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G. Pisent and M. Pusterla: PHYSICAL FOUNDATIONS OF NON RELATIVISTIC QUARK MODELS: BOSONIC SPECTRUM AND REGGE TRAJECTORIES. -

SUMMARY. -

The bosonic spectrum and the Regge trajectories related with two specific non relativistic models (volume and surface models) for the quark-antiquark system are computed in detail.

A connection between the model that gives the best predictions and a more fundamental field theory for the quarks is also considered and discussed.

Within the framework of the SU(3) symmetry the scalar or pseudoscalar field theories are favoured by our analysis in comparison with the vector one.

1. - INTRODUCTION. -

The idea of a dynamical picture of the elementary particles based on the hypothesis of three subconstituents (quarks), considered as the true elementary objects, continues being, after some time since its first appearance, a central point in particle physics[1].

Many authors[2] proposed, with some success, very simple models in which the relative motion of these constituents is non-relativistic, both for describing the bosons (one quark and one antiquark) and

the fermions (three quarks). It may, however, appear puzzling to accept such a non-relativistic description without considering it as an approximate derivation of a more fundamental approach based on field theory. It makes, therefore, sense to ask under which conditions the non-relativistic picture is justified as a meaningful approximation of a more correct dynamics and which are the implications, in a more basic theory, of the features that might arise from the phenomenological models.

A program that tends to this connection, involving quarks, is nowadays too ambitious, both for the weak experimental support that the existence of quarks has, and for the lack of a correct field theory dealing with fields considered elementary. One is then compelled to restrict the problem within more modest limits. In this paper we concentrate our interests on the bosonic particles and their Regge trajectories as physical predictions of the two-body quark-antiquark system, and compare our phenomenological parameters with simple field theories that can be formulated in terms of local fields.

It appears, thus, unnecessary to take into account delicate points such as parastatistics for the quark local fields, that seems to be unavoidable for the baryonic states of three quarks, and one can assume that the quark fields are three Dyrac fields $q_\alpha(x)\left[\alpha = 1,2,3\right]$ (SU(3)-fundamental representation).

In order to deal with "regular" theories we must introduce a bosonic field, the so-called gluon, besides the three spin 1/2 quarks and arrive at the three possible different interaction densities:

$$(1.1) \qquad L_s = g_s \bar{q}^\alpha q^\alpha \psi; \qquad L_p = g_p \bar{q}^\alpha \gamma^5 q^\alpha \phi; \qquad L_v = g_v \bar{q}^\alpha \gamma^\mu q^\alpha B_\mu,$$

where the index α is supposed to be summed from 1 to 3 and B_μ, ϕ, ψ are the vector, pseudoscalar and scalar gluonic fields respectively. From these choices, at a fundamental level, different models follow. First of all we point out that the quark mass computed from a configurational quark-antiquark system may coincide or differ from the free quark mass according to the type of interaction considered.

It is straightforward to deduce the Heisemberg coupled equations for the local fields q^α, ψ, ϕ, B_μ, which we name with general notation a (x)

$$(1.2) \qquad \frac{\partial L}{\partial a_\nu} - \partial_\mu \frac{\partial L}{\partial a_{\nu\mu}} = 0,$$

where L is the total Lagrangian density and $a_{\nu\mu}(x)$ is $\partial_\mu a_\nu(x)$.

A reasonable approximation, that seems to be confirmed by the approximate validity of SU(6) and the absence of "exotic" states consists in neglecting the virtual quark-antiquark pairs and treating the gluon field as an external field.

It can be easily shown that under such conditions the wave configurational equation for the quark becomes formally similar to the free particle equation for the scalar or the pseudoscalar case with the introduction of an effective mass $m_q^x = m - U(x)$ where U(x) is the potential of the equation related with the external field. On the contrary for the vector coupling the classical field limit does not influence the mass term of the quark which keeps being m ($m_q^x = m$) but changes the kinetical momenta as in electrodynamics.

Such a behaviour seems to permit the statement that if we accept the hypothesis of the existence of free quarks with large masses (order 10 GeV), we may find a much smaller quark mass in their interactions, if the basic field theoretical coupling is scalar (or pseudoscalar). This, under the assumption that the field theoretical models introduced above, are realistic.

