

G. A. Viano: ON THE EXPANSION OF THE SCATTERING AMPLITUDE AND THE FOURIER AND LAPLACE TRANSFORMS.

SUMMARY. -

In this paper we investigate some geometrical properties of the Bessel, MacDonalD, Legendre and Gegenbauer functions, which are useful in the expansions of the relativistic scattering amplitude. More precisely we can obtain, using these properties, a geometric interpretation of the Fourier-Bessel and the corresponding Laplace-like transform; with the same procedure we can also consider the Fourier analysis on the non-euclidean disk  $SU(1, 1)/SO(2)$  the corresponding Laplace-like transform and the extension of the non-euclidean Fourier analysis on the hyperbolic space  $H^n$ .

1. - INTRODUCTION. -

There has been considerable interest, in the last decade, in the harmonic analysis of the scattering amplitude on the Lorentz groups. The reader, interested to a survey of the vast amount of literature on this subject, is referred to the review papers of Toller<sup>(1)</sup> and Winternitz<sup>(2)</sup>.

The conventional theory of group-theoretical expansions requires that the functions being expanded be square-integrable over the group manifold, and gives a decomposition of such functions in terms of the matrix elements of the irreducible unitary representations of the group under consideration. More precisely, let us consider the expansion of the scattering amplitude  $F(s, t)$  of two scalar particles (where  $s$  and  $t$  are the usual Mandelstam variables) for a fixed negative value of  $t$ , in terms of the singled valued irreducible unitary representations of the group  $SU(1, 1)$  (i. e. the group of all two-dimensional complex pseudounitary matrices of unit determinant, which is homomorphic with respect to  $O(2, 1)$  in a two-to-one correspondence); in this case the square-integrability requirement translates into power decrease of amplitudes faster than  $s^{-1/2}$  as  $s \rightarrow \infty$ , which is well below the asymptotic bound indicated by experiments and by the Froissart bound<sup>(3, 4)</sup>.

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If one likes to use the complex angular momentum language, then one can roughly say that the harmonic analysis on the group  $SU(1, 1)$ , with the restriction to amplitudes square-integrable over the group manifold, leads to the so-called background integral but misses the leading Regge pole or cut behaviour<sup>(5)</sup>.

There are two different approaches in order to solve this problem. The first one, simply consists in extending the domain of the Fourier integral operator to include polynomially bounded functions. This type of extension has been extensively investigated in the classical Fourier analysis, with the aid of the distribution theory. Subsequently Ruhl<sup>(6, 7)</sup> developed these distribution-valued-transform methods with the intent of giving a foundation to the Fourier transform on the Lorentz groups for polynomially bounded functions.

An alternative approach consists in looking for an expansion, which can be obtained through a suitable analytic continuation of the matrix elements of the irreducible unitary representations to the matrix elements of the non-unitary ones. In fact let us return, for a moment, to the classical Fourier transform; in this case one has the expansion of a function in terms of the irreducible unitary representations of the additive group of real numbers; moreover, through an analytic continuation of these terms so to involve the non-unitary representations, one can get an extension of the harmonic analysis (i. e. the Laplace transform), which allows an expansion of functions which are not square-integrable on the line. Thus many authors (see, for instance, ref. (3)) followed the same procedure also for the harmonic analysis on the group  $SU(1, 1)$ ; in other words they generalize the conventional  $SU(1, 1)$  expansion by deforming the contour of integration. However a mere shift of the contour of integration is not sufficient, as it is admitted by the authors themselves<sup>(3)</sup>, since it is necessary to modify the formula which gives the projection of the scattering amplitude; formula which now involves the use of the so-called second-kind Legendre functions<sup>(8)</sup>. Therefore we shall follow a somewhat different approach; more precisely, we shall investigate the geometrical structure of the Fourier and Laplace transforms. For this reason, in order to make more direct and evident our analysis, we shall work in the symmetric space associated to the group  $SU(1, 1)$ , i. e. the non-euclidean disk  $SU(1, 1)/SO(2)$ . Then we shall consider in Sect. 2, the non-euclidean Fourier analysis on this manifold. We analyze the geometrical properties of the first and second-kind Legendre functions and we shall try to understand the role played by the latter in the Laplace analysis. At this purpose, we observe that the progress, which has been done by the mathematicians, in the theory of special functions, can be of great help in our case. In conclusion the aim of the present paper consists in trying a geometrical analysis of the genesis of the Laplace-like transforms, starting from the corresponding Fourier integrals. Therefore we think that it is necessary to do a preliminary effort in order to understand the geometrical structure of the Fourier transforms.

