}$$

The renormalization constant are actually determined by imposing three normalization conditions. We choose a specific value fo the momentum $p^2 = \bar{p}^2 \neq 0$ where we impose

$$\Gamma^{(2)}|_{p^2=\bar{p}^2} = -\bar{p}^2 + \mu_R \qquad \Gamma^{(4)}|_{\text{s.p.}\bar{p}^2} = -g_R/(2\pi)^{4-\epsilon} =$$

$$\partial \Gamma^{(2)}/\partial p^2|_{p^2=\bar{p}^2} = -1 \qquad = -u \bar{p}^\epsilon/(2\pi)^{4-\epsilon}$$

s.p. \bar{p}^2 means the symmetric point $p_i \cdot p_j = \frac{1}{3} \bar{p}^2 (4 \delta_{ij} - 1)$.

Let us notice that we have introduced for later convenience a dimensionless coupling u . We have carefully taken $\bar{p}^2 \neq 0$ since we want to be free of infrared divergences in our normalization conditions and also in our definition of the renormalization constants when the inverse propagator goes to zero for $p^2 = 0$. Apart from that, \bar{p}^2 is arbitrary. The renormalized quantity $\Gamma^{(n)}$ depends explicitly on \bar{p}^2 as well as implicitly through the dependence of u, μ_R on \bar{p}^2 at fixed g_0, μ_0 . Also Z_3 depends on \bar{p}^2 at fixed g_0, μ_0 . Since the bare quantities do not know about \bar{p}^2 , by taking the total derivative of eq. (5.2) with respect to \bar{p}^2 , at fixed μ_0, g_0 we get:

$$0 = \left(\bar{p}^2 \frac{\partial}{\partial \bar{p}^2} + \bar{p}^2 \frac{\partial u}{\partial \bar{p}^2} \frac{\partial}{\partial u} + \bar{p}^2 \frac{\partial \mu_R}{\partial \bar{p}^2} \frac{\partial}{\partial \mu_R} - \frac{n}{2} \bar{p}^2 \frac{\partial \ln Z_3}{\partial \bar{p}^2} \right) \Gamma^{(n)} .$$

Remembering the canonical dimensions of $\Gamma^{(n)}$ and that $\dim [\mu_R] = 2$, we can derive the following dimensional equation:

$$\left(\bar{p}^2 \frac{\partial}{\partial \bar{p}^2} + \bar{p}^2 \frac{\partial}{\partial \bar{p}^2} + \mu_R \frac{\partial}{\partial \mu_R} - \frac{1}{2} \dim [\Gamma^{(n)}] \right) \Gamma^{(n)} = 0 .$$

Combining the two we obtain:

$$\left(\bar{p}^2 \frac{\partial}{\partial \bar{p}^2} + (\mu_R - \tilde{\vartheta}) \frac{\partial}{\partial \mu_R} - \beta \frac{\partial}{\partial u} + \frac{n}{2} \gamma - 2 + \frac{\epsilon}{2} + \frac{2-\epsilon}{4} n \right) \Gamma^{(n)} = 0, \quad (5.3)$$

where obviously

$$\gamma = \bar{p}^2 \frac{\partial \ln Z_3}{\partial \bar{p}^2}, \quad \tilde{\vartheta} = \bar{p}^2 \frac{\partial \mu_R}{\partial \bar{p}^2}, \quad \beta = \bar{p}^2 \frac{\partial u}{\partial \bar{p}^2} = -\frac{\epsilon}{2} u + \tilde{\beta} .$$

This is the desired renormalization group equation.

In taking the derivative of u we have explicitly separated the derivative of the trivial factor $\bar{p}^{-\epsilon}$ introduced in order to render u adimensional.

The various quantities γ , $\tilde{\vartheta}$, $\tilde{\beta}$ are calculated by applying the renormalization group equation to the Green functions on which the normalization conditions have been imposed and taking $p = \bar{p}$. Taking the equation for $\frac{\partial \Gamma^{(2)}}{\partial \bar{p}^2}$, and noticing that this Green function has normal dimensions equal to zero, we get:

$$\gamma = \bar{p} \left. \frac{\partial^2 \Gamma^{(2)}}{\partial (\bar{p}^2)^2} \right|_{p^2 = \bar{p}^2}$$

Taking the equation for $\Gamma^{(2)}$ we get:

$$\tilde{\vartheta} = \gamma (\mu_R - \bar{p}^2)$$

Finally, taking the equation for $\Gamma^{(4)}$:

$$\tilde{\beta} = 2 u \gamma - \bar{p}^{2-\epsilon} \left. \frac{\partial \Gamma^{(4)}}{\partial \bar{p}^2} \right|_{p = \bar{p}},$$

where the derivative means here the derivative on the whole dependence

on p^2 at the symmetric point.

