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P. Butera, G. M. Cicuta and E. Montaldi : RENORMALIZATION AND SPACE-TIME DIMENSION.

ABSTRACT. -

A renormalization procedure is developped for arbitrary graphs regularized by analytic interpolation in the space-time dimension. This approach, which is of course equivalent to the conventional one, has some intermediate features between the analytic renormalization and the B. P. H. procedure, but appears to be simpler than both.

1. - INTRODUCTION. -

Recently a number of authors(2, 8, 12, 26) proposed to regularize the Feynman amplitudes by analytic interpolation in the dimension of space-time. This procedure appears to be particularly convenient in the renormalization of quantum electrodynamics and gauge theories, since gauge invariance is preserved by the regularization(4, 13, 16, 27, 29). Furthermore, the regularized Feynman amplitudes have a very simple integral representation in terms of the usual parametric functions and this is convenient also for studying their analyticity properties or asymptotic behaviour.

In this work we develop the technique of the interpolation in the space-time dimension to deal with an arbitrary graph of a renormalizable theory. We have in mind Lagrangian models such as $\phi^3)_4$, $\phi^3)_6$, $\phi^4)_4$ (the lower index is the space-time dimension) and the usual quantum electrodynamics.

While this work was being prepared, other authors have developped similar approaches to deal with arbitrary graphs[3, 14]. In Ref. (3) the general Feynman amplitude is regularized by a set of complex dimension-like variables. The amplitude turns out to be a meromorphic function of such variables and, by suitably extending the analysis of the theory of analytic renormalization[7, 18, 19, 25], a generalized evaluator is introduced to obtain the renormalized amplitude.

In Ref. (13) a single complex dimension parameter is used to regularize the Feynman amplitudes whose renormalization is then performed by introducing analytic methods within the recursive Bogoliubov-Parasiuk-Hepp[6, 21] (B. P. H.) scheme of subtractions.

In the present work we suggest an intermediate approach, since a single complex dimension regularizing parameter is used in the frame of a non recursive subtraction procedure developped in Ref. (1). We feel that this proposal is the simplest and the most convenient for applications.

The work is organized as follows : in Section 2 we deal with a simple class of scalar graphs which can be renormalized by a single operation. By first examining this case the essential features of the approach can be discussed in a simple fashion. In Section 3 the generic Feynman graph of a renormalizable scalar theory is dealt with. In Section 4 the procedure is applied to quantum electrodynamics. Some technical details are confined in the Appendices to keep the paper readable.

2. - RENORMALIZATION FOR CLASS $\mathcal{A}$ GRAPHS. -

In order to exhibit the peculiarities of the interpolation in the space-time dimension, we restrict ourselves in this section to the renormalization of a simple class of irreducible graphs, we call the $\mathcal{A}$ class. They include the one loop graphs with any number of vertices and the graphs with two vertices and any number of loops. A simple characterization of the $\mathcal{A}$ class will be given in the next section.

Let us start from the usual parametric integral representation for a Feynman amplitude in a n-dimensional space-time, n being an integer equal to $s+1$, where s is the number of space dimensions. It has the form (see e. g.)[23]:

$$(2.1)\qquad I_{(n)}(p_i) = \int_0^{\infty} \left[C(\alpha_i)\right]^{-\frac{n}{2}} \exp\left[i\,\frac{D(\alpha_i, p_i, m^2)}{C(\alpha_i)}\right] \prod_{i=1}^{l} d\alpha_i$$

except for a factor independent of the external momenta p_i and finite for every value of n; $C(\alpha_i)$ and $D(\alpha_i, p_i, m^2) = W(\alpha_i, p_i) - m^2 C(\alpha_i) \sum_{i=1}^{l} \alpha_i + i\varepsilon$

are the familiar parametric functions. Let us now consider the case of one loop graphs, then $C(\alpha_i) = \sum_{i=1}^{1} \alpha_i$ and, by performing a scale transformation (see Appendix A), the integral (2.1) becomes:

$$(2.2) \qquad I_{(n)}(p_i) = \int_0^1 \delta(1 - \sum_{i=1}^{1} \alpha_i)(\prod_{i=1}^{1} d\alpha_i) \int_0^{\infty} \rho^{1-\frac{n}{2}-1} \exp\left[i\rho D(\alpha_i, p_i, m^2)\right] d\rho$$

It is easily seen that the integral (2.2) is well defined for all integers (negative and positive) such that $n < 21$.

Our procedure is to define an analytic interpolating function $I_{(d)}(p_i)$ which coincides with $I_{(n)}(p_i)$ for integer $d < 21$.

As it was shown in Refs. (2, 8, 12, 26), the renormalized value $I_{ren}(p_i)$ of the integral (2.2) is obtained by simply taking the regular part of the Laurent expansion of $I_{(d)}(p_i)$ around $d = 4$.[1] Some comments concerning the non-uniqueness of this procedure are in order. In fact, if $\mathrm{Reg}(\rho^{1-d/2-1})$ is a particular regularization of $\rho^{1-d/2-1}$ as a generalized function[17] on the space of infinitely differentiable functions of fast decrease, then any other regularization is obtained by adding to it a functional with support in the origin, say $g(\rho) = \sum_{i=0}^{M} c_i \delta^{(i)}(\rho)$. All these regularizations have to coincide in the submanifold of test functions where the functional $\rho^{1-d/2-1}$ exists. This requirement fixes the degree M of the highest derivative of the delta function to be the largest integer $\leq \mathrm{Re}(\frac{d}{2} - 1)$. The arbitrariness in the definition of the regularization, as it was shown in Ref. (12), corresponds to the familiar arbitrariness in the choice of the subtraction point in the Bogoliubov formulation. A further way to exhibit the arbitrariness of the present renormalization procedure is to explicitly perform the ρ-integration in (2.2). One obtains

