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SOME COMMENTS ON THE POSSIBILITY OF DERIVING THE PION CHARGE RADIUS FROM LOW ENERGY PION ELECTROPRODUCTION AT $K^2 = 0$

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SUMMARY. We discuss a sum rule which relates the pion charge radius to low energy pion electroproduction parameters evaluated at zero virtual photon mass. The conclusion is that extremely accurate data would be necessary to allow an unambiguous prediction to be drawn.

Experiments on electron-pion scattering are at present being performed at Serpukhov, and it is hoped that their analysis will provide, with reasonable accuracy, the value of the pion charge radius $\langle r_\pi^2 \rangle^{1/2} \equiv \sqrt{6 \dot{F}_\pi(0)}$. In view of this possibility, we think it might be of interest to reexamine a sum rule which was originally proposed in a rather particular version by Nicolò and Rossi (1), and which relates the value of the pion charge radius to low-energy pion electroproduction parameters evaluated at K^2 (the virtual photon mass) equal to zero. This sum rule is only based on the electromagnetic current conservation and on the axioms of dispersion theory, and it might therefore in principle represent a more modest but useful alternative to the general Chew-Low extrapolation procedure to extract the pion form factor at any K^2 from high energies pion electroproduction data (2). Actually, it is known that for some unfortunate kinematical accident this procedure does not seem to give too much information in this case, so that to derive the pion form factor from high energies electroproduction data (3),(4),(5) one is forced to resort to various models (2),(6),(7), all of which introduce however some assumptions which might prove rather crucial [8].

To derive the sum rule one can start from the pion electro production process which is treated as usually in the one-photon exchange approximation represented in fig. 1, where also the necessary kinematics is sketched. The invariant amplitude is defined as

$$T_\mu^\alpha \equiv i < N'(\vec{p}_2), \pi^\alpha(\vec{q}) | V_\mu(0) | N(\vec{p}_1) > \tag{1}$$

where V_μ is the electromagnetic current, and α is the pion isotopic spin index. The isospin decomposition and our normalization convention for the amplitudes are the same as in ref. (8). The spin decomposition we choose is the following:

$$T_\mu = \bar{u}(\vec{p}_2)\gamma_5 \{ P_\mu T_1 + \hat{k} P_\mu T_2 + q_\mu T_3 + \hat{k} q_\mu T_4 + \\ + k_\mu T_5 + \hat{k} k_\mu T_6 + \gamma_\mu T_7 + \tfrac{1}{2}[\hat{k}, \gamma_\mu] T_8 \} u(\vec{p}_1). \tag{2}$$

$$\hat{k} \equiv k_\mu \gamma^\mu.$$

The amplitudes T_i are actually not all independent, due to current conservation which imposes the two constraints:

$$\nu T_1 + k.q\, T_3 + k^2 T_5 = 0. \\ \nu T_2 + k.q\, T_4 + k^2 T_6 + T_7 = 0. \tag{3}$$

$$\nu = \tfrac{1}{4}(s-u).$$

One can easily see that the elementary pion exchange in the t-channel contributes to the amplitudes $T_3^{(-)}$ and $T_5^{(-)}$, but not to the combination

$$\tau^{(-)} = \frac{1}{2} T_3^{(-)} + T_5^{(-)}. \tag{4}$$

which is therefore a regular function of t at $t = m_\pi^2 \equiv \mu^2$. The same is true, as one would expect, if one considers the pion as a Regge trajectory. In this case $T_1^{(-)}$ also receives a contribution which remains finite at $t = \mu^2$ [9], and this allows Reggeized pion exchange to be gauge invariant by itself, in spite of the fact that elementary pion exchange is not.

Following Nicolò and Rossi[1], we now rewrite the first of the constraints, eq. (3), for the isotopic (-) configuration in the form

$$\nu T_1^{(-)} + \frac{1}{2}(\mu^2 - t)\, T_3^{(-)} + k^2 \tau^{(-)} = 0. \tag{5}$$

From the definition of the pion form factor $F_\pi(k^2)$, we have in the limit $t \to \mu^2$,

$$F_\pi(k^2) = \frac{1}{e\, g_{\pi N}} \left[\nu T_1^{(-)} + k^2 \tau^{(-)} \right] (\nu; t = \mu^2; k^2). \tag{6}$$

where $g_{\pi N}$ is the pion-nucleon coupling constant, $g_{\pi N}^2/4\pi \simeq 14.7$ and $e^2/4\pi = 1/137$.