After reducing the field theoretical problem to a potential problem, we notice that the validity of a "static" approximation such as SU(6), favuours a non-relativistic description of the motion of the quark--antiquark system in its center of mass and a potential U(r) regular at $r = 0$ with finite range R. In fact the regular potential allows high binding energies, as we must expect, and at the same time, non-relativistic motion.

The non-relativistic condition is $R^{-1} \ll m$ where m is the effective mass of the quark during its interaction with the antiquark.

The square well potential raises the relevant question on the mechanism that can produce a function of this sort starting from a Yukawa field interaction. This possibility is denied in perturbation theory and can be explained in strong coupling physics because higher order terms may be as important as the first order one.

The presence and the role of non-central forces in the static potential limit of the coupled equation is also a good question to examine. A vector field theory would favour, as it does in Electrodynamics, a spin-orbit term of the Thomas type[3], thus an experimental answer on the shape of the spin-orbit interaction based on the best fit parameters, is quite relevant in order to draw remarks on the fundamental interaction.

As far as the internal symmetries present in our models are concerned, the vector coupling lagrangian considered above, is SU(3)xSU(3)

chiral invariant while the other two violate the chialrity, being of course always SU(3) invariant.

Before closing these notes of introduction we like to mention that different couplings considered may provide different electromagnetic properties for the quarks and consequently for the nucleon and the bosons.

A "large" effective mass, for instance, requires an enormous anomalous magnetic moment for the quark (in order to produce the physical magnetic moment of the nucleon) whereas the "small" mass may not require it[4].

A final remark we like to point out refers to the gluon particle: it may be considered as elementary as the quarks or a bound state. In both cases it is interesting to see how it is located in the SU(3) bosonic representations, if it is a singlet or not, and whether it coincides with an experimental bosonic resonance already known or not.

2. - THE NON RELATIVISTIC QUARK-ANTIQUARK MODEL. -

We assume here that the motion of the quark-antiquark system, in their center of mass, is non-relativistic and that the forces can be represented by a static potential, regular at the origin and characterized by a finite range.

The consistency of our point of view is going to be checked "a posteriori" once we have fitted the phenomenological parameters of the potential with the experimental data.

The input data are essentially the bosonic mass spectrum. Our potential is assumed to be SU(3) invariant. The SU(3) breaking effects are therefore introduced as a mass difference between the n, p quarks and the λ quark. At this point we deal just with one potential for all our octets and are now going to consider the different possible choices for it(x).

We describe with a central square well the main dynamical effects of the attractive quark-antiquark forces: we think in this way, we are taking into account the peculiarities that the quark-antiquark interaction must exhibit, namely the relevant binding energy, the non-relativistic motion and the finite range.

(x) - Further physical assumptions are necessary in order to extend the same phenomenological potential to the SU(3) singlets and we are not discussing this point in the paper.

In addition to the central square well interaction, we have considered three types of spin-dependent potentials, described by spin-orbit ($\bar{\ell}\,\bar{s}$), spin-spin ($\bar{s}_1\,\bar{s}_2$) and tensor (Q) operators respectively.

As far as the radial dependence of these spin-dependent terms is concerned, owing to the phenomenological character of the approach we have taken into consideration two limit cases, namely either the volume square well potential of the same form of the central one, or the surface potential proportional to the radial derivative of the square well.

Both choices lead to solution of the Schrödinger equation in compact form only when the orbital angular momentum is conserved. When tensor forces are effective, only the delta radial form allows exact solution of the Schrödinger equation.

This is the reason why we have tested separately both radial forms (referred so shortly as "surface" and "volume" model in the following), only for the spin-spin and spin-orbit potentials, while the tensor interaction is assumed to be of the "surface" type (and the central interaction of the volume type) in any case. The restriction is justified by the fact that we expect the tensor strength to be much smaller than the spin-orbit and spin-spin strength (and of course of the central one), because its typical role is to mix the orbital angular momentum states in the coupled triplet states, and it looks very unlikely that this effect is relevant in the deepest bound states, which are the most interesting ones. Our conjecture is confirmed by the experimental fit (see section 3).