Sect. 2 is completely devoted to this problem; more precisely we first analyze the Fourier-Bessel integral (which presents analogies very illuminating for our case), with the aid of the group representation theory, then

we discuss the non-euclidean Fourier analysis, in the sense of Helgason<sup>(10)</sup>, on the disk  $SU(1,1)/SO(2)$ . In Sect. 2 we use and extend the results of a previous paper<sup>(11)</sup>, which shall be referred hereafter as (I). In Sect. 3 we see how one can go from the Fourier-Bessel integral to the corresponding Laplace-like transform and then, with an analogous procedure, we shall pass from the non-euclidean Fourier analysis to the corresponding Laplace-like expansion.

For what concerns the physics of the problem, we consider a spinless two-body scattering amplitude  $F(s, t)$ , in the physical region of the  $s$ -channel.

One can suppose  $s > 0$  fixed and expand the scattering amplitude in functions of  $t$  (in this case one obtains the usual partial-wave expansion), or viceversa it is possible to take  $t < 0$  fixed and decompose the scattering amplitude in functions of  $s$ . In the latter case, the scattering amplitude  $F(s, t)$  can be expressed by means of a hyperbolic function  $\cosh r$  which, if the masses  $m_i$  of the particles are arbitrary, can be written as follows<sup>(3)</sup>:

$$(1) \quad \cosh r = - \frac{t^2 + 2st - t(m_1^2 + m_2^2 + m_3^2 + m_4^2) + (m_1^2 - m_3^2)(m_2^2 - m_4^2)}{\lambda(t, m_1^2, m_3^2) \lambda(t, m_2^2, m_4^2)}$$

where the function  $\lambda(x, y, z)$  is the square root of the usual triangle function

$$\lambda^2(x, y, z) = x^2 + y^2 + z^2 - 2(xy + yz + zx)$$

In the equal mass case the expression (1) reduces to

$$(2) \quad \cosh r = \frac{2s}{4m^2 - t} - 1 \quad (m_i = m; i = 1, 2, 3, 4)$$

Finally, in the present paper, we shall not discuss the complications due to the spin, nor the expansions of multiparticle amplitudes. This simplification makes much more direct our geometrical analysis; since, if we analyze the scattering of spinless particles, then we can consider the Fourier expansion on the space associated to the group, e. g. on the space  $SU(1, 1)/SO(2)$ .

## 2. - THE FOURIER TRANSFORM. -

Let us start by recalling the so-called eikonal expansion of the scattering amplitude, which can be written as follows:

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$$\begin{aligned}
 (3) \quad F(s, t) &= \int_0^{+\infty} f(b) J_0(b \sqrt{-t}) b db = \\
 &= \frac{1}{2\pi} \int_0^{+\infty} f(b) b db \int_0^{2\pi} e^{i b \sqrt{-t} \cos \varphi} d\varphi
 \end{aligned}$$

where  $\underline{b}$  denotes a two-dimensional impact parameter in the plane perpendicular to the incoming beam, and  $\underline{k}$  is a two-dimensional momentum transfer vector, such that

$$(4) \quad \begin{cases} \underline{b}^2 = b^2 \\ \underline{k}^2 = -t \end{cases}$$

and finally  $\varphi$  denotes the angle between  $\underline{k}$  and  $\underline{b}$ .

As it is well-known the eikonal representation can be obtained from the usual direct-channel partial-wave expansion, at the limit for high value of the energy and for small values of the scattering angle ( $I$ ); therefore it can be used in the analysis of the high energy forward scattering (see, for instance, ref. (12)).