$$(2.3) \qquad I_{(n)}(p_i) = \Gamma(1 - \frac{n}{2})\, e^{\frac{i\pi}{2}(1-\frac{n}{2})} \int_0^1 \delta(1 - \sum_{i=1}^{1} \alpha_i) \left[D(p_i, \alpha_i, m^2)\right]^{\frac{n}{2}-1} \prod_{i=1}^{1} d\alpha_i$$

As $n \to -\infty$, $I_{(n)}(p_i)$ exceeds the asymptotic bound $e^{k|n|}$ with $k < \pi$, then a unique interpolating function can not be defined[5]. If $I_{(d)}(p_i)$ is an interpolating function, one may consider as well:

$$(2.4) \qquad \overline{I}_{(d)}(p_i) = I_{(d)}(p_i) + \Gamma(1 - \frac{d}{2})\, g(d)\, P(p_i)$$

where $g(d)$ is an entire function of d, which vanishes at all integers (say $\sin \pi d$) and $P(p_i)$ is a polynomial in the external momenta with arbitrary coefficients and satisfies the following reasonable requirements :

1) It is a Lorentz invariant function of the external momenta.

2) $P(p_i)$ does not grow for large momenta faster then the regularized Feynman integral. This fixes the degree of the polynomial in (2.4) to be less than or equal to $d/2 - 1$, in the quadratic Lorentz invariants.

It is easy to check that the arbitrariness here described corresponds to the different possible choices of regularization as previously described, then leading again to the familiar arbitrariness in the Bogoliubov formulation/2/.

A general graph of the $\mathcal{A}$ class can be renormalized in a similar way, by a single operation. Multiple poles of the interpolating function may appear in the complex d plane : for instance, the self-energy graph in Fig. 1 has a double pole in $d = 4$.

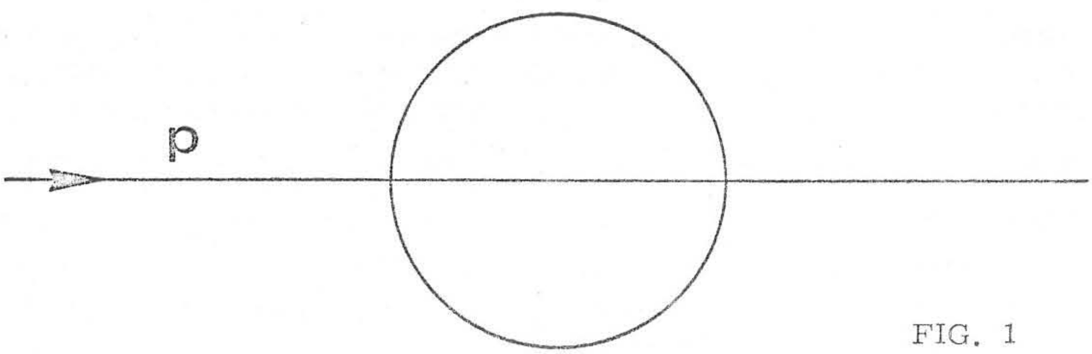

FIG. 1

The associated interpolating function may be taken as

$$(2.5) \qquad I_{(d)}(p^2) = \int_0^\infty \left[C(\alpha_i)\right]^{-\frac{d}{2}} \exp\left[i\frac{D(p^2,\alpha_i,m^2)}{C(\alpha_i)}\right] d\alpha_1\, d\alpha_2\, d\alpha_3$$

where

$$C(\alpha_i) = \alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3$$

$$D(p^2,\alpha_i,m^2) = \alpha_1\alpha_2\alpha_3 p^2 - m^2 C(\alpha_i) \sum_{i=1}^{3} \alpha_i + i\varepsilon \ .$$

The renormalized integral $I_{ren}(p^2)$ is obtained by taking the finite part of the Laurent expansion of $I_{(\lambda)}(p^2)$ at $\lambda = 4$.

One finds (see Appendix A) :

$$(2.6) \qquad \oint \frac{d\lambda}{\lambda - 4}\, I_{(\lambda)}(p^2) = a + bp^2 + R_3(4) .$$

By using the previously described arbitrariness, the renormalized integral may be taken as :

$$(2.7) \qquad I_{ren}(p^2) = R_3(4) = i\int_0^1 d\alpha_1\, d\alpha_2\, d\alpha_3 \frac{\delta(1 - \sum_{i=1}^{3} \alpha_i)}{\left[C(\alpha_i)\right]^3} \left\{ \alpha_1\alpha_2\alpha_3 p^2 \cdot \left[1 - \log\left(1 - \frac{\alpha_1\alpha_2\alpha_3 p^2}{C(\alpha_i)m^2}\right)\right] + C(\alpha_i)\, m^2 \log\left(1 - \frac{\alpha_1\alpha_2\alpha_3 p^2}{C(\alpha_i)m^2}\right)\right\}$$

which corresponds to subtracting at $p^2 = 0$.