The formal solution of the partial differential eq. (5.3) is obtained by considering two auxiliary functions, $u(t)$ and $\mu(t)$, the effective coupling constant and mass, where $t = |\ln p^2 / \bar{p}^2|$, which satisfy the equations

$$du(t)/dt = \frac{\epsilon}{2} u(t) - \tilde{\beta}(u(t), \mu(t))$$

$$d\mu(t)/dt = \mu(t) - (\mu(t) - \bar{p}^2) \gamma(u, \mu(t))$$

with the initial conditions $u(0) = u$, $\mu(0) = \mu_R$. We are interested in the situations in which a fixed point is reached, that is

$$u(t) \xrightarrow{t \rightarrow \infty} u^*, \quad \mu(t) \xrightarrow{t \rightarrow \infty} \mu^* .$$

In this case the behaviour of the Green functions is given by

$$\Gamma^{(n)}(p) \xrightarrow{p \rightarrow 0} p^{4 - \epsilon - n \frac{2 - \epsilon}{2}} - n \gamma(u^*, \mu^*) \quad (5.4)$$

The quantity $\gamma(u^*, \mu^*)$ is called the anomalous dimension of the field.

For dimensions $D = 4$, i.e. $\epsilon = 0$, we have a fixed point at the origin

$$u^* = 0, \quad \mu^* = 0.$$

This point is attractive in u , meaning that $u(t) \rightarrow u^*$, and repulsive in μ , meaning that it is only reached if μ is chosen to be zero the beginning. The special value of $\mu(0) = \mu_R$ is the critical value. The fact that u^* is attractive is seen from the lowest contribution to $\tilde{\beta}$:

$$\tilde{\beta} = c u^2 + \dots$$

with $c > 0$. The theory at $\epsilon = 0$ is asymptotically infrared free.

The idea of the ϵ -expansion is to consider ϵ as a small quantity and look for a fixed point nearby $u = 0$, $\mu = 0$. At the lowest order, for instance, we will have

$$u^* = \epsilon/2 c$$

$$\mu^* = -\bar{p}^{-2} \gamma(u^*, 0)$$

where $\gamma = A u^{*2}$, with $A > 0$.

A couple of remarks. First, since u^* and μ^* are finite we can assume that $\Gamma^{(n)}(p, u^*, \mu^*)$ is also finite. Second in the solution of the renormalization group equation the anomalous dimensions come from the integral

$$\int_0^t dt' \gamma(u(t'), \mu(t')) = t\gamma(u^*, \mu^*) + \int_0^t dt' [(\gamma(u(t'), \mu(t')) - \gamma(u^*, \mu^*))]$$

The last integral is convergent for $t \rightarrow \infty$, as one can verify.

Again the fixed point is repulsive in μ . We have in general a critical line in the (u, μ) plane along which the fixed point is reached.

We assume that we are on the critical line, i.e. we assume that our parameters are at the critical point value. We do not attempt the calculation of the value of the parameters for which this situation is obtained.

6. - SOME STANDARD THERMODYNAMICS

We want to study the specific heat at constant pressure. Here we derive its expression in term of the thermodynamical potential Ω (*). Calling S the entropy, the specific heat at constant volume is given by:

$$C_V = T \left. \frac{\partial S}{\partial T} \right|_N$$

where N is the particle number, which is held fixed in this derivative. The entropy is related to Ω by:

$$S = - \left. \frac{\partial \Omega}{\partial T} \right|_{\mu}$$

The particle density N is related to Ω by:

$$N = - \left. \frac{\partial \Omega}{\partial \mu} \right|_T$$

(*) Since for us the volume V is a constant, we take $V = 1$.

We get for C_V ¹²⁾:

$$C_V = T \frac{\Omega_{\mu\mu}^2 - \Omega_{TT} \cdot \Omega_{\mu\mu}}{\Omega_{\mu\mu}}$$

where we use the notation $\Omega_{xy} = \partial^2 \Omega / \partial x \partial y$. In order to calculate C_P we use the relation ¹²⁾:

$$C_P - C_V = \frac{-T \left(\frac{\partial P}{\partial T} \right)_N^2}{\frac{1}{N} \left(\frac{\partial P}{\partial N} \right)_T}$$

Since $P = -\Omega$:

$$\left. \frac{\partial P}{\partial T} \right|_N = \left. \frac{\partial P}{\partial T} \right|_\mu - \left. \frac{\partial P}{\partial \mu} \right|_T \cdot \left. \frac{\partial N}{\partial T} \right|_\mu \cdot \left(\left. \frac{\partial N}{\partial \mu} \right|_T \right)^{-1} = S - N \frac{\Omega_{\mu T}}{\Omega_{\mu\mu}}$$

On the other hand

$$\frac{1}{N} \left. \frac{\partial P}{\partial \left(\frac{1}{N} \right)} \right|_T = -N \left. \frac{\partial P}{\partial N} \right|_T = -N \left. \frac{\partial P}{\partial \mu} \right|_T \cdot \left. \frac{\partial \mu}{\partial N} \right|_T = N^2 \frac{1}{\Omega_{\mu\mu}}$$