The expansion (3) is a Fourier-Bessel transform, which is usually written in the following way:

$$(5) \quad F(r) = \int_0^{+\infty} f(\varrho) J_0(\varrho r) \varrho d\varrho = \frac{1}{2\pi} \int_0^{+\infty} f(\varrho) \varrho d\varrho \int_0^{2\pi} e^{i\varrho r \cos \varphi} d\varphi$$

and is self-reciprocal in the sense that

$$(6) \quad f(\varrho) = \int_0^{+\infty} F(r) J_0(\varrho r) r dr .$$

Furthermore it is possible to relate the Bessel function  $J_0(\varrho r)$  to the representations of the group of motions of the euclidean plane. Let us recall that by motions of the euclidean plane, one means transformations which preserve the distance between points and do not change the orientation of the plane. As examples of motions we have parallel translations of the plane and rotations of the plane around some point<sup>(13)</sup>. Now, if the motion  $g$  considered is a shift by  $r > 0$  along the axis  $0x$  of the frame of reference, then the irreducible representations  $T_R(g)$  are given by (see ref. (13), p. 206):

$$(7) \quad T_R(g) f(\varphi) = e^{R r \cos \varphi} f(\varphi)$$

where the functions  $f$  belong to the space  $X$  of infinitely differentiable functions on the unit circle. The matrix elements  $t_{mn}^R(g)$  of the irreducible representations can be evaluated as follows:

$$(8) \quad t_{mn}^R(g) = (T_R(g) e^{in\varphi}, e^{im\varphi})$$

where the functions  $\{e^{im\varphi}\}$  form an orthonormal basis in the Hilbert space  $H$ , obtained by completing the space  $X$  with respect to the scalar product

$$(9) \quad (f_1, f_2) = \frac{1}{2\pi} \int_0^{2\pi} f_1(\varphi) \overline{f_2(\varphi)} d\varphi$$

Of course, we have also, for the functions  $f$ , that

$$(10) \quad (f, f) = \frac{1}{2\pi} \int_0^{2\pi} |f(\varphi)|^2 d\varphi < \infty$$

From (8) and (9) one obtains

$$(11) \quad t_{mn}^R(g) = \frac{1}{2\pi} \int_0^{2\pi} e^{\{Rr \cos\varphi + i(n-m)\varphi\}} d\varphi = i^{n-m} J_{n-m}(-iRr)$$

If we put  $n = m$  and  $R = i\varrho$  ( $\varrho$  real), then from (11) one has

$$(12) \quad \frac{1}{2\pi} \int_0^{2\pi} e^{i\varrho r \cos\varphi} d\varphi = J_0(\varrho r)$$

which corresponds to the unitary representation since  $R = i\varrho$  ( $\varrho$  real). Finally we recall that  $J_0$  is invariant under the rotation group of the plane. A very elegant proof of this statement can be found in the Ehrenpreis book on Fourier Analysis in Several Complex Variables (ref. (14), p. 383).

Next we want to consider the Fourier analysis on the non-euclidean disk  $SU(1, 1)/SO(2)$ . At this purpose the first step is to find the analog of the "plane-wave"  $e^{i\varrho r \cos\varphi}$ , in the case of the Lobacevskij space. This method has been discovered by Helgason<sup>(10)</sup>, and it has already been used in (I); here we limit ourselves to recall the main formulas and results for making readable this paper.

The Riemannian structure on the hyperbolic plane  $H^2$  can be written in geodesic polar coordinates  $(\vartheta, r)$  as follows (see ref. (15), p. 405):

$$(13) \quad dr^2 + (\sinh r)^2 d\vartheta^2$$



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Then it is convenient to introduce the following coordinates (Beltrami coordinates):

$$(14) \quad \begin{cases} y_1 = \tanh\left(\frac{r}{2}\right) \sin \vartheta \\ y_2 = \tanh\left(\frac{r}{2}\right) \cos \vartheta \end{cases}$$

From (14) we obtain

$$(15) \quad |z|^2 = y_1^2 + y_2^2 = \tanh^2\left(\frac{r}{2}\right)$$

where  $z = |z|e^{i\theta}$ . With easy computation we have

$$(16) \quad ds^2 = \frac{4(dy_1^2 + dy_2^2)}{(1 - |z|^2)^2} = dr^2 + (\sinh r)^2 d\vartheta^2$$

