Istituto Nazionale di Fisica Nucleare
Sezione di Genova

INFN/AE-75/4
10 Aprile 1975

E. Di Salvo: ANALYSIS ON THE $N^*(1700)$ RESONANCE.

ABSTRACT - We will present some model independent tests to analyze the production mechanism and the decay of the $N^*(1700)$. Some of them are of general interest.

INTRODUCTION. -

The $N^*(1700)$ resonance has been studied in several reactions :

$$\begin{aligned} \pi^{\pm} p &\rightarrow \pi^{\pm}(p\,\pi^+\pi^-) \\ K^{\pm} p &\rightarrow K^{\pm}(p\,\pi^+\pi^-) \\ p\,p &\rightarrow p(p\,\pi^+\pi^-) \\ \bar{p}\,p &\rightarrow \bar{p}(p\,\pi^+\pi^-)\,. \end{aligned} \tag{1}$$

The analyses(1-28) on the experimental data relative to the system suggest $\frac{5}{2}^+$ as the most likely spin-parity assignement for the $N^*(1700)$; they give also some indications in favour of diffractive production. Most of the spin-parity tests consist in the study of the angular distribution for the normal to the N^* decay plane(29). In other tests one assumes that the N^* decays totally into $\Delta^{++}\pi^-$: however, it is very difficult to show this, because the N^* is very close to the threshold(25): in the various papers very different decay percentages are given.

Also the diffractive hypothesis, which is often assumed in the tests, should be verified.

Our purpose is to present some model independent tests to be applied to the $\pi^+\pi^- p$ system, and particularly in the 1700 MeV region. Only some very general hypotheses are formulated and some of the above assumptions on the N^* could be tested. We also outline that many of the arguments we will present are of general interest.

In section 1 we shall discuss some tests, with their limitations, on the density matrix elements.

In section 2 we shall write some functions to be used in the fit to the decay angular distribution, in order to determine the value of the N^* spin, the matrix elements and the decay parameters to be inserted in the Dalitz plot Analysis. We also indicate a test for s-channel and t-channel helicity conservation.

In section 3 we discuss the Dalitz plot analysis. We show that it is possible to write a function depending on 7 free parameters, with only a general hypothesis about the background. Also the branching ratios of the N^* decay modes can be determined.

1. - TESTS ON THE DENSITY MATRIX ELEMENTS. -

At first, we observe that there are some limitations on the density matrix elements for the diffractively produced resonances(26). For example, in the N^* production induced by proton-proton collision, we have

$$p\,p \longrightarrow p\,N^*$$

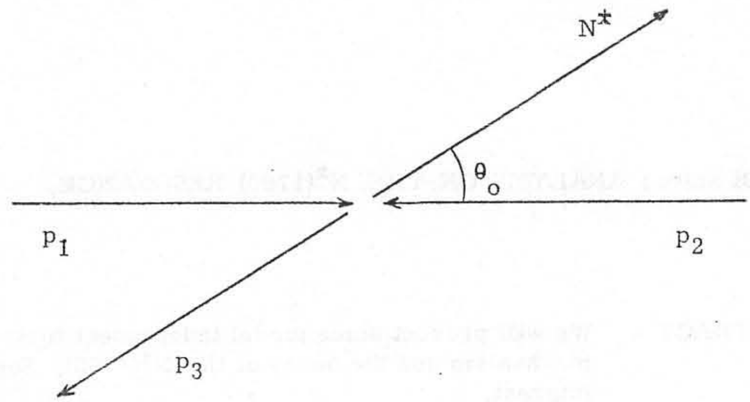

FIG. 1

where the N^* is produced in the forward direction. Let us consider the consequences on the z component of the N^* spin. The initial system is described by a plane wave, which can be decomposed into spherical waves(29)

$$\left| \Omega_o, \lambda_1 \lambda_2 \right\rangle = \sum_M \sum_I \sqrt{\frac{2J+1}{4\pi}}\, \mathcal{D}^{J^*}_{M\lambda}(\Omega^o) \left| JM\, \lambda_1\, \lambda_2 \right\rangle. \qquad (2)$$

By assuming the z axis lying along the direction of $\hat{p}_1$ (see Fig. 1), it is $\Omega_o \equiv (0,0)$, and

$$\mathcal{D}^{J^*}_{M\lambda}(0,0) = \delta_{M\lambda}$$

so that, in the expression (2), the third component of the total spin turns out to be λ. This result can be intuitively understood by the following argument.

For each partial wave J, consider the component of the spin along the incident proton direction $\hat{p}_1$

$$\vec{J} \cdot \hat{p}_1 = \vec{s}_1 \cdot \hat{p}_1 + \vec{s}_2 \cdot \hat{p}_2 = \lambda \qquad (3)$$

where $\vec{s}_1$ and $\vec{s}_2$ are the spins of the two initial protons. So the third component along $\hat{p}_1$ is independent of J. Now, because the N^* direction $\hat{p}'$ in the CMS is very close to $\hat{p}_1$, we can approximate with

$$\vec{J} \cdot \hat{p}_1 \cong \vec{J} \cdot \hat{p}' = \vec{s}_{N^*} \cdot \hat{p}' + \vec{s}_3 \cdot \hat{p}' = \Lambda - \lambda_3 . \qquad (4)$$