Therefore we get:

$$C_P = T \left[-\frac{S^2}{N^2} \Omega_{\mu\mu} + \frac{2S}{N} \Omega_{\mu T} - \Omega_{TT} \right] \quad (6.1)$$

7. - RELEVANT OPERATORS FOR THE CONSTANT PRESSURE SPECIFIC HEAT

We have seen from the previously derived formula that C_P is expressed in terms of first and second derivatives with respect to μ and T of $\Omega(\mu, T)$ (S and N are in fact first derivatives). Those contributions can be put in form of matrix elements of composite operators remembering that

$$\Omega = -T \ln \int d\psi d\psi^+ \exp \frac{1}{T} \int d^D x [\psi^+ (\vec{\nabla}^2 + \mu_0) \psi - \frac{1}{4} g_0 \psi^{+2} \psi^2]$$

Therefore we find, considering the first and second partial derivatives, the following expectation values:

$$\langle \int d^D x \psi^+ \psi \rangle, \quad \langle \int d^D x \psi^+ \vec{\nabla}^2 \psi \rangle, \quad \langle \int d^D x \psi^{+2} \psi^2 \rangle,$$

$$\langle \int d^D x \psi^+ \psi, \int d^D y \psi^+ \psi \rangle_{\text{connected}}, \langle \int d^D x \psi^+ \nabla^2 \psi, \int d^D y \psi^+ \psi \rangle_{\text{connected}}$$

and so on.

Each of those expressions contains an infinite factor in the form of a δ -function, which corresponds to the infinite volume. Since we are interested in quantities per unit volume that factor has to be dropped. The resulting quantities will be called the Green functions related to those operators. A simple dimensional analysis gives for the Green function related to $\langle \int d^4 x \psi^+ \psi \rangle$ dimensions p^2 , for those related to $\langle \int d^4 x \psi^+ \nabla^2 \psi \rangle$ and $\langle \int d^4 x \psi^{+2} \psi^2 \rangle$ dimensions p^4 whereas for $\langle \int d^4 x \psi^+ \psi, \int d^4 y \psi^+ \psi \rangle$ we have dimensions p^0 , and the others have dimensions p^2 or p^4 .

First we notice that when we find a Green function behaving like a positive power of p the actual result is that it goes to a constant and the positive power concerns the correction terms; the constant part (that is expected to be, in general, different from zero) comes from the contribution of the high frequency modes.

Essentially for the same reason we will see that a behaviour p^0 means actually $\ln p$.

Second, the consistency of the ϵ -expansion requires that the anomalous dimensions of the operators cannot change too much from their normal dimensions and therefore the leading infrared contribution is given by the Green functions of lowest canonical dimensions. This indeed can be checked by evaluating explicitly the corrections in the ϵ -expansion. We conclude that the dominant contribution to C_p comes from the Green function corresponding to $(\bar{\psi})$

$$\langle \int d^D x \psi^+ \psi, \int d^D y \psi^+ \psi \rangle .$$

8. - ANOMALOUS DIMENSIONS OF COMPOSITE OPERATORS

Now we have to extend the renormalization group formalism in order to be able to compute the anomalous dimensions of composite operators; in particular we are interested in the $\int d^D x \psi^+ \psi$ operator. We have just seen at the beginning that the study of the Green functions with $\psi^+ \psi$ insertions can be done by introducing in \mathcal{L} a coupling $[a, \nabla \psi^+ \psi]$, where at

$D=4, a_0$ has dimension p^2 . The connected Green functions are thus given by

$$\begin{aligned} a_0^\ell &< T(\psi^+(y_1)\psi(y_1)\dots\psi^+(y_\ell)\psi(y_\ell)\psi^+(x_1)\dots\psi(x_n)) > = \\ &= -\frac{1}{T} \frac{\delta}{\delta V(y_1)} \dots \frac{\delta}{\delta V(y_\ell)} \frac{\delta}{\delta J(x_1)} \dots \frac{\delta}{\delta J(x_n)} + \Omega(\mu, T; J, J^+, V) \end{aligned}$$

From the connected Green functions the one particle irreducible Green functions $\Gamma^{(\ell, n)}(p_1 \dots p_\ell; q_1 \dots q_n)$ are obtained in the usual way ⁽⁴⁾. The p-momenta are carried by the V's and the q-momenta by the ψ 's. Those quantities require new independent renormalizations when $D = 4$. It is easy to see that for $n > 0$ there is only one superficially divergent quantity that is $\Gamma^{(1, 2)}$ with a logarithmic divergence. This leads formally to a renormalization of the coupling constant a_0 which we achieve by imposing, for the renormalized constant a ,

$$\Gamma^{(1, 2)} \Big|_{\bar{p}^2} = -a \quad .$$

To be explicit, we take the squared momentum related to the V to be $p^2 = \frac{4}{3} \bar{p}^2$, and the squared momenta carried by the ψ^+, ψ fields to be equal to \bar{p}^2 . We avoid in this way infrared divergences for $\mu \rightarrow 0$.