It is interesting to examine the asymptotic behaviour, $\sigma \equiv -p^2 \to \infty$, of the regularized integral $I_{(d)}(p^2)$ given in (2.5). Its Mellin transform is :

$$F_{(d)}(\beta) = \int_0^\infty I_{(d)}(\sigma)\, \sigma^{-\beta-1}\, d\sigma = \Gamma(-\beta)\, e^{i\frac{\pi}{2}\beta} A_{(d)}(\beta)$$

where

$$(2.8) \qquad A_{(d)}(\beta) = \int_0^\infty \frac{(\alpha_1\alpha_2\alpha_3)^\beta\, e^{-im^2 \sum_{i=1}^{3} \alpha_i}}{\left[C(\alpha_i)\right]^{\beta + \frac{d}{2}}} \prod_{i=1}^{3} d\alpha_i$$

One easily finds that $A_{(d)}(\beta) = \Gamma(3 + \beta - d)$ times a function which has double poles at $\beta = -1, -2, \ldots$

According to the usual analysis[15], the rightmost poles of $A_{(d)}(\beta)$ in the β plane determine the asymptotic behaviour of $I_{(d)}(\sigma)$. For $d = 4$, the dominant pole is in $\beta = 1$. We have :

$$(2.9) \qquad F_{(4)}(\beta) \underset{\beta \sim 1}{\approx} i\,\Gamma(-\beta)\,\Gamma(\beta - 1) \int_0^1 \frac{\alpha_1\alpha_2\alpha_3\, \delta(1 - \sum_{i=1}^{3} \alpha_i)}{\left[C(\alpha_i)\right]^3} \prod_{i=1}^{3} d\alpha_i$$

By inverting the Mellin transform one obtains the asymptotic behaviour of the renormalized integral:

$$(2.10) \qquad I_{(4)}(\sigma) \sim i\sigma \log \frac{\sigma}{m^2} \int_0^1 \frac{\alpha_1\alpha_2\alpha_3\,\delta(1-\sum_{i=1}^{3}\alpha_i)}{\left[C(\alpha_i)\right]^3} \prod_{i=1}^{3} d\alpha_i$$

which is checked by looking at (2.7).

We like to remark that the poles of $A_{(d)}(\beta)$ in the β plane which are independent of the space-time dimension d, are those which can be predicted by the usual considerations about shortest paths(15, 28)/3/.

We conclude this Section by noting that for the $\mathcal{A}$ class of graphs a single subtraction is sufficient in the B.P.H. renormalization theory. Therefore an operator $O = \frac{1}{2\pi i}\oint \frac{d\lambda}{\lambda - 4}$ which extracts the finite part of a Laurent expansion from a regularized amplitude $I_{(\lambda)}(p_i)$ may either be considered as an evaluator of the analytic renormalization theory(7, 18, 25) or as an operator that extracts a remainder of a Taylor series in the external momenta. For a general graph, different formulations are possible. In order to obtain a prescription similar to analytic renormalization, a set of dimension-like parameters have to be introduced(3). On the other hand, within the B.P.H. recursive subtraction scheme, the theory may be formulated with a single dimension parameter(14). As it was anticipated in the Introduction, our technique, we describe in the next Section, has some intermediate features/4/.

3. - RENORMALIZATION OF A GENERAL SCALAR GRAPH. -

Let G be general graph. A basic notion is the non recursive characterization(1) of the class $\{S(G)\}$ of the dominant (or complete) divergent subgraphs S_i. These are the irreducible subgraphs S_i of the graph G which:

1) are superficially divergent, $\mu_i \geq 0$;

2) cannot be formed from another superficially divergent graph by simply opening one line.

$\{S(G)\}$ consists of those graphs of the class singled out by the recursive B.P.H. procedure which are associated with Taylor subtractions.

The $\mathcal{A}$ class of graphs considered in the previous section is merely the class of graphs G whose $\{S(G)\}$ class contains a single element, the graph G itself.

The renormalization of a general Feynman integral is performed by applying the procedure of the previous Section to each subgraph S_i of the class $\{S(G)\}$.

The regularized integral has the form :

$$I_{(d)}(p_i) = \int_0^{\infty} \left[C(\alpha_i)\right]^{-\frac{d}{2}} \exp\left[i\,\frac{D(p_i, \alpha_i, m^2)}{C(\alpha_i)}\right] \prod_{i=1}^{1} d\alpha_i \,. \tag{3.1}$$

The formal, possibly divergent Feynman integral is recovered by letting the complex variable d be equal to four.

As a consequence of Theor. 1 in Ref. (24), $I_{(d)}(p_i)$ is a meromorphic function in the complex d plane/5/.