We have indicated the spin of the N^* as $\vec{s}_{N^*}$ and Λ as its third component in the Jackson reference frame (JRF)(29); on the other hand $\vec{s}_3$ is the spin of the outgoing proton and λ_3 its helicity. From (3) and (4) we can deduce $\lambda \simeq \Lambda - \lambda_3$ and, since $|\lambda| \leq 1$, it follows that $|\Lambda| \leq 3/2$. This argument, which is also applicable in the helicity reference frame (HRF)(29), can be made more rigorous and quantitative by esamining the experimental θ_o distribution for the N^*. From (2) we have seen that the third component of the total spin in the forward direction in λ; we want now to compute the probability to have the same value λ in a direction $\Omega_o \equiv (\theta_o, \varphi_o)$. Since(29)

$$R(\Omega_o)|0\,0\,\lambda_1\,\lambda_2\rangle = |\theta_o, \varphi_o, \lambda_1, \lambda_2\rangle = \sum_M \sum_J N^J \mathcal{D}^J_{M\lambda}(\theta_o, \varphi_o)|J\,M\,\lambda_1\,\lambda_2\rangle$$

the probability density of having λ as third component is

$$\frac{dp}{d\,\Omega_o} = \left|\sum_{J=|\lambda|}^{\infty} N^J \mathcal{D}^J_{\lambda\lambda}(\theta_o, \varphi_o)|J\,\lambda\,\lambda_1\lambda_2\rangle\right|^2 = \sum_{J=|\lambda|}^{\infty} \frac{2J+1}{4\pi}\left[d^J_{\lambda\lambda}(\theta_o)\right]^2 .$$

By considering the average angle $\overline{\theta}_o$ of the N^* direction in the CMS, the total probability that λ represents the 3rd component of the spin is

$$\Delta p = \sum_{J=|\lambda|}^{\infty} (2J+1)\int_0^{\overline{\theta}_o} \left[d^J_{\lambda\lambda}(\theta_o)\right]^2 d\cos\theta_o .$$

Therefore the density matrix for the N^* produced in proton-proton collision is a 4x4 matrix; similarly, in pion-pion collision, we will have a 2x2 matrix. However, we must still take some symmetry properties into account:

a) Owing to parity invariance (both in the HRF and in the JRF) we have

$$\rho^J_{\Lambda\Lambda'} = (-)^{\Lambda-\Lambda'}\rho^J_{-\Lambda\,-\Lambda'}$$

which is true in every frame with the y axis normal to the decay plane.

b) The density matrix is a hermitian operator:

$$\rho^J_{\Lambda\Lambda'} = (\rho^J_{\Lambda'\Lambda})^* .$$

c) Last

$$\mathrm{tr}\,\rho = \sum_\Lambda \rho_{\Lambda\Lambda} = 1 .$$

Then the density matrix for pp collisons has 7 independent parameters

$$\begin{pmatrix} a & b+ic & d+ie & if \\ b-ic & \frac{1}{2}-a & ig & d-ie \\ d-ie & -ig & \frac{1}{2}-a & -b+ic \\ -if & d+ie & -b-ic & a \end{pmatrix} \qquad (5)$$

The a, b, c, d, e, f are real parameters, with $0 \le a \le \frac{1}{2}$.

Now we examine what are the consequences of assuming spin 0 or 1 particle exchange in the crossed t-channel:

$$\overline{p}\,p \to s \to \overline{p}\,N^*$$

where s is an intermediate virtual particle.

We are in the JRF, where the quantization axis is along the direction of the momentum transfer $\vec{\Delta}$ in the N^* center of mass:

a) By assuming that only a spin 0 virtual particle is exchanged, we expect that the z component of the N^* spin Λ is $\pm 1/2$, so the total production amplitude contains only two terms. We can factorize the reaction and keep only

$$0 \to \overline{p}\,N^*$$

for which we have $A_{\frac{1}{2}\,-\frac{1}{2}} = A_{-\frac{1}{2}\,\frac{1}{2}}$ (6)

(due to parity conservation). In this hypothesis the density matrix is

$$\rho = \begin{pmatrix} \frac{1}{2} & \rho_{1-1} \\ -\rho_{1-1} & \frac{1}{2} \end{pmatrix}$$

b) If also a spin 1 virtual particle is exchanged, other matrix elements must be taken into account, but with some limitations. The amplitudes to be considered are, in this case

$$B_{-\frac{1}{2}\frac{3}{2}} \quad B_{\frac{1}{2}\frac{1}{2}} \quad B_{\frac{1}{2}-\frac{1}{2}} \quad B_{-\frac{1}{2}\frac{1}{2}} \quad B_{-\frac{1}{2}-\frac{1}{2}} \quad B_{\frac{1}{2}-\frac{3}{2}} \tag{7}$$

Then from (6) and (7) we have

$$\rho_{3-3} = \sum_{\Lambda} F_{\lambda\frac{3}{2}} F^{*}_{\lambda-\frac{3}{2}} = F_{\frac{1}{2}\frac{3}{2}} F^{*}_{\frac{1}{2}-\frac{3}{2}} + F_{-\frac{1}{2}\frac{3}{2}} F^{*}_{-\frac{1}{2}-\frac{3}{2}} .$$

So if the experimental value of ρ_{3-3} is not consistent with 0, we can deduce that there is an exchange of spin > 1. Similarly, if we assume that the main contribution comes from spin 0 exchange (i.e. B << A), we have, at first order for the B amplitudes, only three independent nonvanishing matrix elements : ρ_{11}, ρ_{1-1} and ρ_{31}, as one can see by writing these matrix elements and inserting the amplitudes (6) and (7) into their expressions.

2. - DECAY ANGULAR DISTRIBUTION. -

The angular distribution for the normal to the N^{*} decay plane is[29]

$$I(\theta,\varphi) = \frac{2J+1}{4\pi} \sum_{\Lambda\Lambda'} \rho^{J}_{\Lambda\Lambda'} \sum_{\mu} \mathcal{D}^{J*}_{\Lambda\mu}(\theta,\varphi)\, \mathcal{D}^{J}_{\Lambda'\mu}(\theta,\varphi)\, g^{J}_{\mu} \tag{8}$$

where (θ, φ) is the direction of the normal with respect to the JRF or to the HRF, and Λ, Λ' values of the third component of the N^{*} spin (either in HRF or in JRF).