At this point, Following Poincaré, we can construct a model of Lobacevskij geometry in the unit disk  $D$ . First of all it turns out that the non-euclidean distance between the center of the unit disk  $D$  and the point  $z \in D$  is given by

$$(17) \quad d(0, z) = \log \frac{1 + |z|}{1 - |z|}$$

Next we write the Poisson-kernel:

$$(18) \quad P(z, b) = \frac{1 - |z|^2}{1 + |z|^2 - 2|z| \cos(\vartheta - \chi)}$$

where  $z = |z|e^{i\vartheta}$ ,  $b = e^{i\chi}$ . One of the properties of the Poisson-kernel  $P(z, b)$  is that its level lines are the circles tangent to the unit circle at the point  $b$ ; they are the euclidean images of oricycles. In fact the euclidean images of oricycles are circles tangent to the horizon  $|z| = 1$  (i. e. the boundary  $B$  of the unit disk  $D$ ) from within. Moreover  $P(z, b)$ , being the real part of  $(b+z)/(b-z)$ , is a harmonic function of  $z$  and  $(P(z, b))^\mu$ ,  $\mu \in \mathbb{C}$  is an eigenfunction of the Laplace-Beltrami operator on the non-euclidean disk  $D$ . In fact the Laplace-Beltrami operator on  $D$  reads as follows:

$$(19) \quad \Delta = \frac{1}{4} \left\{ 1 - (y_1^2 + y_2^2) \right\}^2 \left( \frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2} \right)$$

and one can verify by direct computation that<sup>(10)</sup>

$$(20) \quad \Delta(P(z, b))^\mu = \mu(\mu - 1)(P(z, b))^\mu, \quad \mu \in \mathbb{C}$$

Finally  $P(z, b)$  is invariant with respect to any transformation that preserves the unit disk.

Therefore we can conclude that:

$$(21) \quad e^{\mu \langle z, b \rangle} = \left( \frac{1 - |z|^2}{1 + |z|^2 - 2|z| \cos(\vartheta - \chi)} \right)^\mu, \quad \mu \in \mathbb{C}$$

is the non-euclidean analog of the "plane-wave"  $e^{i\varrho r \cos \varphi}$ . In fact  $(P(z, b))^\mu$  is constant on each oricycle of normal  $b$ , it is an eigenfunction of the Laplace-Beltrami operator on  $D$  and finally  $\langle z, b \rangle$  gives the non-euclidean distance from the center of the unit disk to the oricycle with normal  $b$  and passing through  $z$  ( $\langle z, b \rangle$  is negative if the center of the unit disk falls inside the oricycle). Thanks to these considerations we can speak of a Fourier transform on the disk  $SU(1, 1)/SO(2)$ ; in fact the following theorem, due to Helgason (see, for instance, ref. (10), p. 9), can be proved

Theorem (Helgason): - For  $f \in C_c^\infty(D)$  set

$$(22) \quad \tilde{f}(\lambda, b) = \int_D e^{(-i\lambda + 1/2) \langle z, b \rangle} f(z) dz \quad ; \quad \lambda \in \mathbb{R}, \quad b \in B$$

where

$$dz = \frac{4 dy_1 dy_2}{\{1 - (y_1^2 + y_2^2)\}^2}$$

is the volume element on  $D$ ; then

$$(23) \quad f(z) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}} \int_B e^{(i\lambda + 1/2) \langle z, b \rangle} \tilde{f}(\lambda, b) \tanh(\pi\lambda) \lambda d\lambda db$$

where  $db$  is the usual angular measure on  $B$

Moreover it holds the following Plancherel formula:

$$(24) \quad \int_D |f(z)|^2 dz = \int_{\mathbb{R} \times B} |\tilde{f}(\lambda, b)|^2 d\mu(\lambda, b)$$

where:

$$d\mu(\lambda, b) = \frac{1}{(2\pi)^2} \tanh(\pi\lambda) \lambda d\lambda db.$$

The formulae (22), (23) can be reduced to the classical Mehler-transform formulae (see ref. (16), vol. I, p. 175); at this purpose it is enough to recall the integral representation of the first-kind Legendre functions:

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$$(25) \quad P_{\alpha}(\cosh r) = \frac{1}{2\pi} \int_0^{2\pi} (\cosh r + \sinh r \cos \varphi)^{\alpha} d\varphi$$

and observe that:

$$(26) \quad e^{\alpha \langle z, b \rangle} = \left( \frac{1 - |z|^2}{1 + |z|^2 - 2|z| \cos(\vartheta - \chi)} \right)^{\alpha} = \left\{ \cosh r - \sinh r \cos(\vartheta - \chi) \right\}^{-\alpha}$$

More specific details can be found in I.