At the perturbative order we are interested in, the graphs which contribute to $\Gamma^{(1, 2)}$ are expressed by the same integrals, with the same renormalization prescription, as those which contribute to $\Gamma^{(4)}$.

Combining as before dimensional analysis with the arbitrariness of \bar{p} we get the renormalization group equation for $\Gamma^{(\ell, n)}$

$$\begin{aligned} &\left(p^2 \frac{\partial}{\partial p^2} + (\mu_R^{-\tilde{\theta}}) \frac{\partial}{\partial \mu_R} + \left(\frac{\epsilon}{2} u - \tilde{\beta} \right) \frac{\partial}{\partial u} + \right. \\ &\quad \left. + \frac{n}{2} \gamma - 2 + \frac{\epsilon}{2} + \frac{2-\epsilon}{4} n + \ell(1-\tilde{\eta}) \right) \Gamma^{(n, \ell)} = 0 \end{aligned} \quad (8.1)$$

where $\tilde{\eta} = \bar{p}^2 \partial a / \partial \bar{p}^2$ and we have used the fact that $\Gamma^{(n, \ell)} \sim a^\ell$. Taking eq. (8.1) for $\Gamma^{(1, 2)}$ at the normalization point of $\Gamma^{(1, 2)}$ we obtain

$$\tilde{\eta} = -\bar{p}^2/a \left. \frac{\partial \Gamma^{(2, 1)}}{\partial p^2} \right|_{\bar{p}^2} + \gamma \quad .$$

It is clear that $\tilde{\eta}$ is the anomalous dimension of the composite operator $\psi^+\psi$; incidentally, we have seen that the necessity of new distinct subtractions leads to the fact that the anomalous dimension of $\psi^+\psi$ is not, as naively it might be expected, twice the anomalous dimension of the field ψ .

We are also interested in the case $n = 0$ and in particular in $\Gamma^{(2)}$, which in fact is related to $\langle \int d^D x \psi^+ \psi, \int d^D y \psi^+ \psi \rangle$, the leading contribution to C_p . To be precise we wish to study the behaviour of $\Gamma^{(2,0)}$ for $p=$ in terms of $T-T_c$ as $T \rightarrow T_c$. First we observe that $\Gamma^{(2,0)}(p^2)$ is logarithmically divergent in 4 dimensions. To renormalize it we introduce a subtractive counterterm by imposing that for $p^2 = \bar{p}^2$

$$\Gamma^{(2,0)}(\bar{p}^2) = 0 .$$

The renormalization group equation is in this case inhomogeneous and take the form

$$\left(p^2 \frac{\partial}{\partial p^2} + (\mu_R - \tilde{\vartheta}) \frac{\partial}{\partial \mu} + \left(\frac{\epsilon}{2} u - \tilde{\beta} \right) \frac{\partial}{\partial u} - 2 + \frac{\epsilon}{2} + 2(1 - \tilde{\eta}) \right) \Gamma^{(2,0)} = C$$

And indeed specializing $p^2 = \bar{p}^2$ in this equation we find:

$$C = \bar{p}^2 \left. \frac{\partial \Gamma^{(2,0)}}{\partial p^2} \right|_{\bar{p}^2} .$$

The constant C does not correspond to any anomalous dimension and its actual value is irrelevant for the subsequent treatment. We then con-

We then consider the dependence of $\Gamma^{(2,0)}(p=0)$ on the unrenormalized parameter μ_0 , when $\mu_0 \rightarrow \mu_{0c}$ and μ_{0c} corresponds to the critical value, i.e. the particular value of μ_0 which at a given temperature and coupling constant allows to reach the fixed point.

Let us suppose we are nearby the critical point $\mu_R = \mu^*$ for which $\mu_R - \tilde{\vartheta} = 0$ and introduce $\Delta = \mu_R - \mu^*$, and $\mu_R - \tilde{\vartheta} = x\Delta$. Setting for simplicity $u = u^*$, at $p^2 = 0$ we get the equation

$$\left(x \Delta \frac{\partial}{\partial \Delta} + \frac{n}{2} \gamma - 2 + \frac{\epsilon}{2} + \frac{2-\epsilon}{4} n + \ell (1 - \tilde{\eta}) \right) \Gamma^{(\ell,n)} = 0 .$$

This gives

$$\Gamma(\ell, n) \sim \Delta^{(2-\epsilon/2-n/2 \gamma - \frac{2-\epsilon}{4} n - \ell(1-\tilde{\eta}))} / x \quad (8.2)$$

for $\Delta \rightarrow 0$. We then find the relation between Δ and $\omega = \mu_0 - \mu_{0c}$. The derivative of a Green function with respect to ω corresponds to the insertion, in every graph, of the operator $\psi^+ \psi$. Let us consider $\Gamma^{(0,2)}(p^2=0)$. Its derivative $\partial \Gamma^{(0,2)} / \partial \omega$ will obey to the same equation as $\Gamma^{(1,2)}$, therefore $\Gamma^{(1,2)} \sim \frac{\partial \Gamma^{(0,2)}}{\partial \omega} \sim \Delta^{(\tilde{\eta}-\gamma)} / x$ whereas $\Gamma^{(0,2)} \sim \Delta^{(1-\gamma)} / x$.