The g^{J}_{μ} are defined in Chung[29]: it must be

$$\sum_{\mu=-J}^{J} g^{J}_{\mu} = 1 . \tag{9}$$

By fitting the experimental distribution with the above function, one could determine, in principle :

a) the spin of the N^{*}, by fitting the distribution with different values of J ;

b) the matrix elements of the resonance (7 independent real parameters for the N^{*}, as stated before).

However, in most cases it looks quite difficult to fit the experimental data with 2J+7 parameters. In the following, we will present some alternative methods to determine both the spin and the production and the decay parameters.

2.1. - Fitting method. -

By integrating (8) over the azimuthal angle, one obtains :

$$\begin{aligned} I(\theta) &= \frac{2J+1}{2} \sum_{\mu} \sum_{\Lambda} \rho^{J}_{\Lambda\Lambda} \left[d^{J}_{\Lambda\mu}(\theta) \right]^2 g^{J}_{\mu} = \\ &= \frac{2J+1}{2} \sum_{\mu} \left\{ a \left[d^{J}_{3\mu}(\theta) \right]^2 + \left(\frac{1}{2} - a\right) \left[d^{J}_{1\mu}(\theta) \right]^2 \right\} (g^{J}_{\mu} + g^{J}_{-\mu} 0) \end{aligned} \tag{10}$$

where a is the parameter defined in (5).

So we have a function with 2J+1 free parameters, taking into account the condition (9). Then, we can determine the nondiagonal matrix elements by fitting the distribution in θ, φ with the expression (8), or by fitting the integrated expression $I(\varphi)$:

$$I(\varphi) = \frac{2J+1}{4\pi} \sum_{\Lambda\Lambda'} \rho^J_{\Lambda\Lambda'} \sum_\mu e^{i(\Lambda-\Lambda')\varphi} g^J_\mu \int d\cos\theta \, d^J_{\Lambda\mu}(\theta)\, d^J_{\Lambda'\mu}(\theta) \,. \tag{11}$$

We have to deal with 6 real parameters, since the remaining have been already determined.

Also the hypothesis that the helicity is conserved in diffractive production can be tested experimentally by analizing the distribution $I(\varphi)$ in the HRF (in this case we test SCHC) or in the JRF (then we test TCHC). Helicity conservation implies[30] that $\Lambda = \Lambda' = \lambda_1$. From (11) it follows that the angular distribution is independent of the azimuthal angle. It can be useful to evaluate the moments of (11):

$$I(\varphi) = \sum_\nu G_\nu e^{i\nu\varphi} \qquad\qquad G_\nu = \frac{1}{2\pi N} \sum_{k=1}^{N} e^{-i\nu\varphi_k}$$

where N is the total number of events. The defined moments allow us to evaluate helicity conservation violations. We note that this holds even if there is more than one (partial) decaying wave.

2.2. - Moments method. -

The distribution (10) can be written as[29]

$$I(\theta) = \sum_{L=0}^{2J} \sqrt{\pi(L+1)} \sum_{\mu=-J}^{J} C^{JLJ}_{\mu o \mu} (g^J_\mu + g^J_{-\mu}) \left[a\, C^{JLJ}_{3 o 3} + (\tfrac{1}{2} - a)\, C^{JLJ}_{1 o 1} \right] Y^L_0(\theta) \,.$$

We define the moments

$$M_L = \frac{1}{N} \sum_{i=1}^{N} Y^L_0(\theta_i) = \sqrt{\pi(L+1)} \sum_\mu C^{JLJ}_{\mu o \mu} (g^J_\mu + g^J_{-\mu}) \left[a\, C^{JLJ}_{3 o 3} + (\tfrac{1}{2} - a)\, C^{JLJ}_{1 o 1} \right] . \tag{12}$$

Since the number of moments equals the number of unknown parameters, we can solve the linear system (12).

Once the g^J_μ and "a" have been determined, we can apply the moments method also to the distribution (8), for which the moments are defined as[29]:

$$H(L,m) = \frac{1}{N} \sum_{k=1}^{N} \mathcal{D}^L_{mo}(\theta_k, \varphi_k)$$

with

$$H(L,m) = \Big(\sum_{\Lambda\Lambda'} \rho_{\Lambda\Lambda'} C^{JLJ}_{\Lambda' m \Lambda} \Big) \Big(\sum_\mu C^{JLJ}_{\mu o \mu} g^J_\mu \Big) .$$

2.3. - The decay angular distribution can be parametrized in more convenient ways if we developed g^J_μ[29, 31]:

$$g^J_\mu = \frac{1}{2^4 (2\pi)^8} \sqrt{\frac{s-M^2}{2s}} \int dW\, dE_1\, dE_2 \, \frac{W \sum_\mu |F^J_\mu|^2}{(W-W_o)^2 + \Gamma_{tot} W_o^2} \tag{13}$$

where $F^J_\mu(\lambda)$ is the invariant decay amplitude of the $N^* \to p\pi^+\pi^-$, λ being the helicity of the proton and μ the Z component of the N^* spin along the body - fixed frame[29]. Hence, developing in partial waves

$$F^J_\mu(\lambda) = \sum_m d^{\frac{1}{2}}_{\lambda m}(\tfrac{\pi}{2})\, F^J_\mu(m) = \sqrt{\frac{4\pi(2L+1)}{2J+1}}\ \sum_m d^{\frac{1}{2}}_{\lambda m}(\tfrac{\pi}{2})\, C^{L\ \frac{1}{2}\ J}_{M\,m\,\mu} \sum_{k=0}^{L} S_k \ \cdot$$

$$\cdot \sum_{m_1 m_2} C^{k\ L-k\ L}_{m_1\, m_2\, m} = \sum_{LMm} d^{\frac{1}{2}}_{\lambda m}(\tfrac{\pi}{2})\, C^{L\ \frac{1}{2}\ J}_{M\ m\ \mu}\ A^L_M \ .$$

We have expressed the helicity state as a function of the states with the third component m with respect to the Z axis in the body - fixed frame. L is the total orbital angular momentum, and due to parity conservation it must be either $J - 1/2$ or $J + 1/2$, but not a mixture of the two waves.