Let us choose a subgraph $S_k \in \{S(G)\}$. By performing a scale transformation on the variables α_j associated with the lines belonging to S_k we obtain :

$$\begin{aligned} I_{(d)}(p_i) = \int_0^{\infty} \prod_{\alpha_i \notin S_k} d\alpha_i \int_0^{1} \prod_{\alpha_j \in S_k} d\alpha_j\, \delta(1 - \sum_{\alpha_j \in S_k} \alpha_j) \;\cdot \\ \cdot \int_0^{\infty} \rho^{-r_k - 1}\, \hat{C}^{-\frac{d}{2}} \exp(i\hat{D}/\hat{C})\, d\rho \,. \end{aligned} \tag{3.2}$$

Here $r_k = -n_k + \frac{d}{2} l_k$, n_k being the number of lines of the subgraph S_k and l_k the order of the zero of $C(\rho\alpha_j)$ for $\rho = 0$, that is also the number of loops of S_k; furthermore

$$\hat{C}(\alpha_i, \alpha_j, \rho) = \rho^{-l_k}\, C(\alpha_i, \alpha_j \rightarrow \rho\,\alpha_j)$$

$$\begin{aligned} \hat{D}(\alpha_i, \alpha_j, \rho, p_i, m^2) &= \rho^{-l_k} D(\alpha_i, \alpha_j \rightarrow \rho\,\alpha_j, p_i, m^2) = \\ &= \rho^{-l_k}\left\{ W(\alpha_i, \alpha_j \rightarrow \rho\alpha_j, p_i) - m^2 C(\alpha_i, \alpha_j \rightarrow \rho\alpha_j)\left[\Sigma\alpha_i + \rho\,\Sigma\,\alpha_j\right]\right\} \end{aligned}$$

Let us now introduce an operator $\mathcal{F}_{(S_k)}$ to remove the divergence from the ρ-integration in (3.2)

$$\mathcal{F}_{(S_k)} I_{(d)}(p_i) = \int_0^\infty \prod_{\alpha_i \notin S_k} d\alpha_i \int_0^1 \prod_{\alpha_j \in S_k} d\alpha_j \, \delta(1 - \sum_{\alpha_j \in S_k} \alpha_j) \cdot$$

$$\cdot \int_0^\infty \rho^{-r_k - 1} \hat{C}^{-\frac{d}{2}} e^{-im^2(\Sigma \alpha_i + \rho \Sigma \alpha_j)} \left[1 - \mathcal{M}^{(\rho)}_{(r_k)}\right] \exp(iW/\hat{C}) d\rho$$

$\mathcal{M}^{(\rho)}_{(r_k)} f(\rho)$ being the Taylor expansion of $f(\rho)$ around the point $\rho = 0$, truncated at order $r_k|_{d=n}$. The regularized amplitude $I_{(d)}(p_i)$ may be writen as

$$I_{(d)}(p_i) = I^{(G)}_{(d)} \left[I^{(S_k)}_{(d)} (q_1, \ldots . q_r) \right] ,$$

where the functional dependence of $I^{(G)}_{(d)}$ upon $I^{(S_k)}_{(d)}$ is denoted by the square brackets and $q_1, \ldots, q_r$ are the external momenta of S_k.

We are now ready to describe our general procedure by the following Lemma and Theorem.

Lemma. The parameter r_k, evaluated for $d = 4$, equals $\frac{1}{2}\mu_k$ where μ_k is the superficial divergence, usually defined by $\mu_k = 2(2l_k - n_k)$, n_k being the number of internal lines and l_k the number of loops for the subgraph S_k. Furthermore

$$\mathcal{F}_{(S_k)} I^{(G)}_{(d)} \left[I^{(S_k)}_{(d)} (q_1, \ldots, q_r) \right] = I^{(G)}_{(d)} \left[\int_0^1 d\rho \, \frac{(1-\rho)^{\mu_k}}{\mu_k !} \left(\frac{\partial}{\partial \rho}\right)^{\mu_k + 1} \cdot \right.$$

$$\left. \cdot \; I^{(S_k)}_{(d)} (\rho q_1, \ldots, \rho q_r) \right]$$

Let $\mathcal{F} = \Pi \mathcal{F}_{(S_k)}$, where the product is the successive application of $\mathcal{F}_{(S_k)}$ for each element S_k of the class $\{S(G)\}$ in a given order.

Theorem. $\mathcal{F} I_{(d)}(p_i)$ is finite at $d = 4$ and equals the renormalized Feynman integral:

$$\mathcal{F} I_{(d)}(p_i) \Big|_{d=4} = I_{ren}(p_i)$$

This procedure does not depend on the ordering of the operators $\mathcal{F}_{(S_k)}$.

Lemma and Theorem are proved in Appendix B as a straightforward consequence of the scheme developped in Ref. (1).

As a final remark, we recall that the choice of the origin in momentum space as a subtraction point poses no restriction on this formalism. Subtraction in a different point, say $p_i = a$, may be performed by applying the convenient integral representation for the operator $\mathcal{M}_{(r_k, a)}$ given in Ref. (12) to each subgraph of the class $\{S(G)\}$.

4. - QUANTUM ELECTRODYNAMICS. -

As it is well known[1, 23], Feynman amplitudes may be given a parametric integral representation also in theories involving spinors, by applying derivative operators to properly modified parametric functions.

We only define explicitly our interpolation of the set of Dirac matrices for space-time dimension different from four/6/. With the aim of describing a class of models having an arbitrary number of space-time dimensions, we consider the set of n Dirac matrices γ_i satisfying $\{\gamma_i, \gamma_j\} = 2 g_{ij}$, where $g_{oo} = 1$, $g_{ii} = -1$ if $i = 1, \ldots, n-1$, $g_{ij} = 0$ if $i \neq j$. The computation of traces and products follows then trivially as it is indicated in Appendix C. The interpolation to complex values of the dimension has to be done after products or traces of Dirac matrices have been performed.

Gauge invariance at second order has been exhibited in studying the vacuum polarization tensor for arbitrary dimension[2, 9, 26]. Here we confine ourselves to check the validity of the usual Ward identity for an arbitrary value of the dimension of space-time, at the lowest non trivial order.