We call the attention on the fact that the renormalization constants which are involved in the computation of $\frac{\partial \Gamma^{(0,2)}}{\partial \omega}$ and $\Gamma^{(0,2)}$ are free of infrared divergences, since we have chosen the normalization point at $p^2 = \bar{p}^2 \neq 0$. Therefore the infrared singular behaviour is the same for renormalized and unrenormalized quantities. From this we infer that

$$\Delta \sim \omega^x \frac{1}{1-\tilde{\eta}} \quad (8.3)$$

In particular we get

$$\Gamma^{(2,0)} \sim \omega^{(-\frac{\epsilon}{2} + 2\tilde{\eta})} \frac{1}{1-\tilde{\eta}}, \quad \Gamma^{(1,0)} \sim \omega^{(1-\frac{\epsilon}{2} + \tilde{\eta})} \frac{1}{1-\tilde{\eta}} \quad (8.4)$$

In general, the equations (8.2) and (8.3) give the answer for the behaviour $\omega \rightarrow 0$ at $p = 0$. We see that this behaviour does not depend on the unspecified quantity x .

If it happens that the exponent of ω in eq. (8.4) is zero (for instance, because $\frac{\epsilon}{2} = 2\tilde{\eta}$), then the behaviour of $\Gamma^{(2,0)} \sim \ln \omega$. This follows from the inhomogeneous equation which $\Gamma^{(2,0)}$ satisfies. A final step can be made in order to relate the behaviour in ω with the behaviour as $T \rightarrow T_c$ at fixed density number N .

Let us remember first the definition of μ_{0c} : for $\mu_0 = \mu_{0c}$

$$\Gamma_{n=0}^{(0,2)}(p=0) = 0 \quad .$$

This is solved for $\mu_{0c} = \mu_{0c}(T)$, because the effective value of μ_0 receives contributions also from loops containing the $n \neq 0$ degrees of freedom. The

definition of the critical temperature T_c at fixed density N is the following:

$$N = \sum_n \Gamma_n^{(1,0)}(p=0, \mu_c(T_c), T_c) .$$

As a function of T , $\mu_{0c}(T)$ is expected to have no singularities, in particular for $T = T_c$, and therefore we write

$$\mu_{0c}(T) = \mu_{0c}(T_c) + B(T-T_c) . \quad (8.5)$$

In general, that is away from the critical line, we can define a function $\mu_0(T)$ from the requirement that the density is N :

$$N = \sum_n \Gamma_n^{(1,0)}(\mu_0(T), T) .$$

The density function $\sum_n \Gamma_n^{(1,0)}(\mu_0, T)$ is an analytic function of T , around T_c , since the dependence in T coming from the $n \neq 0$ contributions is carried by terms which are not infrared singular, and we have for $T \rightarrow T_c$:

$$\sum_n \Gamma_n^{(1,0)}(\mu_0, T) - \sum_n \Gamma_n^{(1,0)}(\mu_0, T_c) \sim A(T-T_c) . \quad (8.6)$$

On the contrary we have a singularity in μ_0 for $\mu_0 \rightarrow \mu_{0c}(T)$:

$$\begin{aligned} \sum_n \Gamma_n^{(1,0)}(\mu_0, T) - \sum_n \Gamma_n^{(1,0)}(\mu_{0c}(T), T) &\sim D(\mu_0 - \mu_{0c}(T))^\sigma + \\ &+ D_1(\mu_0 - \mu_{0c}(T)) . \end{aligned} \quad (8.7)$$

σ represents a previously obtained critical index, see eq. (8.4), with $\omega = \mu_{0c}(T)$, and the other term comes from the non singular $n \neq 0$ terms.

From the eq. (8.7) we get

$$\begin{aligned} 0 &= \sum_n \Gamma_n^{(1,0)}(\mu_0(T), T) - \sum_n \Gamma_n^{(1,0)}(\mu_{0c}(T_c), T_c) \sim \\ &\sim D(\mu_0(T) - \mu_{0c}(T))^\sigma + D_1(\mu_0(T) - \mu_{0c}(T)) + \\ &+ \sum_n \Gamma_n^{(1,0)}(\mu_{0c}(T), T) - \sum_n \Gamma_n^{(1,0)}(\mu_{0c}(T_c), T_c) . \end{aligned}$$