If N^* is diffractively produced, Morrison's rule(32) suggeststhat $L = J - 1/2$: this hypothesis can be tested.

In this connection we observe that the factorization(32) could be usefully tested for the reactions (1), in order to further investigate the production mechanism.

If we set:

$$a = \int dW\, dE_1\, dE_2\, K(W) \left| A^L_{\mu-\frac{1}{2}} \right|^2 \qquad a > 0$$

$$b = \int dW\, dE_1\, dE_2\, K(W) \left| A^L_{\mu+\frac{1}{2}} \right|^2 \qquad b > 0$$

$$c + id = \int dW\, dE_1\, dE_2\, K(W)\, A^L_{\mu-\frac{1}{2}}\, A^{L*}_{\mu+\frac{1}{2}}$$

(see Chung(29) fon expression of K(W)),

we can express the g^J_μ in the distributions(8), (10), (11) as functions of a, b, c, d. Because of condition (9), we have 3 free parameters in all.

This method is the most convenient, since we can test the two different orbital angular momentum apmlitudes with only 3 parameters; however we loose information about the decay parameters. Therefore we consider an alternative method, useful also in the Dalitz plot distribution analysis (see sect. 4). We know that the N^* decays into :

a)	$p\,\pi^+\pi^-$	incoherently	: I^J_μ	
b)	$p\,\rho^o$		: R^J_μ	
c)	$\Delta^{++}\,\pi^-$		: Δ^J_μ	(14)
d)	$N^o\,\pi^+$		: N^J_μ .	

So we can write :

$$F^J_\mu = I^J_\mu + R^J_\mu + \Delta^J_\mu + N^J_\mu$$

and developing in partial waves, we obtain :

$$F^J_\mu = \sqrt{\frac{4\pi(L+1)}{2J+1}}\ \sum_m d^{\frac{1}{2}}_{\lambda m}(\tfrac{\pi}{2})\, C^{L\ \frac{1}{2}\ J}_{M\,m\,\mu} \left[\sum_{K=0}^{L} \alpha^K_M\, S_k + \alpha^1_M (b_1 + c_1 + d_1) \right]$$

$$\alpha^K_M = \sum_{m_1 m_2} C^{K\,L-K\,L}_{m_1\,m_2\,M}$$

$$\sum_{\lambda} \left|F^J_{\mu}(\lambda)\right|^2 = \sum_{\lambda m m'} G^{\mu}_{\lambda m m'} \left[\sum_k \alpha^K_M \alpha^K_{M'} \left|S_k\right|^2 + \alpha^1_M \alpha^1_{M'} \left|b_1 + c_1 + d_1\right|^2 \right] \tag{15}$$

where S_k are the partial amplitudes of the N^* incoherent decay. Of course they don't interfere neither among them, nor with the partial decay modes. On the other hand, we have:

$$R^J_{\mu} = \text{cost} \sum_{m=-\frac{1}{2}}^{\frac{1}{2}} d^{\frac{1}{2}}_{\lambda m}\left(\frac{\pi}{2}\right) C^{L\,\frac{1}{2}\,J}_{M\,m\,\mu}\, \alpha^1_M\, b_1$$

and similar expressions for Δ^3_{μ} and N^J_{μ}.

The sum over partial waves runs from 0 to L, because only the lowest possible angular momenta are supposed to contribute to the amplitude. More precisely, for a given total orbital angular momentum L, if l_1 is the angular momentum of the system and l_2 the orbital angular momentum of the proton with respect to the same system, we take into account only the values for which $L = l_1 + l_2$.

The order of magnitude for the neglected amplitudes has been evaluated with non relativistic scattering formulas[33]: the greatest of these amplitudes results to be $\approx 10\%$ of the amplitudes taken into account.

In conclusion it results:

$$g^J_{\mu} = \sum_{\lambda m m'} G^{\mu}_{\lambda m m'} \left(\alpha^1_M \alpha^1_{M'} R + \sum_k \alpha^K_M \alpha^K_{M'} A_K \right)$$

where

$$R = \int dW\, dE_1\, dE_2\, K(W) \left|b_1 + c_1 + d_1\right|^2$$

$$A_K = \int dW\, dE_1\, dE_2\, K(W) \left|S_k\right|^2 .$$

So g^J_{μ} can be expressed as functions of R and A_K; they are L+2 positive parameters which can be related to same parameters of the Dalitz plot distribution (See the following section).

3. - DALITZ PLOT ANALYSIS. -

If W is the effective mass of the $p\,\pi^+\pi^-$ system and E_1, E_2 the pions energies in the CMS, the Dalitz plot fitting function is[29, 31]:

$$I(W, E_1, E_2) = \frac{1}{2^4 (2\pi)^8} \sqrt{\frac{s-M^2}{s}}\; W \left[\frac{\varepsilon \sum_{\mu} \left|F^J_{\mu}\right|^2}{(W^2 - W_o^2) + \Gamma^2_{tot} W_o^2} + (1-\varepsilon) f(W, E_1, E_2)\, \alpha (W - W_{min})(W_{max} - W) \right] \tag{16}$$

where s is the Mandelstam variable of the whole system, M is the mass of the nucleon, ε the percentage of the resonant amplitude ($0 < \varepsilon < 1$), F^J_{μ} the decay amplitudes of the N^* and $f(E_1, E_2, \overline{W})$ a function to be defined below. In the formula (16), we assume that the Breit-Wigner resonance and the background add incoherently. The background is supposed to be a smooth function vanishing at the minimum (W_{min}) and at the maximum (W_{max}) value of W, with a similar hypothesis for $f(E_1, E_2, W)$, as we shall see; α is a positive parameter.