At second order, $\Sigma(p)$, the fermion self energy may be written (see Appendix C):

$$\Sigma(p) = -\frac{4\pi\alpha}{(2\pi)^n} i \sum_1 g^{ll} \int dq \frac{\gamma^l [\gamma \cdot (p-q) + m] \gamma^l}{[(p-q)^2 - m^2](q^2 - \mu^2)} = -i \frac{4\pi\alpha}{(2\pi)^n} e^{-\frac{i\pi n}{4}} \cdot \tag{4.1}$$

$$\cdot\ \pi^{\frac{n}{2}} e^{i\frac{\pi}{2}(2-\frac{n}{2})} \Gamma(2 - \frac{n}{2}) \int_0^1 \frac{(2-n)\alpha\gamma \cdot p + nm}{B^{2-\frac{n}{2}}} d\alpha ,$$

where $B = \alpha(1-\alpha)p^2 - \alpha\mu^2 - (1-\alpha)m^2$, and the photon has been given mass μ.

The insertion of an external photon, carrying zero momentum, gives the fermion vertex function at third order :

$$\Gamma^{\mu}(p,0) = -i\frac{4\pi\alpha}{(2\pi)^n}\sum_{1} g^{11}\int dq\,\frac{\gamma^1\left[\gamma\cdot(p-q)+m\right]\gamma^{\mu}\left[\gamma\cdot(p-q)+m\right]\gamma^1}{\left[(p-q)^2-m^2\right]^2(q^2-\mu^2)} =$$

$$(4.2)\qquad = -i\frac{4\pi\alpha}{(2\pi)^n}\,e^{-\frac{i\pi n}{4}}\,\pi^{\frac{n}{2}}\left\{-e^{i\frac{\pi}{2}(2-\frac{n}{2})}\,\Gamma(2-\tfrac{n}{2})\,\frac{(2-n)^2}{2}\,\gamma^{\mu}\int_0^1\frac{d\alpha}{B^{2-\frac{n}{2}}}\,+\right.$$

$$\left.+\,i\,e^{i\frac{\pi}{2}(3-\frac{n}{2})}\,\Gamma(3-\tfrac{n}{2})\int_0^1\frac{d\alpha}{B^{3-\frac{n}{2}}}\,Y^{\mu}\right\},$$

where $Y^{\mu} = (2-n)\,\alpha^2\gamma\cdot p\gamma^{\mu}\gamma\cdot p + 2\,m\,n\,\alpha p^{\mu} + (2-n)m^2\gamma^{\mu}$.

By using the identity, proved in Appendix C :

$$(4.3)\qquad \Gamma(3-\tfrac{d}{2})\int_0^1\frac{(1-\alpha)(m^2-\alpha^2p^2)}{B^{3-\frac{d}{2}}}\,d\alpha = \Gamma(2-\tfrac{d}{2})\int_0^1\frac{(\frac{d}{2}-1)(1-\alpha)-\alpha}{B^{2-\frac{d}{2}}}\,d\alpha,$$

valid for arbitrary d, the Ward identity obtains

$$(4.4)\qquad \Gamma^{\mu}(p,0) = -\frac{\partial\Sigma(p)}{\partial p_{\mu}},$$

which also holds for arbitrary d.

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APPENDIX A. -

In dealing with parametric functions it is often convenient to perform a scale transformation (see e. g. Ref. (15), ch. 3). It is a transformation from a set of n variables α_i to a set of n+1 variables ρ, $\bar{\alpha}_i$ with the constraint $\sum_{i=1}^{n} \bar{\alpha}_i = 1$. One has :

$$\text{(A. 1)} \quad \int_0^\infty \left(\prod_{i=1}^{n} d\alpha_i\right) f(\alpha_i) = \int_0^1 \left(\prod_{i=1}^{n} d\bar{\alpha}_i\right) \delta\left(1 - \sum_{i=1}^{n} \bar{\alpha}_i\right) \int_0^\infty f(\rho\,\bar{\alpha}_i)\, \rho^{n-1}\, d\rho$$

To perform the inverse transformation, the following relation may be used :

$$\text{(A. 2)} \quad \int_0^1 \left(\prod_{i=1}^{n} dx_i\right) \delta\left(1 - \sum_{i=1}^{n} x_i\right) \int_0^\infty d\rho\, g(\rho, x_i) = \int_0^\infty \left(\prod_{i=1}^{n} dy_i\right) \left(\sum_{i=1}^{n} y_i\right)^{1-n} g\left(\sum_{i=1}^{n} y_i, \frac{y_i}{\sum_{i=1}^{n} y_i}\right).$$

The computation of the finite part of the Laurent expansion of $I_{(d)}(p^2)$ given in (2.4) is straight forward. After a scale transformation one obtains :

$$\text{(A. 3)} \quad I_{(d)}(p^2) = \Gamma(3-d)\, e^{\frac{i\pi(3-d)}{2}} \int_0^1 \left(\prod_{i=1}^{3} d\alpha_i\right) \delta\left(1 - \sum_{i=1}^{3} \alpha_i\right) \cdot \left[C(\alpha_i)\right]^{3\left(1-\frac{d}{2}\right)} \left[D(\alpha_i, p^2, m^2)\right]^{d-3}$$