The asymptotic behaviour of the amplitudes T_i is given by Regge theory,[9] and we find that $T_{1,2,4,6,8} \sim \nu^{\alpha-1}$ as $\nu \to \infty$ whilst $T_{3,5,7} \sim \nu^{\alpha}$. By writing an unsubtracted dispersion relation for $T_1^{(-)}$, and by noticing that the current conservation constraints are automatically satisfied by the imaginary part of the amplitudes, we are led to the final expression:

$$\frac{1}{k^2}\left[F_\pi(k^2) - F_1^V(k^2)\right] = \frac{k^2}{16\nu^2 - k^4} F_1^V(k^2) + $$

$$+ \frac{1}{e\, g_{\pi N}} \mathrm{Re}\, \tau^{(-)}(\nu; t = \mu^2; k^2) - \frac{2\nu^2}{\pi e\, g_{\pi N}} \int_0^\infty \frac{\mathrm{Im}\, \tau^{(-)}(\nu'; t = \mu^2; k^2)\, d\nu'}{\nu'(\nu'^2 - \nu^2)}. \tag{7}$$

This relation between $F_\pi(k^2)$ and $F_1^V(k^2)$ must be satisfied if the current is conserved and the axioms of dispersion theory are correct.

Since the right-hand side of eq. (7) is independent of ν it can be evaluated at any convenient ν value. In particular, if one chooses $\nu = \infty$ and assumes that $\tilde{\tau}^{(-)}$ vanishes at the point ($\nu = \infty$, $t = \mu^2$, k^2), then eq. (7) can be written, in the limit $k^2 \to 0$, in the particularly simple form*:

$$\dot{F}_\pi(0) - \dot{F}_1^V(0) = \frac{2}{\pi e g_{\pi N}} \int_0^\infty \frac{d\nu'}{\nu'} \operatorname{Im} \tilde{\tau}^{(-)}(\nu', t=\mu^2, k^2=0) \tag{8}$$

$$\dot{F}(0) \equiv \frac{d}{dk^2} F(k^2)\Big|_{k^2=0}.$$

which is the Nicolò and Rossi sum rule [1].

From a formal point of view, eq. (8) is very similar to another famous current algebra sum rule [10] relating the derivate at $K^2 = 0$ of $F_1^V(K^2)$ to that of the axial vector nucleon form factor $G_A(K^2)$, which reads:

$$\dot{g}_A(0) - \dot{F}_1^V(0) = \frac{2M_N}{e\pi g_{\pi N}} \int_0^\infty \frac{d\nu'}{\nu'} \operatorname{Im} T_6^{(-)}(\nu', t = k^2 = 0). \tag{9}$$

$$g_A(k^2) \equiv G_A(k^2)/G_A(0).$$

However, from a practical point of view, whereas eq. (9) seems to be rather satisfactory in the sense that the integral on the r.h.s. appears quickly convergent, so that only a few low energy multipoles are sufficent to saturate it reasonably [11], the same is probably not true of eq. (8). Nicolò and Rossi actually evaluated the dispersive integral in eq. (8) by approximating it with the N^*_{33} resonance contribution treated in the isobaric model approximation and found the result

$$\dot{F}_\pi(0) - \dot{F}_1^V(0) \simeq -0{,}011 \text{ Fermi}^2. \tag{10}$$

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* of course the same result follows substituting for $\tilde{\tau}^{(-)}$ in (7) an unsubtracted dispersion relation.