We impose[31] the condition:

$$\frac{1}{(4\pi)^3(2J+1)}\int dE_1\, dE_2 \sum_\mu \left|F^J_\mu\right|^2 = W_o\, \Gamma(W_o)\,.$$

We note that in this case $\Gamma(W_o)$ is the partial width for the N^* decay into $p\,\pi^+\pi^-$, while in (15) $_{tot}$ is the total N^* with[31].

In order to have the normalization $\int I(W\,E_1\,E_2)dW\,dE_1\,dE_2 = 1$, we must set:

$$\alpha\int f(E_1\,E_2\,W)\left(\frac{1}{2^4(2\pi)^8}\sqrt{\frac{s-M^2}{s}}\,W\,dW\,dE_1\,dE_2 = 1\,. \qquad (17)$$

The percentage ε can be determined by integrating (15) over dE_1 and dE_2 and fitting the effective mass distribution:

$$I(W) = \frac{1}{2^4}\frac{1}{(2\pi)^8}\sqrt{\frac{s-M^2}{s}}\,W\left[\varepsilon\frac{(4\pi)^3(2J+1)W_o\,\Gamma(W)}{(W^2-W_o^2)^2+W_o^2\Gamma^2_{tot}} + (1-\varepsilon)F(W)\,\alpha\,(W-W_{min})(W_{max}-W)\right]$$

where $F(W) = \int dE_1\,dE_2\,f(E_1\,E_2\,W)$.

As we shall see (cfr. (21)), $F(W)$ does not depend on W. We now check the function $\sum_\mu\left|F^J_\mu\right|^2$ in detail. Taking into account the previous hypothesis about the decay modes (see end of section 3), we may write $\sum_\mu\left|F^J_\mu\right|^2$ following Zemach[34] notation. The sum over the spin components results to be the contaction of an irreducible J-th rank tensor with its coniugate, the irreducible tensor being constructed by the π^+, π^- momenta. It results:

$$\sum_\mu\left|F^J_\mu\right|^2 = \sum_{k=0}^{L} a_k\, T^J_k : T^{J*}_k + \tau^J : \tau^{J*} \qquad (18)$$

where

$$T^J_o = T^J(\vec{p}_1) \qquad\qquad T^J_1 = T^J(\vec{p}_1 \ldots \vec{p}_1\,\vec{p}_2)$$

$$T^J_k = T^J(\vec{p}_1 \ldots \vec{p}_1,\,\vec{p}_2 \ldots \vec{p}_2) \qquad (\vec{p}_2 \text{ appears k times}).$$

In ref. (18) we have indicated with p_1 and p_2 the π^+ and π^- momenta in the N^* rest frame. As in ref. (14) formula (18) contains L+1 partial waves which add incoherently and which are described by the T^J_k; furthermore, it contains the partial decay modes we have previously seen in (13), described by τ^J:

$$\tau^J = b_\Delta \mathcal{B}_\Delta(W_{13})\,T^J(\vec{p}_2\ldots\vec{p}_2,\vec{p}_1-\vec{p}_3) + e^{i\varphi}\,b_N\mathcal{B}_N(W_{23})\,T^J(\vec{p}_1\ldots\vec{p}_1,\vec{p}_2-\vec{p}_3) +$$
$$+\, e^{i\psi}\,b_\rho\,\mathcal{B}_\rho(W_{12})\,T^J(\vec{p}_3\ldots\vec{p}_3,\vec{p}_1-\vec{p}_2)$$

where p_3 is the proton momentum in the N^* rest frame, W_{12} the effective mass of the $\pi^+\pi^-$ system, W_{13} the mass of $p\pi^-$ and W_{23} the mass of $p\pi^-$. $\mathcal{B}_\Delta(W_{13})$ is the Δ^{++} propagator:

$$\mathcal{B}_\Delta(W_{13}) = \left[W^2_{13} - M^2_\Delta - i\,\Gamma_\Delta M_\Delta\right]^{-1}$$

with analogous formulas for $\mathcal{B}_\rho(W_{12})$ and $\mathcal{B}_N(W_{23})$. The b_ρ, b_Δ, b_N are real positive parameters, φ and ψ are the relative phases of the ρ and the $N^{*o}(1470)$ decay amplitudes with respect to the Δ^{++} amplitude.

In Appendix I we illustrate a differential technique to calculate the contraction of the irreducible tensors, and we present a general formula for $T^J : T^{J*}$. Here we summarize the results for the most likely spin assignements for the N^*: $J = 5/2$ and $7/2$.

$$J = \frac{5}{2} \qquad \sum_\mu \left| F_\mu^{\frac{5}{2}} \right|^2 = a_o T^{\frac{5}{2}}(\vec{p}_1) : T^{\frac{5}{2}*}(\vec{p}_1) + a_1 T^{\frac{5}{2}}(\vec{p}_1, \vec{p}_2) : T^{\frac{5}{2}*}(\vec{p}_1 \vec{p}_2) + a_2 T^{\frac{5}{2}}(\vec{p}_2) : T^{\frac{5}{2}*}(\vec{p}_2) + \mathcal{T}^{\frac{5}{2}} : \mathcal{T}^{\frac{5}{2}*}$$