A new change of variables is performed by using the formula :

$$\text{(A. 4)} \quad \int_0^1 d\alpha_1 \int_0^{1-\alpha_1} d\alpha_2\, f(\alpha_1 + \alpha_2, \alpha_1\alpha_2) = \frac{1}{2} \int_0^1 \frac{dx}{\sqrt{1-x}} \int_0^1 dy\, y\, f\left(y, \frac{1}{4} x y^2\right)$$

We obtain :

$$I_{(d)}(p^2) = \Gamma(3-d)\,\frac{e^{\frac{i\pi(3-d)}{2}}\,m^{2(d-3)}}{2}\int_0^1 \frac{dx}{\sqrt{1-x}}\,\cdot \tag{A.5}$$

$$\cdot\int_0^1 dy\, y^{1-\frac{d}{2}}\left(\frac{1}{4}xy+1-y\right)^{-\frac{d}{2}}(A-1)^{d-3}$$

where

$$A \equiv \frac{\frac{p^2}{4m^2}\,xy(1-y)}{\frac{1}{4}xy+1-y}$$

To identify the finite part of $I_{(d)}(p^2)$ at $d\sim 4$, it is convenient to add and subtract in the y integrand the first two terms of the Taylor expansion of the last factor, around $p^2=0$. Then $I_{(d)}(p^2)$ is decomposed as :

$$I_{(d)}(p^2) = \Gamma(3-d)\,\Gamma(2-\tfrac{d}{2})\,R_1(d) + \Gamma(4-d)p^2 R_2(d) + R_3(d)$$

where the three functions $R_i(d)$ are regular at $d=4$.

$(2\pi i)^{-1}\oint \frac{d\lambda}{\lambda-4}\, I_{(\lambda)}(p^2)$ is only determined as being $= \alpha+\beta p^2 +$ $+ R_3(4)$, where α and β are finite constants, because of the arbitrariness described in Section 2.

Explicitly, one finds

$$R_3(4) = -\frac{im^2}{2}\int_0^1 \frac{dx}{\sqrt{1-x}}\int_0^1 \frac{dy}{y}\,\frac{1}{(\frac{1}{4}xy+1-y)^2}\Big[(1-A)\log(1-A)+A\Big]$$

By inverting the transformation (A. 4), this may be written

$$R_3(4) = i\int_0^1 \Big(\prod_{i=1}^{3} d\alpha_i\Big)\,\delta\Big(1-\sum_{i=1}^{3}\alpha_i\Big)\Big[C(\alpha_i)\Big]^{-3}\Big\{p^2\alpha_1\alpha_2\alpha_3\,\cdot$$

$$\cdot\left[1-\log\Big(1-\frac{\alpha_1\alpha_2\alpha_3 p^2}{C(\alpha_i)m^2}\Big)\right] + m^2 C(\alpha_i)\log\Big(1-\frac{\alpha_1\alpha_2\alpha_3 p^2}{C(\alpha_i)m^2}\Big)\Big\}\,.$$

APPENDIX B. -

We prove here the Lemma and Theorem in Sect. 3. The first part of Lemma follows the definition of superficial divergence. We remark that it also holds in a model with integer space-time dimension $n \neq 4$, then $\mu_k = 2(\frac{n}{2} l_k - n_k)$. Furthermore :

$$\left[1 - \mathcal{M}^{(\rho)}_{(r_k)}\right] \exp(iW/\hat{C}) = \int_0^1 d\xi \, \frac{(1-\xi)^{\frac{\mu_k}{2}}}{(\frac{\mu_k}{2})!} \left(\frac{\partial}{\partial \xi}\right)^{\frac{\mu_k}{2}+1} \tag{B.1}$$

$$\cdot \exp\left[iW(\rho \to \xi\rho)/\hat{C}\right]$$

It is now possible to exchange the ρ and the ξ integrations and perform the inverse of a scale transformation (see eq. (A. 2)) :

$$\mathcal{F}_{(S_k)} I_{(d)}(p_i) = \int_0^\infty \prod_{\alpha_i \notin S_k} d\alpha_i \int_0^1 d\xi \, \frac{(1-\xi)^{\frac{\mu_k}{2}}}{(\frac{\mu_k}{2})!} \left(\frac{\partial}{\partial \xi}\right)^{\frac{\mu_k}{2}+1} \cdot$$

$$\cdot \int_0^1 \prod_{\alpha_j \in S_k} d\alpha_j \, \delta(1 - \sum_{\alpha_j \in S_k} \alpha_j) \int_0^\infty d\rho \, \rho^{-\mu_k - 1} \, \hat{C}^{-\frac{d}{2}} \cdot \tag{B.2}$$

$$\cdot \exp\left[i \frac{W(\rho \to \xi\rho)}{\hat{C}} - (\dots)\right] = \int_0^\infty \prod_{i=1}^{1} d\alpha_i \cdot$$

$$\cdot \int_0^1 d\xi \, \frac{(1-\xi)^{\frac{\mu_k}{2}}}{(\frac{\mu_k}{2})!} \left(\frac{\partial}{\partial \xi}\right)^{\frac{\mu_k}{2}+1} \exp\left[i \frac{\overline{W}(\alpha_i, \xi, p_i)}{\overline{C}(\alpha_i, \xi)} - im^2 \sum_{i=1}^{1} \alpha_i\right]$$

where $\overline{C}(\alpha_i, \xi) = C(\alpha_j \to \xi\alpha_j)$ for all $\alpha_j \in S_k$

$$\overline{W}(\alpha_i, \xi, p_i) = W(p_i, \alpha_j \to \xi\alpha_j) \text{ for all } \alpha_j \in S_k .$$