From eq. (8.6) considered at $\mu_0 = \mu_{0c}(T_c)$ we get, using eq. (8.5),

$$\begin{aligned} & \sum_n \Gamma_n^{(1,0)}(\mu_{0c}(T), T) - \sum_n \Gamma_n^{(1,0)}(\mu_{0c}(T_c), T_c) = \\ & = \sum_n \Gamma_n^{(1,0)}(\mu_{0c}(T_c) + B(T-T_c), T) - \sum_n \Gamma_n^{(1,\gamma)}(\mu_{0c}(T_c), T) + A(T-T_c) = \\ & = A(T-T_c) + D(B(T-T_c))^\sigma + D_1 B(T-T_c). \end{aligned}$$

Finally: $0 = D(\mu_0(T) - \mu_{0c}(T))^\sigma + D_1(\mu_0(T) - \mu_{0c}(T)) +$
 $+ (A + D_1 B)(T - T_c) + D B^\sigma (T - T_c)^\sigma .$

If $\sigma < 1$ the terms with the critical exponent σ dominate, if $\sigma > 1$ the leading terms are the regular ones. But in any case the answer is the same, namely

$$\mu_0(T) - \mu_{0c}(T) \sim T - T_c .$$

Therefore eq. (8.4) gives the behaviour of the specific heat substituting ω with $(T - T_c)$.

9. - CALCULATION OF THE RENORMALIZATION GROUP STRUCTURE FUNCTIONS

In this section we will calculate the anomalous dimensions, namely the values of γ , η at the critical point (u^*, μ^*) . It was shown that u^* and μ^* respectively the zeros of $\beta = -\frac{\epsilon}{2}u + \tilde{\beta}$ and of $\vartheta = \mu - \tilde{\vartheta}$. Actually μ^* is $O(u^{*2})$, and at the order we work, we can put in the propagators $\mu^* = 0$.

Since we wish to calculate the anomalous dimensions up to the order ϵ^2 , also u^* will be calculated up to this order; from this it follows that β must be calculated up to the order u^3 since it begins at the order u .

In fact, we have that:

$$\beta = -\frac{\epsilon}{2}u + 2u\gamma - \frac{-2-\epsilon}{p} \left. \frac{\partial \Gamma^{(4)}}{\partial p^2} \right|_{s.p.}$$

and so we need first γ :

$$\gamma = \bar{p}^2 \frac{\partial^2 \Gamma^2}{\partial (\bar{p}^2)^2} \Big|_{\bar{p}^2} = \frac{\partial^2 \Sigma}{\partial (\bar{p}^2)^2} \Big|_{\bar{p}^2}, \text{ if } \Gamma^{(2)} = -p^2 + \Sigma(p).$$

Σ is the integral corresponding to the graph of Fig. 1 and it is sufficient to know its value for $\epsilon = 0$.



Fig. 1

Thus we have

$$\gamma = \frac{u^2}{2} \bar{p}^2 f''(p) = u^2 \frac{\pi^4}{4} \quad \text{with } f(p) = \int \frac{d^4 k d^4 q}{(p-k-q)^2 k^2 q^2} - \text{counter term}$$

and the factor $\frac{1}{2}$ is the statistical weight of the graph.

The other contributions to β come from $\Gamma^{(4)}$. We need $\Gamma^{(4)}$ up to order u^3 : the graph at this order can be calculated for $\epsilon = 0$ while the graphs of the order u^2 must be calculated in $4-\epsilon$ dimensions:

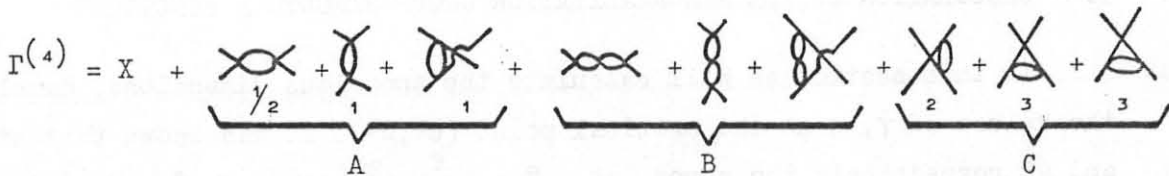


Fig. 2

where we put the statistical weights explicitly. Clearly the 1st graph does not contribute to $\partial \Gamma^{(4)} / \partial p^2$ and, as it easy to see, also the graphs of class B do not contribute to it. From this is follows:

$$\bar{p}^{-2} \frac{\partial \Gamma^4}{\partial p^2} \Big|_{s.p. \bar{p}^2} = \frac{5}{2} u^2 p^{2+2\epsilon} \frac{\partial h}{\partial p^2} \Big|_{s.p. \bar{p}^2} - 8 u^3 \frac{\partial \ell}{\partial p^2} \Big|_{s.p. \bar{p}^2} .$$

Here $h = \int \frac{d^D k}{k^2 (p-k)^2}$ - counterterm and

$$u^2 \bar{p}^{-2+2\epsilon} \frac{\partial h}{\partial p^2} \Big|_{s.p. \bar{p}^2} = u^2 \bar{p}^{-\epsilon} (K \epsilon \pi^2) .$$

We have put

$$K = \frac{\pi^2}{2} [\ln \pi + 2 - \gamma] , \gamma \text{ is the Euler-Mascheroni constant.}$$