where

$$\mathcal{T}^{\frac{5}{2}} : \mathcal{T}^{\frac{5}{2}*} = b_\Delta^2 \left| \mathcal{B}_\Delta(W_{13}) \right|^2 T^{\frac{5}{2}}(\vec{p}_2, \vec{p}_1 - \vec{p}_3) : T^{\frac{5}{2}*}(\vec{p}_2, \vec{p}_1 - \vec{p}_3) + b_N^2 \left| \mathcal{B}_N(W_{13}) \right|^2 \cdot$$
$$\cdot\, T^{\frac{5}{2}}(\vec{p}_1, \vec{p}_2 - \vec{p}_3) : T^{\frac{5}{2}*}(\vec{p}_1, \vec{p}_2 - \vec{p}_3) + b_\rho^2 \left| \mathcal{B}(W_{12}) \right|^2 T^{\frac{5}{2}}(\vec{p}_3, \vec{p}_1 - \vec{p}_2) : T^{\frac{5}{2}*}(\vec{p}_3, \vec{p}_1 - \vec{p}_2) +$$
$$+\, 2 b_\Delta b_N \operatorname{Re}\left[e^{-i\varphi} \mathcal{B}_\Delta(W_{13})\, \mathcal{B}_N^*(W_{23}) \right] T^{\frac{5}{2}}(\vec{p}_2, \vec{p}_1 - \vec{p}_3) : T^{\frac{5}{2}*}(\vec{p}_1, \vec{p}_2 - \vec{p}_3) +$$
$$+\, 2 b_N b_\rho \operatorname{Re}\left[e^{-i(\varphi+\psi)} \mathcal{B}_N(W_{23})\, \mathcal{B}_\rho^*(W_{12}) \right] T^{\frac{5}{2}}(\vec{p}_1, \vec{p}_2 - \vec{p}_3) : T^{\frac{5}{2}*}(\vec{p}_3, \vec{p}_1 - \vec{p}_2)$$

$$J = \frac{7}{2} \qquad \sum_\mu \left| F_\mu^{\frac{7}{2}} \right|^2 = c_o T^{\frac{7}{2}}(\vec{p}_1) : T^{\frac{7}{2}*}(\vec{p}_1) + c_1 T^{\frac{7}{2}}(\vec{p}_1 \vec{p}_1 \vec{p}_2) : T^{\frac{7}{2}*}(\vec{p}_1 \vec{p}_1 \vec{p}_2) + c_2 T^{\frac{7}{2}}(\vec{p}_1 \vec{p}_2 \vec{p}_2) : T^{\frac{7}{2}*}(\vec{p}_1 \vec{p}_2 \vec{p}_2) + c_3 T^{\frac{7}{2}}(\vec{p}_2) : T^{\frac{7}{2}*}(\vec{p}_2) + \mathcal{T}^{\frac{7}{2}} : \mathcal{T}^{\frac{7}{2}*}$$

where last term is similar to $\mathcal{T}^{\frac{5}{2}} : \mathcal{T}^{\frac{5}{2}*}$.

To have explicitly the tensor contractions, the following expressions are useful (J half-integer, $L = J - 1/2$). (See Appendix).

$$T^J(\vec{p}) : T^{J*}(\vec{p}) = \frac{C_L\, p^{2L}}{2L+1}\,(L+1)$$

$$T^J(\vec{p} \ldots \vec{p}\, \vec{a}) : T^{J*}(\vec{p} \ldots \vec{p}\, \vec{a}) = \frac{C_L\, p^{2L}(L+1)}{2L+1} \left\{ \left(\frac{\hat{p} \cdot \vec{a}}{p}\right)^2 + \frac{P_L'(1)}{p^2} \left[a^2 - (\hat{p} \cdot \vec{a})^2 \right] \right\}$$

$$T^J(\vec{p} \ldots \vec{p}\, \vec{b}) : T^{J*}(\vec{q} \ldots \vec{q}\, \vec{a}) = \frac{C_L\, p^L q^L (L+1)}{2L+1} \left\{ \frac{\hat{q} \cdot \vec{a}}{q} \left[\frac{\hat{p} \cdot \vec{b}}{p} P_L(x) + \frac{P_L'(x)}{L}\, dy \right] + \frac{1}{L} \frac{\hat{p} \cdot \vec{b}}{p} P_L'(x)\, dx + \frac{1}{L^2} P_L''(x)\, dx\, dy + P_L'(x)\, \delta(dx) \right] \right\}$$

where $P_L(x)$ is the L-th Legendre Polynomial, $P_L'(x)$ and $P_L''(x)$ are its first and second derivatives ; $x = \hat{p} \cdot \hat{q}$, where $\hat{p}$ and $\hat{q}$ are the unit vectors of $\vec{p}$ and $\vec{q}$. The dx, dy, δ(dx) are defined in the Appendix.

The parameters a_o, a_1, a_2, c_o, c_1, c_2, c_3 are real and positive and can be considered as free parameters; however they can be related to the decay parameters appearing in expression (14). By comparing this expression with (18), we obtain:

$$a_k T_k^J : T_k^{J^*} = \sum_{\mu} \sum_{\lambda MM'} G^{\mu}_{\lambda MM'} \alpha^k_M \alpha^k_{M'} \left| S_k \right|^2 \tag{19}$$

$$\zeta^J : \zeta^{J^*} = \sum_{\mu} \sum_{\lambda MM'} G^{\mu}_{\lambda MM'} \alpha^1_M \alpha^1_{M'} \left| b_1 + c_1 + d_1 \right|^2 . \tag{20}$$

Therefore

$$\int dW\, dE_1\, dE_2\, K(W)\, a_k T_k^J : T_k^{J^*} = \sum_{\mu} \sum_{\lambda MM'} G^{\mu}_{\lambda MM'} \alpha^k_M \alpha^k_{M'} A_k$$

$$\int dW\, dE_1\, dE_2\, K(W)\, \zeta^J : \zeta^{J^*} = \sum_{\mu} \sum_{\lambda MM'} G^{\mu}_{\lambda MM'} \alpha^1_M \alpha^1_{M'} R .$$

So we determine the parameters a_k and a relation among b_{Δ}, b_N, b_{ρ}, φ, ψ .

Last, we try to describe the background with a phenomenological function. The system $p\pi^+\pi^-$ can contain the following states:

a) N^* resonance

b) $\Delta^{++}\pi^-$

c) $N^{*o}\pi^+$

d) $p\rho$

e) 3 uncorrelated particles.