Next we consider the form:  $x_0^2 - x_1^2 - x_2^2 = 1$ ,  $x_0 > 0$ ; the  $\{x_i\}$ , ( $i = 0, 1, 2$ ) are the so-called Weierstrass coordinates of the points of the non-euclidean plane (see ref. (17), vol. I, p. 407), and are simply related to the Beltrami coordinates  $y_1, y_2$  of formula (14) as follows  $y_1 = (x_1/x_0)$ ,  $y_2 = (x_2/x_0)$  therefore the  $\{x_i\}$  can be written in the following way:

$$(27) \quad \begin{cases} x_0 = \cosh\left(\frac{r}{2}\right) \\ x_1 = \sinh\left(\frac{r}{2}\right) \sin \vartheta \\ x_2 = \sinh\left(\frac{r}{2}\right) \cos \vartheta \end{cases}$$

Then consider the group M of rotations about the  $x_0$ -axis, which is a subgroup of the tridimensional Lorentz group, which leaves invariant the form  $x_0^2 - x_1^2 - x_2^2$ . The elements of M can be represented as follows:

$$(28) \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos \varphi & -\sin \varphi \\ 0 & \sin \varphi & \cos \varphi \end{pmatrix}$$

It can be explicitly proved that the Legendre function (25) is invariant under M. The proof can be found in the Ehrenpreis book which we have quoted before (ref. (14), p. 383). From an intuitive point of view we can roughly say that this rotational invariance is reflected by the factor  $\cos \varphi$  in the integrand of formula (25) and, of course, by the rotationally invariant measure  $d\varphi$ .

In order to be as clear as possible on these questions, which are essential for our analysis, it can be useful to discuss these results from a different point of view. Let us return, for a moment, to the group SU(1, 1). It is well-known that the Lie algebras of SU(1, 1) and of O(2, 1) are the same, and that there is a two-to-one homomorphism from SU(1, 1) to O(2, 1). The Lie algebra of SU(1, 1) is spanned by the three elements  $J_0, J_1, J_2$  obeying the following commutation rules (see, for instance, ref. (18)):



$$(29) \quad \begin{cases} -i [J_0, J_1] = J_2 \\ -i [J_0, J_2] = -J_1 \\ -i [J_1, J_2] = -J_0 \end{cases}$$

$J_0$  is the generator of the maximal compact subgroup of  $SU(1, 1)$  while  $J_1$  and  $J_2$  are noncompact generator. In the language of Lorentz transformation in three dimensions, namely the group  $O(2, 1)$   $J_0$  generates spatial rotations in a plane (the subgroup  $O(2)$ ), while the transformations generated by  $J_1$  and  $J_2$  form an  $O(1, 1)$  subgroup of  $O(2, 1)$ . Following the classical paper of Bargmann<sup>(19)</sup>, one can analyze the structure of the irreducible unitary representations of  $O(2, 1)$ , using a basis in the representation space, in which the generator of the compact subgroup  $O(2)$  of  $O(2, 1)$  is diagonal. More precisely this representation space can be realized as the Hilbert space of Lebesgue square-integrable functions on the unit circle, and in this space one can choose a basis consisting of the functions  $\{e^{im\varphi}\}$ , such that  $J_0 e^{im\varphi} = m e^{im\varphi}$  (see formula (2.1) of ref. (18)). Then, using this representation space, one can write the matrix elements of the irreducible representations (see ref. (13)) and furthermore derive the integral representations for the zonal spherical functions, obtaining the conical functions  $P_{-(1/2)+i\lambda}(\cosh r)$ , which enter in the Mehler transform, in correspondence to the principal series of the irreducible unitary representations.

Finally we recall that the Joos representation<sup>(20)</sup> of the scattering amplitude

$$(30) \quad F(s, t) = \frac{1}{i} \int_0^{+\infty} \frac{P_{-1/2+i\lambda}(\cosh r) f(\lambda, t)}{\cosh(\pi\lambda)} \lambda d\lambda$$

(where  $\cosh r$  is given by formulae (1) or (2)), is a non-euclidean Fourier expansion of the type (23), (for details the reader is referred to (I)). The representation (30) can be identified with the background integral of the Sommerfeld-Watson transform<sup>(\*)</sup> analytically continued from the crossed  $t$ -channel. In formula (30)  $t < 0$  is fixed and one has a decomposition of the scattering amplitude in functions of  $s$ . Of course, from (30) one cannot obtain the asymptotic Regge pole behaviour; for this reason one must look for a Laplace-like extension of (30), which shall be considered in the next Section.