By using the formula :

$$\text{(B. 3)} \qquad \int_0^1 d\xi \, \frac{(1-\xi)^n}{n\,!} \left(\frac{\partial}{\partial \xi}\right)^{n+1} f(\xi) = \int_0^1 dx \, \frac{(1-x)^{2n}}{(2n)\,!} \left(\frac{\partial}{\partial x}\right)^{2n+1} f(x^2) \, ,$$

the expression (B. 2) becomes :

$$\text{(B. 4)} \qquad \begin{aligned} \mathcal{F}_{(S_k)} I_{(d)}(p_i) &= \int_0^\infty \prod_{i=1}^{1} d\alpha_i \int_0^1 d\xi \, \frac{(1-\xi)^{\mu_k}}{\mu_k\,!} \left(\frac{\partial}{\partial \xi}\right)^{\mu_k+1} \cdot \\ &\cdot \exp\left[i \, \frac{\overline{W}(\alpha_i, \xi^2, p_i)}{\overline{C}(\alpha_i, \xi^2)} - im^2 \sum_{i=1}^{1} \alpha_i \right] \end{aligned}$$

This proves the second part of Lemma, as shown in Sect. 3 of Ref. (1).

A new operator $\mathcal{F}_{(S_j)}$ corresponding to a different subgraph $S_j \in \{S(G)\}$ can now be applied to $\mathcal{F}_{(S_k)} I_{(d)}(p_i)$. By the same procedure it is again transformed into an operator of the type :

$$\text{(B. 5)} \qquad \int_0^1 d\xi_j \, \frac{(1-\xi_j)^{\mu_j}}{\mu_j\,!} \left(\frac{\partial}{\partial \xi_j}\right)^{\mu_j+1}$$

acting on properly modified parametric functions.

The theorem to be proved is a simple consequence of the results of Ref. (1), where it is proved that the familiar subtractions in the external momenta of the subgraphs can be performed by operators of the type (B. 5), acting on properly modified parametric functions. In the same work, it is also shown that the operators $\mathcal{F}_{(S_k)}$ commute.

APPENDIX C. -

We first recall some useful formulae valid for Dirac matrices γ_i forming a Clifford algebra in an n-dimensional space-time. From the basic relation $\{\gamma_i, \gamma_j\} = 2\, g_{ij} I$, one easily obtains :

$$\sum_1 g^{11} \gamma^1 \gamma^k \gamma^1 = (2-n)\gamma^k$$

$$\sum_1 g^{11} \gamma^1 \gamma^a \gamma^b \gamma^1 = (n-4)\gamma^a \gamma^b + 4\, g^{ab}$$

$$\sum_{1} g^{11}\gamma^{1}\gamma^{a}\gamma^{b}\gamma^{c}\gamma^{1} = (4-n)\gamma^{a}\gamma^{b}\gamma^{c} - 2\gamma^{c}\gamma^{b}\gamma^{a} .$$

Let us now consider the second-order fermion self energy

$$\Sigma(p) = -\frac{4\pi\alpha}{(2\pi)^n}\, i \sum_{1} g^{11}\int dq\, \frac{\gamma^1\left[\gamma\cdot(p-q)+m\right]\gamma^1}{\left[(p-q)^2-m^2\right](q^2-\mu^2)}$$

and the third-order fermion vertex function

$$\Gamma^{\mu}(p,0) = -i\frac{4\pi\alpha}{(2\pi)^n} \sum_{1} g^{11}\int dq\, \frac{\gamma^1\left[\gamma\cdot(p-q)+m\right]\gamma^{\mu}\left[\gamma\cdot(p-q)+m\right]\gamma^1}{(q^2-\mu^2)\left[(p-q)^2-m^2\right]^2}$$

By using the familiar exponential parametrization, and then by performing the q-integration with the aid of the formulae :

$$I = \int d^n p\, e^{i(ap^2+bp\cdot k)} = i\, e^{-i\frac{\pi n}{4}} \left(\frac{\pi}{a}\right)^{\frac{n}{2}} e^{-i\frac{b^2k^2}{4a}}$$

$$I_{\mu} = \int d^n p\, p_{\mu}\, e^{i(ap^2+bp\cdot k)} = -\frac{ib}{2a}k_{\mu}\, e^{-i\frac{\pi n}{4}} \left(\frac{\pi}{a}\right)^{\frac{n}{2}} e^{-i\frac{b^2k^2}{4a}}$$

$$I_{\mu\nu} = \int d^n p\, p_{\mu}p_{\nu}\, e^{i(ap^2+bp\cdot k)} = \frac{1}{2a}\left(\frac{ib^2}{2a}k_{\mu}k_{\nu} - g_{\mu\nu}\right)e^{-i\frac{\pi n}{4}} \left(\frac{\pi}{2}\right)^{\frac{n}{2}} e^{-i\frac{b^2k^2}{4a}}$$

we get, after some standard manipulations :

$$\Sigma(p) = -i\frac{4\pi\alpha}{(2\pi)^n}\, e^{-\frac{i\pi n}{4}}\, \pi^{\frac{n}{2}}\, e^{i\frac{\pi}{2}(2-\frac{n}{2})}\, \Gamma(2-\tfrac{n}{2})\int_0^1 \frac{(2-n)\gamma\cdot p\,\alpha + nm}{B^{2-\frac{n}{2}}}\, d\alpha$$