For ℓ we have:

$$\ell = \int \frac{d^4 k_1}{k_1^2 (2p-k_1)^2} - \int \frac{d^4 k_2}{k_2^2 (k_1+p-k_2)^2} - \text{counterterms} ,$$

and

$$\bar{p}^{-2} \frac{\partial \ell}{\partial p^2} \Big|_{s.p. \bar{p}^2} = -\pi^4$$

Adding the various contributions

$$\beta = -\frac{\epsilon}{2} u + \frac{5}{2} (\pi^2 - K \epsilon) u^2 - \frac{15}{2} \pi^4 u^3$$

which has the trivial zero $u = 0$ and the infrared stable fixed point u^* :

$$u^* = \frac{\epsilon}{5 \pi^2} + \left(\frac{K}{5 \pi^4} + \frac{3}{25 \pi^2} \right) \epsilon^2 .$$

From this we get:

$$\gamma = \frac{\epsilon^2}{100} .$$

Now we need

$$\tilde{\eta} = -\frac{\bar{p}}{a} \frac{\partial \Gamma^{21}}{\partial p^2} \Big|_{\bar{p}^2 + \gamma} .$$

At the order we work the following diagrams contribute to $\Gamma^{(2,1)}$:



Fig. 3

The 1st and 3rd graph do not contribute to $\left. \frac{\partial \Gamma^{(2,1)}}{\partial p^2} \right|_{p^2}$, the 2nd is computed in $4-\epsilon$ dimensions, the 4th in 4 dimensions.

Its clear that these diagrams are essentially those calculated for $\Gamma^{(4)}$, but with different statistical weights. Thus we have:

$$\tilde{\eta} = \pi^2 u - K \epsilon u - \frac{3}{2} \pi^4 u^2 + \gamma .$$

Substituting in it the value of u^* we get up to the order ϵ^2 :

$$\tilde{\eta} = \frac{\epsilon}{5} + \frac{7}{100} \epsilon^2 .$$

Now we have all the ingredients which are necessary in the calculation of the critical index α of the heat capacity at constant pressure C_p , defined by:

$$C_p \sim (T - T_c)^{-\alpha} .$$

It was previously shown that:

$$\alpha = \left(\frac{\epsilon}{2} - 2 \tilde{\eta} \right) \frac{1}{1 - \tilde{\eta}}$$

and if we insert into this formula the values of $\gamma, \tilde{\eta}$ and expand up to order ϵ^2 , we get:

$$\alpha = \frac{\epsilon}{10} - \frac{6 \epsilon^2}{50} .$$

Further taking for ϵ the physical value $\epsilon = 1$ we get

$$\alpha = -\frac{1}{50}$$

at the order ϵ^2 .

10. - COMPARISON WITH EXPERIMENTS

The experimental results on the dependence of C_p on $|T-T_c|$ is well represented by a logarithmic plot

$$C_p = A \ln |T-T_c| + B .$$

Experiments so far have been performed up to $|T-T_c| \sim 10^{-6}$ °K.

In our language a logarithmic dependence means $\alpha = 0$. The result of the ϵ -expansion is that α is small. And indeed the result of the order ϵ^2 , $\alpha = -1/50$ for $\epsilon = 1$, is not in disagreement with the experimental data.

Of course, the ϵ -expansion represents an approximate solution of the renormalization group equation and it is difficult to estimate with sufficient precision the confidence we can put on a result at a given order. There are suggestions that the series in ϵ is an asymptotic one. For the case of the Ising model good results are obtained at $O(\epsilon^2)$ (*).

Let us show explicitly in Fig. 4 the comparison of the result for α calculated up to the order ϵ^2 : $\alpha = -1/50$ for $\epsilon = 1$, with the experimental data. Our plot is of the form

$$C_p = A f + B$$

(*) Actually for the critical exponent η , which is the same as our $\gamma/2$, the next to the lowest order is ϵ^3 . A good result for η is obtained including the $O(\epsilon^3)$ term.

Fig. 4

Experimental data of C_p ref. (3)
ref. (4)

Theoretical prediction:

$$C_p = Af(T-T_c) + B \quad f(T-T_\lambda) = \frac{1-(T-T_c)^{0.02}}{0.02}$$

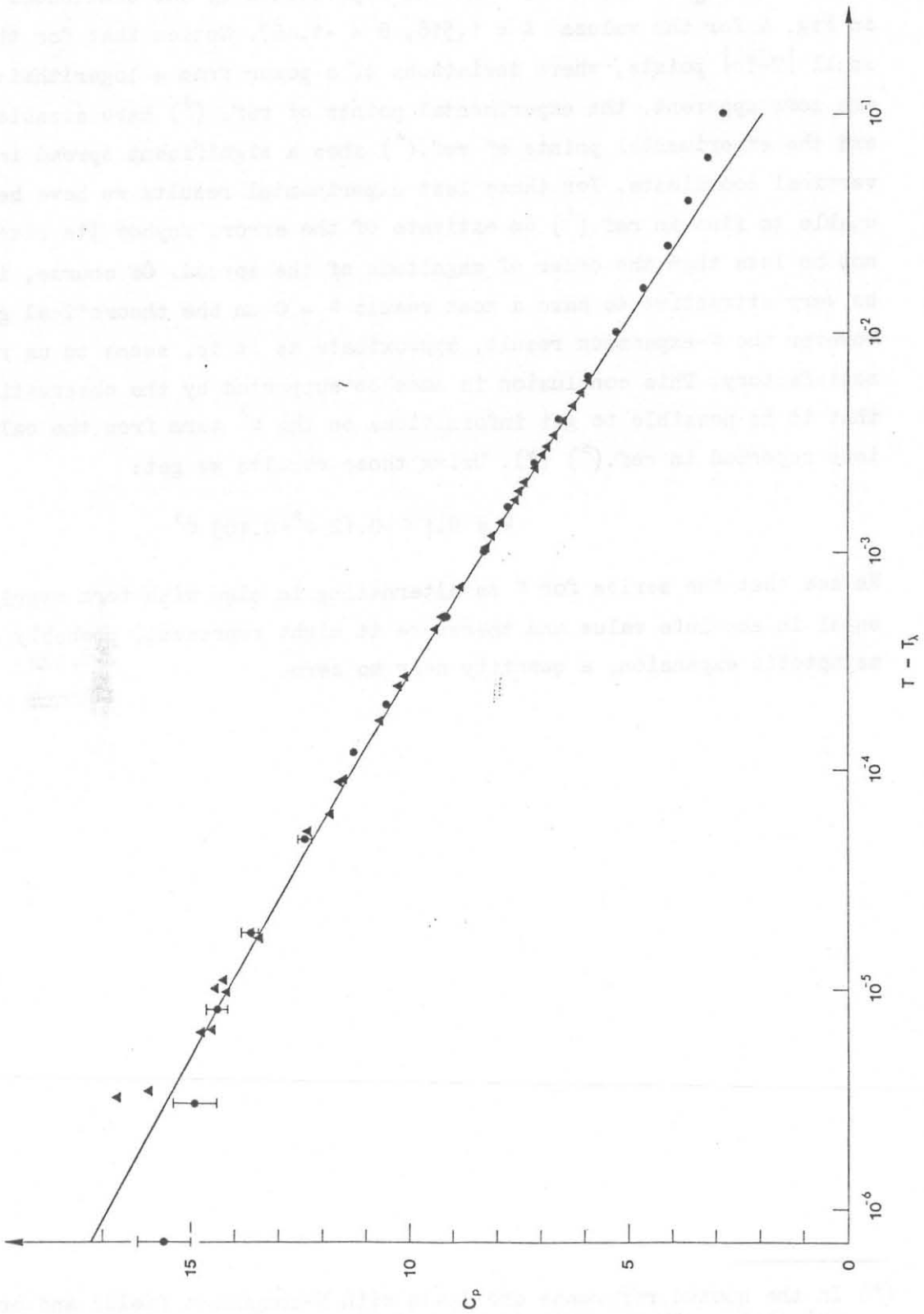
$$A = 1.518$$

$$B = -1.467$$

Empirical determination

The abscissa of experimental points $(T-T_c)^\circ\text{K}$ of ref.(3) have been normalized to those of ref.(4), i.e. from their original value the quantity: $2 \cdot 10^{-2}^\circ\text{K}$ has been subtracted for the reason discussed in ref.(4).

The ordinates are expressed in $(\text{J g}^{-1}\text{K}^{-1})$.



where $f = -\frac{1}{\alpha} (1-|T-T_c|^{-\alpha})$. This is represented by the continuous line in Fig. 4 for the values $A = 1.518$, $B = -1.467$. Notice that for the small $|T-T_c|$ points, where deviations of a power from a logarithmic law are more apparent, the experimental points of ref. (3) have sizable error, and the experimental points of ref. (4) show a significant spread in the vertical coordinate. For those last experimental results we have been unable to find in ref. (4) an estimate of the error, anyhow its size could not be less than the order of magnitude of the spread. Of course, it would be very attractive to have a neat result $\alpha = 0$ on the theoretical grounds. However the ϵ -expansion result, approximate as it is, seems to us rather satisfactory. This conclusion is somehow supported by the observation that it is possible to get informations on the ϵ^3 term from the calculations reported in ref. (5) (*). Using those results we get:

$$\alpha = 0.1 \epsilon - 0.12 \epsilon^2 + 0.105 \epsilon^3 .$$

We see that the series for ϵ is alternating in sign with term nearly equal in absolute value and therefore it might represent, probably as an asymptotic expansion, a quantity near to zero.

(*) In the quoted reference one deals with N-component fields and one calculates other critical exponent that are related through simple and known relations to α . Through those relations, and taking $N=2$, the exponent of our interest is finally obtained.

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