All the details on the model employed, and the solution of the relative Schrödinger equation are given in the Appendix. As a final step of these calculations one finds two kinds of bound state equations (eqs. A20, A.21) for the coupled and uncoupled case respectively.

In fact one has to distinguish between two possibilities, namely:

a) the triplet states (s=1) with orbital angular momenta $\ell = j+1$ and $\ell = j-1$, coupled together by the tensor interaction (parity $(-)^j$).

b) The singlet and triplet pure states with $j=\ell$ and parity $(-)^{j+1}$.

The parity of the mesons in our quark model is in fact $(-1)^{\ell+1}$, ℓ being the orbital angular momentum whereas the charge conjugation number C for the neutral states is $C=(-1)^{\ell+s}$.

The bound state equations (see eqs. A20, A21, A23 of the Appendix) are

$$(2.1)\quad \left[\omega^j_{j-1,1}\, J_{j-3/2}(\omega^j_{j-1,1})/J_{j-1/2}(\omega^j_{j-1,1}) + \Lambda K_{j-3/2}(\Lambda)/K_{j-1/2}(\Lambda) + \right.$$

$$+ V^{j}_{j-1,1;j-1,1}\Big]\Big[\omega^{j}_{j+1,1} J_{j+1/2}(\omega^{j}_{j+1,1})/J_{j+3/2}(\omega^{j}_{j+1,1})$$

(2.1)

$$+ \Lambda K_{j+1/2}(\Lambda)/K_{j+3/2}(\Lambda)+V^{j}_{j+1,1;j+1,1}\Big] - \Big[V^{j}_{j-1,1;j+1,1}\Big]^{2} = 0$$

for the case (a), and

$$\omega^{j}_{js} J_{j-1/2}(\omega^{j}_{js})/J_{j+1/2}(\omega^{j}_{js})+\Lambda K_{j-1/2}(\Lambda)/K_{j+1/2}(\Lambda)+V^{j}_{js,js} = 0 \tag{2.2}$$

for the case (b), with s=0 and s=1 for the singlet and triplet pure state respectively.

The arguments of the J and K functions are the internal (equation A14b) and external (A19) momentum times the interaction radius, namely:

$$\omega^{j}_{\ell s} = R\sqrt{2m(U^{j}_{\ell s,\ell s}-\varepsilon)}\ , \tag{2.3}$$

$$\Lambda = R\sqrt{2m\varepsilon}\ . \tag{2.4}$$

The reduced mass of the system m and the binding energy write as follows

$$m = \frac{m_1 m_2}{m_1+m_2}\ ; \qquad \varepsilon = m_1+m_2-M\ , \tag{2.5}$$

where the choices of the quark masses m_1 and m_2 in the analyzed configurations are explained in Table I.

Finally the connection between the matrix elements $U^{j}_{\ell's',\ell s}$, $V^{j}_{\ell's',\ell s}$ and the parameters U_c, U_{ss}, $U_{\ell s}$, V_{ss} $V_{\ell s}$, V_t (see Table II) are given by equations (A6), here repeated for easy reference

$$U^{j}_{\ell's',\ell s} = \langle \ell's'j\,|\,U_c+U_{ss}\,\bar{s}_1\cdot\bar{s}_2+U_{\ell s}\,\bar{\ell}\cdot\bar{s}\,|\,\ell sj\rangle \tag{2.6}$$

$$V^{j}_{\ell's',\ell s} = \langle \ell's'j|\,V_{ss}\,\bar{s}_1\cdot\bar{s}_2+V_{\ell s}\,\bar{\ell}\cdot\bar{s}+V_t Q\,|\,\ell sj\rangle \tag{2.7}$$

TABLE I

Table of the experimental boson masses M (in GeV); m_p, m_n and m_λ are the quark masses.