The theory developed above, can be generalized to the case of the hyperbolic spaces  $H^n$ .

For the space  $H^n$  the metric can be written, in terms of geodesic polar coordinates, as follows<sup>(21)</sup>:

(\*) - For some remarks on this point see ref. (3).

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$$\begin{aligned}
 ds^2 &= dr^2 + \sum_{i,j=1}^{n-1} g_{ij} d\vartheta_i d\vartheta_j = \\
 (31) \quad &dr^2 + \sinh^2 r d\vartheta_1^2 + \sinh^2 r \sin^2 \vartheta_1 d\vartheta_2^2 + \dots \\
 &\dots + \sinh^2 r \sin^2 \vartheta_1 \sin^2 \vartheta_2 \dots \sin^2 \vartheta_{n-2} d\vartheta_{n-1}^2
 \end{aligned}$$

Moreover one has with obvious notations:

$$(32) \quad g = \det(g_{ij}) = \sinh^{(2n-2)} r \sin^{(2n-4)} \vartheta_1 \sin^{(2n-6)} \vartheta_2 \dots \sin^2 \vartheta_{n-2}$$

The Laplace-Beltrami equation reads as follows:

$$(33) \quad \Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial r} \frac{\partial}{\partial r} + \frac{1}{\sqrt{g}} \sum_k \frac{\partial}{\partial \vartheta_k} \left( \sum_i g^{ik} \sqrt{g} \frac{\partial}{\partial \vartheta_i} \right)$$

where  $g^{ij} = (g_{ij})^{-1}$ ; in our case we have

$$\frac{1}{\sqrt{g}} \frac{\partial \sqrt{g}}{\partial r} = (n-1) \frac{\cosh r}{\sinh r}$$

Therefore from the Laplace-Beltrami equation, through the separation of variables, we obtain

$$(34) \quad \frac{d^2 \psi(r)}{dr^2} + (n-1) \frac{\cosh r}{\sinh r} \frac{d\psi(r)}{dr} = \mu \psi(r)$$

which can be reduced to the usual form of the Gegenbauer equation (see, for instance, ref. (16), vol. I, p. 178, formula (21) (if we put:  $\cosh r = z$ ,  $\mu = \alpha(\alpha + 2\nu)$ ,  $n = (2\nu + 1)$ ). Then the solutions of eq. (34) are the Gegenbauer functions, which can be expressed through the following integral representation:

$$\begin{aligned}
 C_\alpha^{\frac{n-1}{2}}(\cosh r) &= \frac{1}{\sqrt{\pi}} \frac{\Gamma(\alpha + n - 1) \Gamma(\frac{n}{2})}{\Gamma(\alpha + 1) \Gamma(n - 1) \Gamma(\frac{n-1}{2})} \cdot \\
 (35) \quad &\int_0^\pi (\cosh r + \sinh r \cos \vartheta)^\alpha (\sin \vartheta)^{(n-2)} d\vartheta = \\
 &= \frac{\Gamma(\alpha + n - 1)}{\Gamma(\alpha + 1) \Gamma(n - 1)} \frac{\int (\cosh r + \sinh r \cos \vartheta_1)^\alpha \sqrt{g} d\vartheta_1 \dots d\vartheta_{n-1}}{\int \sqrt{g} d\vartheta_1 \dots d\vartheta_{n-1}}
 \end{aligned}$$

since

$$\int_0^\pi \sin^{(n-1)} \theta d\theta = \frac{\sqrt{\pi} \Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}$$

Of course the Gegenbauer functions (35) coincide with the Legendre functions  $P_\alpha(\cosh r)$  for  $n=2$ .