Case a) has already been discussed. For incoherent decay we formulate the same hypothesis used to write the effective mass distribution (11). So we have:

$$\begin{aligned} f(E_1 E_2 W) = {} & \beta I_{\rho}^{-1} \left| \mathcal{B}_{\rho}(W_{12}) \right|^2 q_{12}^2 + \gamma \left| \mathcal{B}_{\Delta}(W_{13}) \right|^2 q_{13}^2 I_{\Delta}^{-1} + \\ & + \lambda I_N^{-1} \left| \mathcal{B}_{N^{o*}}(W_{23}) \right|^2 q_{23}^2 + \sigma I_F^{-1} (E_1 + E_2 - 2m_{\pi})(W - E_1 - E_2) \end{aligned} \tag{21}$$

where

$$I_{\rho} = \int dE_1\, dE_2 \left| \mathcal{B}_{\rho}(W_{12}) \right|^2 q_{12}^2$$

with similar expressions for I_{Δ}, I_N, I_F .

The q_{12} is the two pions momentum in the ρ CMS, q_{13} and q_{23} are defined in a similar way. One can express them as functions, respectively, of W_{12}, W_{13}, W_{23}. The coefficients β, γ, λ , σ are real and positive parameters, with

$$\beta + \gamma + \lambda + \sigma = 1 . \tag{22}$$

In conclusion, taking into account the conditions (16), (17), (19), (20), (22) we have defined a Dalitz plot density distribution with 7 free parameters.

4. - CONCLUSION. -

We have seen some different tests to be applied to the $\pi^+\pi^- p$ system and in particular to the $N^*(1700)$ resonance. Now let us briefly discuss the advantages and the limitations of each method.

As we have seen, the $I(\theta, \varphi)$ in (8), or the integrated distributions $I(\theta)$ and $I(\varphi)$, allows us to determine, in principle, both the spin and the density matrix elements of the $N^*(1700)$. The moments method looks more sure than the fitting method, as in that case we only take the average values of some angular functions (ex. the Legendre functions). However, the first method presented in sect. 2.3 seems to be the most effective, since we need not consider the integrated distributions (we have only 2J+3 parameters): when we fit the $I(\theta)$, we neglect the φ dependence and the parameters so determined are probably not best to fit the $I(\theta, \varphi)$. Last, we have purposed the Dalitz plot analysis, in connection with the latter method in sect. 2.3, as an alternative way to determine the spin for the N^*. This analysis, which generally does not seem to be appreciated in studying the N^*, has some advantages. First, we can also determine the decay modes; secondly, expression (15) is independent of the production process; and last, if we consider more than one partial wave J, we have no interference terms[29].

ACKNOWLEDGEMENTS. -

I am grateful to C. Caso, E. Eberle and D. Teodoro for their indications and for the useful discussions I had with them.

APPENDIX. -

Zemach[34] gives the following formula for contraction of tensors :

$$T^J(\vec{p}) : T^{J*}(\vec{q}) = \frac{C_L q^L p^L}{2L+1} \left[(L+1)\, p_L(x) - i\vec{\sigma}\cdot\vec{p}\wedge\vec{q}\, p_L'(x) \right]$$

If we follow the differential technique suggested in the same reference, we obtain, by substituting $\vec{q} \to \vec{q} + \vec{a}$

$$T^J(\vec{p}) : \left[T^J(\vec{q}) + L T^J(\vec{q}\ldots\vec{q}\,\vec{a}) + \ldots \right]^* = \frac{C_L p^L q^L}{2L+1} \left\{ 1 + L\,\frac{\hat{q}\cdot\vec{a}}{q} + \ldots \right\} \cdot$$

$$\cdot \left\{ (L+1)\left[p_L(x) + dx\, p_L'(x) \right] - i(\vec{\sigma}\cdot\vec{p}\wedge\vec{q})\left[p_L'(x) + dx\, p_L''(x) \right] \right\}$$

where $dx = (\hat{p}\cdot\vec{a} - x\hat{q}\cdot\vec{a})\cdot\frac{1}{q}$.

Now, if we compare the coefficients of $\vec{a}$ in the r.h.s. and l.h.s. of the equation, we have:

$$T^J(\vec{p}) : T^{J*}(\vec{q}\ldots\vec{q}\,\vec{a}) = \frac{\hat{q}\cdot\vec{a}}{q}\, T^J(\vec{p}) : T^J(\vec{q}) + \frac{C_L p^L q^L}{L(2L+1)} \left\{ (L+1)\, p_L'(x)\, dx + \right.$$

$$\left. + i\vec{\sigma} \left[(\hat{p}\wedge\frac{\vec{a}}{q} - \frac{\hat{q}\cdot\vec{a}}{q}\,\hat{p}\wedge\hat{q})\, p_L'(x) + \hat{p}\wedge\hat{q}\, p_L''(x)\, dx \right] \right\} .$$

In a similar way we can substitute $\vec{p} \to \vec{p} + \vec{b}$ in the last expression, obtaining

$$T^J(\vec{p}\ldots\vec{p}\,\vec{b}) : T^{J*}(\vec{q}\ldots\vec{q}\,\vec{a}) = \frac{\hat{q}\cdot\vec{a}}{q} \left\{ \frac{\hat{p}\cdot\vec{b}}{p}\, T^J(\vec{p}) : T^J(\vec{q}) + \frac{C_L p^L q^L}{L(2L+1)} \left[(L+1)\, p_L'(x)\, dy - \right. \right.$$

$$- i\vec{\sigma}\,(\frac{\vec{b}}{p}\wedge\hat{q} - \frac{\hat{p}\cdot\vec{b}}{p}\,\hat{p}\wedge\hat{q})\,p'_L(x) - i\vec{\sigma}\cdot\hat{p}\wedge\hat{q}\,p''_L(x)\,dy\Big] + \frac{C_L p^L q^L}{L(2L+1)}\,\frac{\hat{p}\cdot\vec{b}}{p}\;\cdot$$