State $\|\ell s j\rangle$	$m_1=m_2=m_p=m_n$		$m_1=m_p=m_n$, $m_2=m_\lambda$	
	particle	M (GeV)	particle	M (GeV)
$\|000\rangle$	π	0.140	K	0.497
$\|110\rangle$	π_n	1.016	---	---
$\|111\rangle$	A_1	1.070	K_A	1.242
$\|101\rangle$	B	1.233	K'_A	1.350
$\|011\rangle\|211\rangle$	ϱ	0.774	K^*	0.890
$\|112\rangle\|312\rangle$	A_2	1.310	K_N	1.408

Note that volume and surface interactions are always distinguished by employment of symbols U and V respectively (see eq. A4).

3. - FIT OF THE BOSON SPECTRUM. -

We employ now the model outlined in the preceding section, in both the surface and the volume version, to fit the boson masses shown in Table I. The spectrum is not complete, and some resonances like η, ϕ, ω, are excluded since their mass values require SU(3) breaking in the interaction (see section 5).

Our problem is then to fit the eleven resonances given in Table I, by the seven parameters $m_p=m_n$, m_λ, R, U_c, V_t and U_{ss}, $U_{\ell s}$ or V_{ss} $V_{\ell s}$ in the "volume" and "surface" model respectively.

As a starting point, an orientative analysis on the order of magnitude of the parameters able to fit the data, and on the existence of possible ambiguities, may be carried out through the following steps:

- choose the third column of Table I, which depend on one quark mass only;
- assume the principal role be played by the central interaction U_c, and neglect all spin dependent effects as a first approximation;
- choose the lower angular momentum configuration (S and P respectively),

for the two coupled states ϱ and A_2, since in the above approximation the coupling disappears.

In this simplified situation the problem reduces to the fit of two degenerate states $M_o = 0.457$ GeV (average value between the masses of π and ϱ ; $\ell = 0$) and $M_1 = 1.157$ GeV (average among π_n, A_1, B, A_2; $\ell = 1$) by means of the three parameters R, U_c and m_p. In this particular case the eqs. (2.2, 2.1) become

$$\omega_o J_{-1/2}(\omega_o)/J_{1/2}(\omega_o) + \Lambda K_{-1/2}(\Lambda)/K_{1/2}(\Lambda) = 0, \tag{3.1}$$

$$\omega_1 J_{1/2}(\omega_1)/J_{3/2}(\omega_1) + \Lambda K_{1/2}(\Lambda)/K_{5/2}(\Lambda) = 0, \tag{3.2}$$

with

$$\omega_\ell = R\sqrt{m_p(U_c - 2m_p + M_\ell)}. \tag{3.3}$$

The initial overdetermined problem becomes so undetermined, and we find a continuous set of parameters satisfying eqs. (3.1, 3.2), as shown in Fig. 1.

Now we introduce gradually the second quark mass m_λ (initial value: $m_\lambda = m_p$), able to remove the degeneration between the 3rd and 5th columns of Table I, and the spin dependent parameters (initial values: $U_{ss} = U_{\ell s} = V_{ss} = V_{\ell s} = V_t = 0$), which remove completely the degeneration among the rows of Table I.

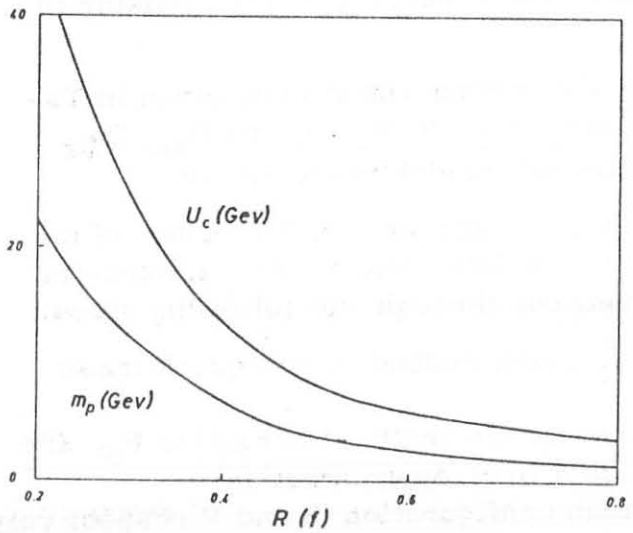

FIG. 1 - The figure shows the ambiguity connected with the preliminary analysis described in sec. 3. All sets U_c m_p R extracted from the figure fit equally well the degenerate experimental spectrum M_o = =0.452 GeV ($\ell = 0$) and M_1 =1.157 GeV ($\ell = 1$).