which is about one tenth of the expected value of $\overset{\cdot}{\overline{f_1}}{}^V(0)$ (12).
Therefore they concluded that their result was in agreement with the idea that the pion form factor was more or less equal to the Dirac isovector electric form factor. But we do not believe that their evaluation provides very strong evidence for $\overset{\cdot}{\overline{f_\pi}}(0) \simeq \overset{\cdot}{\overline{f_1}}{}^V(0)$. Even if $\tau^{(-)}(\nu, t=\mu^2, k^2)$ really does vanish as $\nu \to \infty$, which, on the basis of Regge theory, does not seem likely, the convergence of the right-hand side of eq. (8) would be rather slow, so that a saturation with low-lying resonances is a dubious procedure . We have checked the result eq. (10) by assuming that $\tau^{(-)}(\nu = \infty,\ t = \mu^2,\ k^2 = 0)$ does in fact vanish, and by using the experimental multipoles for the integral . Briefly, the transverse multipoles at $k^2 = 0$ were taken from the recent phenomenological analyses of pion photoproduction (13,14), and the resonant longitudinal multipole $L_1^{(-)}$ was assumed to be a constant proportion of $E_1^{(-)}$, at least in the first resonance region.

As we expected, the cut-off of around $E_\gamma^L \simeq 500$ MeV, imposed by our first resonance region parametrization of the multipoles, was not high enough, and the integral clearly receives important contributions from the second and higher resonances. Furthermore, we found that the integral received an important contribution from $L_1^{(-)}$. This is not, in itself, of great importance, but taken with the slow convergence of the integral, it does mean that any attempt to saturate the sum rule (9) with a $\Delta(1236)$ isobar will not be reliable.

It is interesting to notice that the conclusion $\overline{f_\pi} \simeq \overline{f_1}{}^V$ has also been drawn by Brown [3] et al., who have interpreted their high energy electroproduction data in terms of a model proposed by Berends [6]. This model makes essentially use of the hypotheses thet $\tau^{(-)}$ vanishes at infinity, and that the dispersive integrals are saturated by the N_{33}^{*} resonance. Therefore, it is not surprising that its final answer is in line with the Nicolò and Rossi result. This also shows that the sum rule eq. (7) might be really a simple alternative to, or at least a check of the more general procedures commonly used to determine the pion form factor from high energy pion electroproduction. However, we think that a safer way to exploit it is to give up the assumption of the asymptotical vanishing of $\tau^{(-)}(\nu; t=\mu^2, k^2)$ and choose a finite value of ν to work. This would correspond in the scheme of the models [(2), (7)] to the introduction of an unknown subtraction constant. As a simple example, we choose to evaluate eq. (7) again at $k^2 = 0$ at the ν value which corresponds to the physical threshold, i.e. $\nu \equiv \nu_0 = \mu(2M+\mu)^2/4(M+\mu)$. In this way, the dispersive integral will become very quickly convergent and dominated by the lowest multipoles. The reason why we are limited to the case $k^2 = 0$ is that the only experimental data available for the real part of the multipoles are photoproduction data. Moreover, since the minimum physical value of t raises with k^2, the distance from the unphysical point $t = \mu^2$ to the physical t-range will be minimum at $k^2 = 0$.

We consider first the integral over the absorptive part. The factor $\sim 1/\nu'^3$ ensures rapid convergence and since the d-waves are purely real for some distance above threshold, we ignore this contribution to the integral. Notice that $\mathrm{Im}\, \tau^{(-)}$ is to be calculated at $t = \mu^2$, outside the physical region, but the additional factors of $|\vec{q}|$ in the imaginary part ensure the rapid convergence of the multipole expansion. To evaluate the integral numerically, we use the phenomenological transverse s-and p-waves [(13), (14)] and the following prescription for the longitudinal multipoles:

$$\begin{aligned} \mathrm{Im}\, L_{0+}^{(-)} &\simeq 0{,}6 \;\; \mathrm{Im}\, E_{0+}^{(-)} . \\ \mathrm{Im}\, L_{1+}^{(-)} &\simeq 0{,}6 \;\; \mathrm{Im}\, E_{1+}^{(-)} . \\ \mathrm{Im}\, L_{1-}^{(-)} &\simeq 0 . \end{aligned} \tag{11}$$