$$\Gamma^{\mu}(p,0) = -i\frac{4\pi\alpha}{(2\pi)^n}\, e^{-\frac{i\pi n}{4}}\, \pi^{\frac{n}{2}}\Big\{ -e^{i\frac{\pi}{2}(2-\frac{n}{2})}\, \Gamma(2-\tfrac{n}{2})\,\frac{(2-n)^2}{2}\gamma^{\mu} \,.$$

$$\cdot\int_0^1 \frac{d\alpha}{B^{2-\frac{n}{2}}} + i\, e^{i\frac{\pi}{2}(3-\frac{n}{2})}\, \Gamma(3-\tfrac{n}{2})\int_0^1 \frac{d\alpha}{B^{3-\frac{n}{2}}}\, Y^{\mu} \Big\}$$

where $B = \alpha(1-\alpha)p^2 - \alpha\mu^2 - (1-\alpha)m^2$

and $Y^{\mu} = (2-n)\alpha^2\gamma\cdot p\gamma^{\mu}\gamma\cdot p + 2mn\alpha p^{\mu} + (2-n)m^2\gamma^{\mu}$.

It is now easy to verify that

$$\Gamma^{\mu}(p,0) = -\frac{\partial \Sigma(p)}{\partial p_{\mu}} \;;$$

to this aim, we need the identity

$$\Gamma(3-\frac{d}{2}) \int_0^1 \frac{(1-\alpha)(m^2-\alpha^2 p^2)}{B^{3-\frac{d}{2}}}\, d\alpha = \Gamma(2-\frac{d}{2}) \int_0^1 \frac{(\frac{d}{2}-1)(1-\alpha)-\alpha}{B^{2-\frac{d}{2}}}\, d\alpha$$

which may be established in the following way. Let us start from the elementary formula:

$$k(\lambda+k)_p \int_0^1 d\alpha\, \alpha^{k+p+1}(1-\alpha)^k + (\lambda-1)(\lambda+k)_p \int_0^1 d\alpha\, \alpha^{k+p}(1-\alpha)^{k+1} =$$

$$= (\lambda+k)(\lambda+k+1)_p \int_0^1 d\alpha\, \alpha^{k+p}(1-\alpha)^{k+1} - (\lambda+k)_p \int_0^1 d\alpha\, \alpha^{k+p+1}(1-\alpha)^k .$$

If we multiply both sides by $x^p/p\,!$, and them sum over p from zero to infinity, we have:

$$k \int_0^1 d\alpha\, \alpha^{k+1}(1-\alpha)^k(1-\alpha x)^{-\lambda-k} + (\lambda-1) \int_0^1 d\alpha\, \alpha^k(1-\alpha)^{k+1}(1-\alpha x)^{-\lambda-k} =$$

$$= (\lambda+k) \int_0^1 d\alpha\, \alpha^k(1-\alpha)^{k+1}(1-\alpha x)^{-\lambda-k-1} - \int_0^1 d\alpha\, \alpha^{k+1}(1-\alpha)^k(1-\alpha x)^{-\lambda-k}$$

Next, we multiply both sides by $(-1)^k s^k$, and them sum over s from zero to infinity. This gives:

$$-\lambda s \int_0^1 d\alpha\, \alpha^2(1-\alpha) X^{-\lambda-1} + (\lambda-1) \int_0^1 d\alpha (1-\alpha) X^{-\lambda} =$$

$$= \lambda \int_0^1 d\alpha\, (1-\alpha) X^{-\lambda-1} - \int_0^1 d\alpha\, \alpha\, X^{-\lambda} \quad ,$$

where $X \equiv 1 - \alpha x + s\alpha(1-\alpha)$.

The required identity follows now by taking $x = 1 - \frac{\mu^2}{m^2}$, $s = \frac{p^2}{m^2}$ and $\lambda = 2 - \frac{d}{2}$.

FOOTNOTES. -

/1/ - Of course, a similar procedure applies also to renormalizable field theoretic models with a space-time dimension $n \neq 4$.

/2/ - These remarks may be useful in discussing the number of arbitrary parameters in a lagrangian model. In fact we obtain the conventional results both for renormalizable and for unrenormalizable models. However if a reasonable condition were introduced to select a particular interpolating function, then any theory, whether renormalizable or not, would contain just a single arbitrary parameter(19) which may be identified as a subtraction point.

/3/ - This remark may be relevant in understanding why reggeization has been proved in superconvergent theories. In fact, asymptotic contributions associated to shortest paths which may exponentiate are obscured, in divergent graphs, by dominating contributions depending on the dimension of space-time.

/4/ - We thank Profs. F. Guerra and H. Mitter for useful discussions about this point and Prof. N. Nakanishi for a communication concerning the analytic renormalization.

/5/ - More precisely, $I_{(d)} \left\{ \sum_P \prod_{j=2}^{l} \Gamma(j - \frac{d}{2} N_j(P)) \right\}^{-1}$ is an entire function of d. Here, $\sum_P$ means sum over all permutations of the labels of the l lines of the graph and, for a given permutation P, $N_j(P)$ is the number of loops of the graph consisting of lines 1 through j with their vertices.

/6/ - We thank Prof. Santhanam for a communication concerning CPT in odd dimensional spaces.

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