The fit is performed now by the electronic computer, through minimization of the mean square error Ω, by means of the complete equations (2.1), (2.2).

Since the equations are not linear, the first approximation values of the parameters are obviously important in determining the final results. This is the reason why the calculation has been repeated several times, starting from different sets of R, U_c, m_p within the range shown in Fig. 1.

The best agreement with experiments (see Fig. 2) has been reached by means of the set of parameters (1a) and (1b) of Table II, in the framework of the "volume" and "surface" model respectively.

In order to give an idea of how much the ambiguity studied in Fig. 1 may be considered removed by the complete analysis, we have reported in Fig. 3 the spectra obtained by starting from the region of very low R values (sets 2a, 2b of Table II). The detailed values of the masses calculated by means of the parameters of Table I are given in Table III.

Finally note that in Figs. 2, 3 all resonances theoretically found in the analyzed energy have been reported, i.e. also "spurious" resonances not experimentally observed.

Let us now draw our conclusions:

- the ambiguities found by "central forces" analysis (Fig. 1) cannot be considered completely removed by the final best fit, since the mean square errors relative to solutions (1) are only slightly lower than those of solutions (2):
- the absence of spurious states seem to be a better argument in favour of solutions (1) with respect to solutions (2), than the mean square error magnitude;
- on the basis of the same arguments, one may conclude that solutions (a) (volume model) are clearly better than solutions of type b (surface model);
- solution (1a) is the best in absolute sense;
- the tensor potential is always very low, and is practically zero in the case (1a) (by taking $V_t=0$ the modifications introduced into the spectrum 1a are unappreciable). This means that the coupling is unessential, and that the $\varrho, K^*(A_2, K_N)$ states are almost pure S(P) states. Furthermore the lack of analysis with volume tensor terms is not at all dramatic;
- spin dependent potentials are in general small in comparison with the central one, justifying "a posteriori" the analysis on the degenerate masses $M_0 M_1$, which we were started from.