The reason why we use this prescription is that it is suggested both by Von Gehlen's dispersive evaluation [15] and by current algebra calculations [16]. In this way, we obtain:

$$\frac{2\nu_0^2}{\pi e g_{\pi N}} \int_0^\infty \frac{\mathrm{Im}\, \tau^{(-)}(\nu'; t=\mu^2; k^2=0)\, d\nu'}{\nu'(\nu'^2-\nu_0^2)} = 1{,}8 . 10^{-4} \mu^{-2} . \tag{12}$$

The integral is completely insensitive to the cut-off for $E_\gamma > 400$ MeV, and the result is very small, being approximately 0.5 % of $\dot{F}_1^V(0)$. As was observed in the evaluation of the Nicolò and Rossi sum rule, there is a tendency fôr the dominant $M_1^{(-)}$ and $L_1^{(-)}$ contributions to cancel, so the result is probably sensitive to our assumption eq. (11). However, since eq. (12) is much smaller than $\dot{F}_1^V(0)$, even a factor of two uncertainty in the integral is unimportant, although we would consider our result to be more reliable than this. We conclude, therefore, that <u>if there is any</u> appreciable difference between $\dot{F}_\pi(0)$ and $\dot{F}_1^V(0)$, it must be produced by Re $\tau^{(-)}(\nu_0, t=\mu^2, k^2=0)$.

The evaluation of the real part of $\tau^{(-)}$ presents greater difficulties. Equation (13) involves Re $\tau^{(-)}$ at the unphysical point $t=\mu^2$, and since $\cos\theta(t=\mu^2) = q_0/|\vec{q}|$, multipoles for all ℓ values enter with finite weight, and we cannot evaluate the sum. Since $\tau^{(-)}$ is regular at this point, however, we know that this sum convergences ultimately to a finite limit, and the best that we can do is to assume that $\tau^{(-)}$ does not vary appreciably if we continue away from $t = \mu^2$ to the closest physical value $t = t_o$ at a fixed value of ν. Since we have chosen ν at its photoproduction threshold value, we can evaluate Re $\tau^{(-)}$ at the physical pion photoproduction threshold where $t = t_o = -M\mu^2/(M+\mu)$ and assume that since $\tau^{(-)}$ does not have any poles in t in this neighbourhood, its value is unchanged by this continuation.

The multipole expansion for Re $\tau^{(-)}(\nu_0, t_0, k^2=0)$ is :

$$\begin{aligned} Re\,\tau^{(-)}(\nu_0, t, k^2=0) &= e\mu\sqrt{\frac{M}{M+\mu}}\left\{\left[-\frac{1}{\mu}\right]E_{0+}^{(-)}\right. \\ &+\left[\frac{M\mu}{M+\mu}\right]y^{(-)} - \frac{3}{4}\left[\frac{2M\mu+3\mu^2}{M+\mu}\right]z^{(-)} + \\ &+\left[\frac{3M\mu}{\mu^2}\cdot\frac{2M\mu+3\mu^2}{2M+\mu}\right]\chi^{(-)} + \frac{L_{0+}^{(-)}}{k_0} + \\ &\left.+\left[\frac{M\mu(2M+\mu)}{M+\mu}\right]\frac{u^{(-)}}{k_0}\right\}_{|\vec{q}|=0} \end{aligned} \tag{13}$$

where :

$$y \equiv \frac{2M_{1+}+M_{1-}}{|\vec{k}||\vec{q}|}\,;\quad z \equiv \frac{E_{1+}-M_{1+}}{|\vec{k}||\vec{q}|}\,;\quad u = \frac{L_{1-}-2L_{1+}}{|\vec{k}||\vec{q}|}\,; \tag{14}$$

$$\chi = \frac{M_{2+}-E_{2+}-M_{2-}-E_{2-}}{|\vec{q}|^2}.$$

Among these six parameters, only $E_{0+}^{(-)}$, $y^{(-)}$ and $z^{(-)}$ can be considered as known at the moment (13),(14). To try anyway an evaluation, we have taken $\chi^{(-)}$ in the Born approximation (which is the usual assumption one makes to derive E_{0^+}, y and z from the experimental data) and have used for $u^{(-)}$ and $L_{0+}^{(-)}$ the dispersive evaluation of Von Gehlen (15) which is in line with the current algebra results obtained by Furlan et al.[16]. Neglecting the indetermination on each parameter, we would obtain:

$$\dot{f}_{\pi}(0)-\dot{f}_{1}^{V}(0) = 0{,}0225\,\mu^{-2}. \qquad (15)$$

which, taking the Wilson[12] world average for the charge radius of the proton as

$$\langle r_p^2\rangle^{1/2} = 0{,}814 \pm 0{,}015 \text{ Fermi}. \qquad (16)$$

would correspond to

$$\dot{f}_{\pi}(0) = 0{,}0573\,\mu^{-2}. \qquad (17)$$

to be compared with

$$\dot{G}_E^{P}(0) = 0{,}0553 \pm 0{,}002\,\mu^{-2}. \qquad (18)$$

However, the result eq. (15) is rather unstable with respect to small variations of the six parameters: actually, the small final number comes from the difference between two large and opposite members. Therefore, a measurement of these quantities of an extreme accuracy would be necessary to make the result reasonably safe. This is an unfortunate situation which resembles very strongly what happens at high energies. Even in that case the Chew-Low extrapolation would require data of an extreme accuracy, well beyond those available at the moment[2].

Since we expect some continuity in the properties of the amplitude $\mathcal{C}^{(-)}$ when we move away from threshold to low energies, we think that this instability of the l.h.s. of eq. (7) at $k^2 = 0$ will persist when ν varies in this energy region. Therefore our (unfortunately negative) conclusion is that it appears very unlikely, although in principle possible, to derive any information about the pion charge radius from low energy electroproduction at $k^2 = 0$ through the sum rule eq. (7), based on the most general axioms of dispersion theory.* Since a similar conclusion holds about the possibility of determining in the dispersion theory framework the pion form factor from high energy electroproduction[8], we think that one should seriously start thinking about the possibility that one-dimensional dispersion relations alone might prove unable to solve the problem of the shape of the pion form factor, and that either some extra information would be needed, or perhaps some different technique should be employed. Work along these directions is now in progress.

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*The possibility of using eq(7) at $K^2 \neq 0$ for small ν values appears, from the previous discussion, rather unlikely.

REFERENCES

1) F. Nicolò and G.C. Rossi, Nuovo Cimento 56A, (1968) 207.

2) See the discussion given by :R.C.E. Devenish and D.H. Lyth, Phys. Rev. D5, 47 (1972).

3) C.N. Brown et al. Phys. Rev. Letters 26, (1971) 991.

4) P.S. Kummer et al., Nuovo Cimento Letters 1, (1971) 1026.

5) C. Driver et al.,Phys.Letters 35B, (1971) 77, 81; Nucl. Phys. B30, (1971) 245.

6) F. Berends, Phys. Rev. D1, (1970) 2590.

7) R. Manweiler and W. Schmidt, Phys. Rev. D3, (1971) 2752.

8) B.H. Kellett and C. Verzegnassi : "Can the pion form factor be determined from pion electroproduction?" to be published on "Il Nuovo Cimento".

9) J.S. Ball and M. Jacob, Nuovo Cimento 54A, (1968) 620.

10) G. Furlan, R. Jengo, E. Remiddi, Nuovo Cimento 44A, 427 (1966).

11) S.L.Adler, F.J. Gilman, Phys. Rev. 152, 1460 (1966).

12) R. Wilson, Proc. Vth Int. Symp. on Electron Photon Interactions at High Energies, Cornell, (1971).

13) P. Noelle, W.Pfeil and D. Schwela, Nucl. Phys. B26, (1971) 461; P. Noelle and W. Pfeil, Nucl. Phys. B31, (1971) 1.

14) P. Noelle, Bonn preprint, Bonn PI 2-92 (1971).

15) G.Von Gehlen, Bonn preprints, Bonn PI 2-80 (1970) and Bonn PI 2-94 (1971).

16) G. Furlan, N. Paver and C. Verzegnassi, Nuovo Cimento, 70A, (1970) 247.