Next we introduce the following coordinates  $\{y_i\}$  :

$$(36) \quad \left\{ \begin{array}{l} y_1 = \tanh(r/2) \sin\vartheta_1 \sin\vartheta_2 \dots \sin\vartheta_{n-2} \sin\vartheta_{n-1} \\ y_2 = \tanh(r/2) \sin\vartheta_1 \sin\vartheta_2 \dots \sin\vartheta_{n-2} \cos\vartheta_{n-1} \\ y_3 = \tanh(r/2) \sin\vartheta_1 \sin\vartheta_2 \dots \cos\vartheta_{n-2} \\ \dots \\ \dots \\ \dots \\ y_n = \tanh(r/2) \cos\vartheta_1 \end{array} \right.$$

thus we can use the unit ball model for  $H^n$ , i. e. the unit ball in  $R^n$  with the Riemannian structure

$$(37) \quad ds^2 = \frac{4(dy_1^2 + dy_2^2 + \dots + dy_n^2)}{\left\{1 - (y_1^2 + y_2^2 + \dots + y_n^2)\right\}^2}$$

Furthermore the Poisson-kernel is now given by (see ref. (22), p. 108):

$$(38) \quad P(y, b) = \left\{ \frac{1 - |y|^2}{|y - b|^2} \right\}^{(n-1)}$$

where  $|y|^2 = y_1^2 + y_2^2 + \dots + y_n^2 = \tanh^2(r/2)$ . Let us recall that, in this model, the oricycles are the spheres tangential to the unit sphere  $|y| = 1$ , from within. Therefore, in strict analogy with the bi-dimensional case, we shall write:

$$(39) \quad e^{\langle y, b \rangle} = \frac{1 - |y|^2}{|y - b|^2}$$

If we introduce the following notations:

"arbitrary" functions into "waves"  $e^{\mu\langle z, b \rangle}$ , which can be related to the first-kind Legendre functions  $P_{-(1/2)+i\lambda}(\cosh r)$ , that are invariant under the group  $M$  (of rotations about the  $x_0$ -axis of the quadratic form  $x_0^2 - x_1^2 - x_2^2 = 1$ ); analogously the Laplace transform involves the projection on functions, which must be invariant under  $M^H$ , whose elements can be obtained from the corresponding elements of  $M$  (formula (28)), changing the angle  $\varphi$  into a pure imaginary angle. Therefore in the integral representation of the functions we are looking for, we should have:  $(\cosh r + \sinh r \cosh \varphi)$  in place of  $(\cosh r + \sinh r \cos \varphi)$ . Following an heuristic approach, we can start from the integral representation (25) for the Legendre function  $P_{-\alpha-1}(\cosh r)$  (recall that  $P_{-\alpha-1} = P_\alpha$ ), then do in the integral the substitution indicated above, write properly the integration limits and so obtain:

$$(57) \quad \int_0^{+\infty} (\cosh r + \sinh r \cosh \varphi)^{-\alpha-1} d\varphi = Q_\alpha(\cosh r), \quad \operatorname{Re} \alpha > -1$$

which is the second-kind Legendre function (see ref. (26), vol. I, pag. 157, formula (12)). For a rigorous proof of the invariance of  $Q_\alpha(\cosh r)$  under  $M^H$ , one must follow the Ehrenpreis considerations (ref. (14), pag. 385).

Let us return to the group  $SU(1, 1)$  (or  $O(2, 1)$ ; we have seen in Sect. II that it is possible, following Bargmann<sup>(19)</sup>, to analyze the structure of the irreducible unitary representations of  $O(2, 1)$  in a basis in which the generator of the compact subgroup  $O(2)$  of  $O(2, 1)$  is diagonal, and in this case the representation space is a space of square-integrable functions on the unit circle. On the other hand one can also examine the structure of these representations in a basis, in which the hyperbolic generator of an  $O(1, 1)$  subgroup is diagonalized. This analysis has been explicitly done by Mukunda (refs. (18), (26); see also ref. (27)). This approach requires to introduce a space of functions given on the real line. Mukunda gives explicitly (formula (3.6) of ref. (18)) the change of variables from  $\varphi$ , which varies from 0 to  $2\pi$ , to  $q$  which varies continuously on the real line; moreover he can show that, to a trigonometric function of  $\varphi$  it corresponds a hyperbolic function of  $q$ . Thereafter one can analyze this realization of the representations, and it turns out that the representation functions show a second-kind behaviour.