$$\cdot\Big[(L+1)\,p'_L(x)\,dx - i\vec{\sigma}\,(\hat{p}\wedge\frac{\vec{a}}{q} - \frac{\hat{q}\cdot\vec{a}}{q}\,\hat{p}\wedge\hat{q})\,p'_L(x) - i\vec{\sigma}\cdot\hat{p}\wedge\hat{q}\,p''_L(x)\,dx\Big] +$$

$$+ \frac{C_L p^L q^L}{L^2(2L+1)}\Big\{(L+1)\Big[p''_L(x)\,dx\,dy + p'_L(x)\,\delta(dx) - i\vec{\sigma}\Big[\frac{\vec{b}\wedge\vec{a}}{pq} - \frac{(\hat{p}\cdot\vec{b})\hat{p}\wedge\vec{a}}{pq} -$$

$$- \frac{\hat{q}\cdot\vec{a}}{pq}\,(\vec{b}\wedge\hat{q} - (\hat{p}\cdot\vec{b})\hat{p}\wedge\hat{q})\Big]\,p'_L(x) + i\vec{\sigma}\,(\frac{\hat{p}\wedge\vec{a}}{q} - \frac{\hat{q}\cdot\vec{a}}{q}\,\hat{p}\wedge\hat{q})\,p''_L(x)\,dy\; -$$

$$- i\vec{\sigma}\,(\frac{\vec{b}}{p} - \frac{\vec{b}\cdot\hat{p}}{p}\,\hat{p})\wedge\hat{q}\,p''_L(x)\,dx - i\vec{\sigma}\cdot\hat{p}\wedge\hat{q}\,\Big[p''_L(x)\,\delta(dx) + p'''_L(x)\,dx\,dy\Big]\Big\}$$

where

$$dy = (\hat{q}\cdot\vec{b} - x\,\hat{p}\cdot\vec{b})\,\frac{1}{p}$$

$$\delta(dx) = \frac{1}{pq}\Big[\vec{b}\cdot\vec{a} - (\hat{p}\cdot\vec{b})(\hat{p}\cdot\vec{a}) - (\hat{q}\cdot\vec{a})\,p\,dy\Big]\,.$$

Note that in the text we have omitted all σ_i terms, because they wanish when we sum over the helicities of all the final protons :

$$\sum_m \bar{u}(m)\,\sigma_i\,u(m) = \mathrm{tr}\,\sigma_i = 0\,.$$

By successively applying the differential technique, we can calculate every mixed term of the type

$$T^J(p^{(1)}\ldots p^{(k)}\,q\ldots q) : T^J(p^{(1)}\ldots p^{(r)}\,q\ldots q)\,.$$

REFERENCES. -

(1) - S. L. Adelman, Phys. Rev. Letters 14, 1043 (1965).
(2) - W. Alles-Borelli et al., Nuovo Cimento 50, 1000 (1967).
(3) - S. P. Almeida et al., Nuovo Cimento (1967).
(4) - Y. Y. Lee et al., Phys. Rev. 174, 1638 (1968).
(5) - R. Ehrlich et al., Phys. Rev. Letters 21, 1839 (1968).
(6) - R. A. Iespersen et al., Phys. Rev. Letters 21, 1368 (1968).
(7) - G. Kayas et al., Nuclear Phys. 135, 169 (1968).
(8) - I. W. Lamsa et al., Phys. Rev. 166, 1395 (1968).
(9) - I. G. Rushbrooke, Report HEP 68-20 (1968).
(10) - P. Antich et al., Phys. Rev. Letters 22, 39 (1969).
(11) - V. E. Barnes et al., Phys. Rev. Letters 23, 1516 (1969).
(12) - D. C. Colley et al., Nuovo Cimento 59A, 519 (1969).
(13) - R. I. Plano, Boulder Conference (1969).
(14) - W. E. Ellis et al., Report BNL 13918 (1969).
(15) - I. I. Rhode et al., Phys. Rev. 187, 1844 (1969).
(16) - J. G. Rushbrooke, Phys. Rev. 177, 2367 (1969).
(17) - E. Cirba et al., Nuclear Phys. B23, 533 (1970).
(18) - K. F. Galloway et al., Phys. Rev. D1, 3077 (1970).
(19) - G. Miyiashita et al., Nuovo Cimento 66A, 606 (1970).
(20) - R. B. Willmann et al., Phys. Rev. Letters 24, 1260 (1970).
(21) - K. Boesbeck et al., Nuclear Phys. B33, 445 (1971).
(22) - G. V. Beketov et al., S.J. Nuclear Phys. 13, 605 (1971).
(23) - E. Colton, Phys. Rev. D3, 1063 (1971).
(24) - E. Morse et al., Phys. Rev. D4, 133 (1971).
(25) - J. Rushbrooke, Nuclear Phys. B35, 1 (1971).
(26) - J. Rushbrooke et al., Phys. Rev. D4, 3273 (1971).
(27) - A. Johnstad et al., Scandinavian Coll. (1971).
(28) - J. W. Lamsa, Nuclear Phys. B37, 364 (1972).
(29) - S. U. Chung, Report CERN 71-8 (1971); in particular: Sections 4. 2 and 4. 3 for two particles states; Sections 5. 2 and 5. 3 for decay amplitude and density matrix; Section 6 for three particles system.
(30) - Cohen-Tannoudji et al., Phys. Letters 33B, 183 (1970).
(31) - J. D. Jackson, Nuovo Cimento 34, 1644 (1964).
(32) - D. R. O. Morrison, Report CERN D. Ph. II/PHYS 70-64 (1970).
(33) - S. M. Blatt and V. F. Weisskopf, Theoretical Nuclear Physics (Wiley, New York; Chapman and Hall, London), in particular Ch. VIII.
(34) - C. Zemach, Phys. Rev. 140, 97 (1965).