For what concerns the asymptotic behaviour of  $Q_\alpha(z)$ , one has:

$$(58) \quad Q_\alpha(z) \Big|_{|z| \rightarrow \infty} \sim \frac{\sqrt{\pi} \Gamma(\alpha+1)}{2^{\alpha+1} \Gamma(\alpha + \frac{3}{2})} z^{-\alpha-1} (1 + O(z^{-2})); \quad |\arg z| < \pi$$

and

$$(59) \quad Q_\alpha(\cosh r) \Big|_{|\alpha| \rightarrow \infty} \sim \frac{c}{\sqrt{\alpha}} e^{-(\alpha + \frac{1}{2})r}; \quad r > 0$$

Therefore one can conjecture that the Laplace-like projection formula is simply given by:

$$(60) \quad f(\ell) = \int_1^{+\infty} F(\cosh r) \cdot Q_\ell(\cosh r) d(\cosh r)$$

which coincides exactly with the formula proposed by refs. (3), (9).

At this point, for the sake of completeness, we report two results due to Constrom and Klink<sup>(3)</sup> and to Conström<sup>(9)</sup> respectively. First of all let us recall the following representation of the second-kind Legendre function

$$(61) \quad Q_\ell(\cosh r) = \int_r^{+\infty} \frac{e^{-(\ell + \frac{1}{2})\varphi}}{(2 \cosh \varphi - 2 \cosh r)^{1/2}} d\varphi$$

From (61) it follows that the integral (60) can be formally rewritten as

$$(62) \quad f(\ell) = \int_0^{+\infty} e^{-(\ell + \frac{1}{2})\varphi} g(\varphi) d\varphi$$

where:

$$(63) \quad g(\varphi) = \frac{1}{\sqrt{2}} \int_1^{\cosh \varphi} F(x) (\cosh \varphi - x)^{-1/2} dx$$

Then the following theorem can be proved:

Theorem (Conström and Klink<sup>(3)</sup>). If  $F(x)$ , ( $x \in (1, \infty)$ ) is an arbitrary, locally integrable function, such that the following integral exists

$$(64) \quad \int_1^{+\infty} (x^2 - 1)^{-1/4} x^{-p-\frac{1}{2}} |F(x)| dx < \infty$$

for some real value of  $p > -\frac{1}{2}$ . Then:

- (i)  $f(\ell)$  is a Laplace-transform of the function  $g(\varphi)$ , with abscissa of convergence  $\text{Re } \ell = p$  (so that  $f(\ell)$  is regular analytic in the half-plane  $\text{Re } \ell > p$ );
- (ii) the asymptotic behaviour of  $f(\ell)$  is given by:  $f(\ell) = o\{(\ell + 1/2)^{-1/2}\}$  as  $|\text{Im } \ell| \rightarrow \infty$ .

Furthermore, following Cronström<sup>(9)</sup>, one can insert the inverse of (62) into the inverse of (63) and obtain:



18.

$$(65) \quad F(\cosh r) = \frac{1}{2\pi i} \int_{\text{Re } \ell = p} (2\ell + 1) f(\ell) R_{\ell}(\cosh r) d\ell$$

where the line of integration runs to the right of the singularities of  $f(\ell)$  and

$$(66) \quad R_{\ell}(\cosh r) = \frac{1}{\pi} \int_0^r \frac{e^{(\ell + \frac{1}{2})\varphi}}{(2 \cosh r - 2 \cosh \varphi)^{1/2}} d\varphi$$

The formula (65) gives the inverse of (60).

As it is well-known, in the theory of classical Laplace transformation, it is possible to relate the asymptotic behaviour of the original function to the analytical properties of the image (see ref. (28), pag. 212). This method is often used in system-theory for the investigation of the stability. If, for instance, the physical system is controlled by an ordinary differential equation, then the location of the pole of the image, with the greatest real part, determines if the solution is stable or not (see ref. (28), pag. 217). It would be very interesting to translate such a theory in the case of high-energy scattering. Unfortunately, in the present case, theorems as powerful as in the classical case (see, for instance, theorem 41.1 of ref. (28)) are not available. However it is possible to relate, through a suitable shift of the contour of integration in (65), the asymptotic behaviour of the scattering amplitude to the analytical properties of its Laplace-like image (60); for this problem, as well as for what concerns the relationships between (65) and the Sommerfeld-Watson-Mandelstam representation the interested reader is referred to refs. (3) and (9).

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