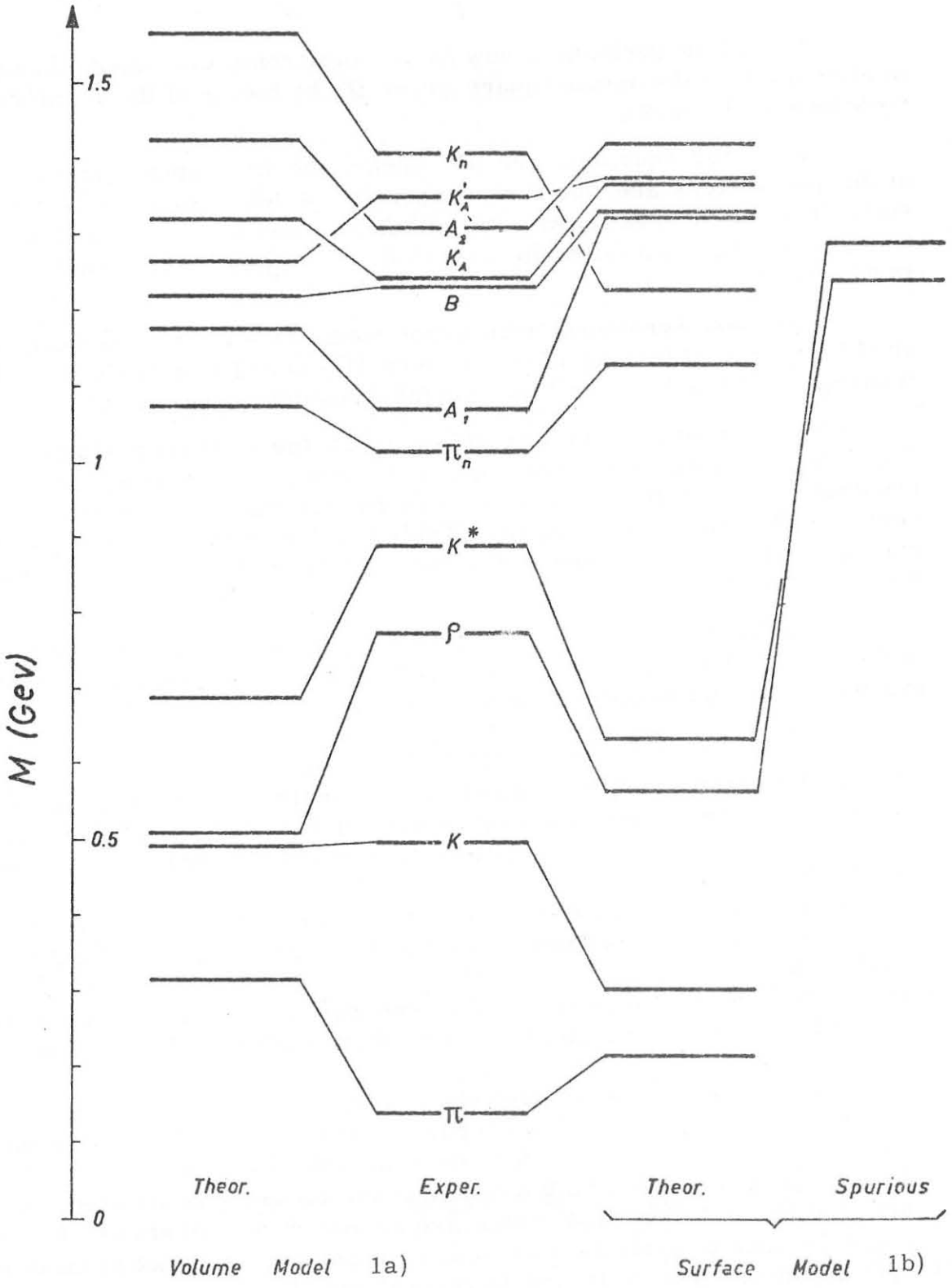

FIG. 2 - Comparison between the experimental spectrum, and the theoretical ones obtained by means of "small" quark masses (solutions 1 of Table II). The "spurious" states obtained by the surface model (solution 1b), are shown separately.

TABLE II

Solution	R	$m_p=m_n$	m_λ	U_c	U_{ss}	$U_{\ell s}$	V_{ss}	V_s	V_t	$\Omega \equiv \sqrt{\Sigma(M_{exp}-M)^2/N}$
	fermis	GeV								
1a	0.4706	1.576	1.795	3.473	-0.201	-0.131	0	0	-0.004	0.043
1b	0.3806	1.128	1.271	3.105	0	0	0.527	0.430	-0.051	0.050
2a	0.2140	18.538	18.732	38.389	-0.100	-0.112	0	0	0.024	0.050
2b	0.2110	18.516	18.738	38.455	0	0	0.010	0.184	0.044	0.060

TABLE III

State $\lvert \ell sj \rangle$	Particle	Experimental masses (GeV)	Theoretical masses (GeV)			
			Fit (1a)	(1b)	(2a)	(2b)
$\lvert 000\rangle$	π	0.140	0.316	0.215	0.304	0.316
	K	0.497	0.493	0.302	0.487	0.529
$\lvert 110\rangle$	π_n	1.016	1.076	1.131	1.032	1.094
$\varepsilon\lvert 111\rangle$	A_1	1.070	1.178	1.325	1.162	1.194
	K_A	1.242	1.318	1.367	1.344	1.402
$\lvert 101\rangle$	B	1.233	1.119	1.328	1.167	1.205
	K'_A	1.350	1.258	1.375	1.348	1.414
$\lvert 011\rangle$ $\lvert 211\rangle$	ϱ	0.774	0.508	0.564	0.402	0.316
	K^*	0.890	0.688	0.635	0.588	0.528
$\lvert 112\rangle$ $\lvert 312\rangle$	A_2	1.310	1.424	1.420	1.378	1.224
	K_N	1.408	1.566	1.228	1.560	1.432

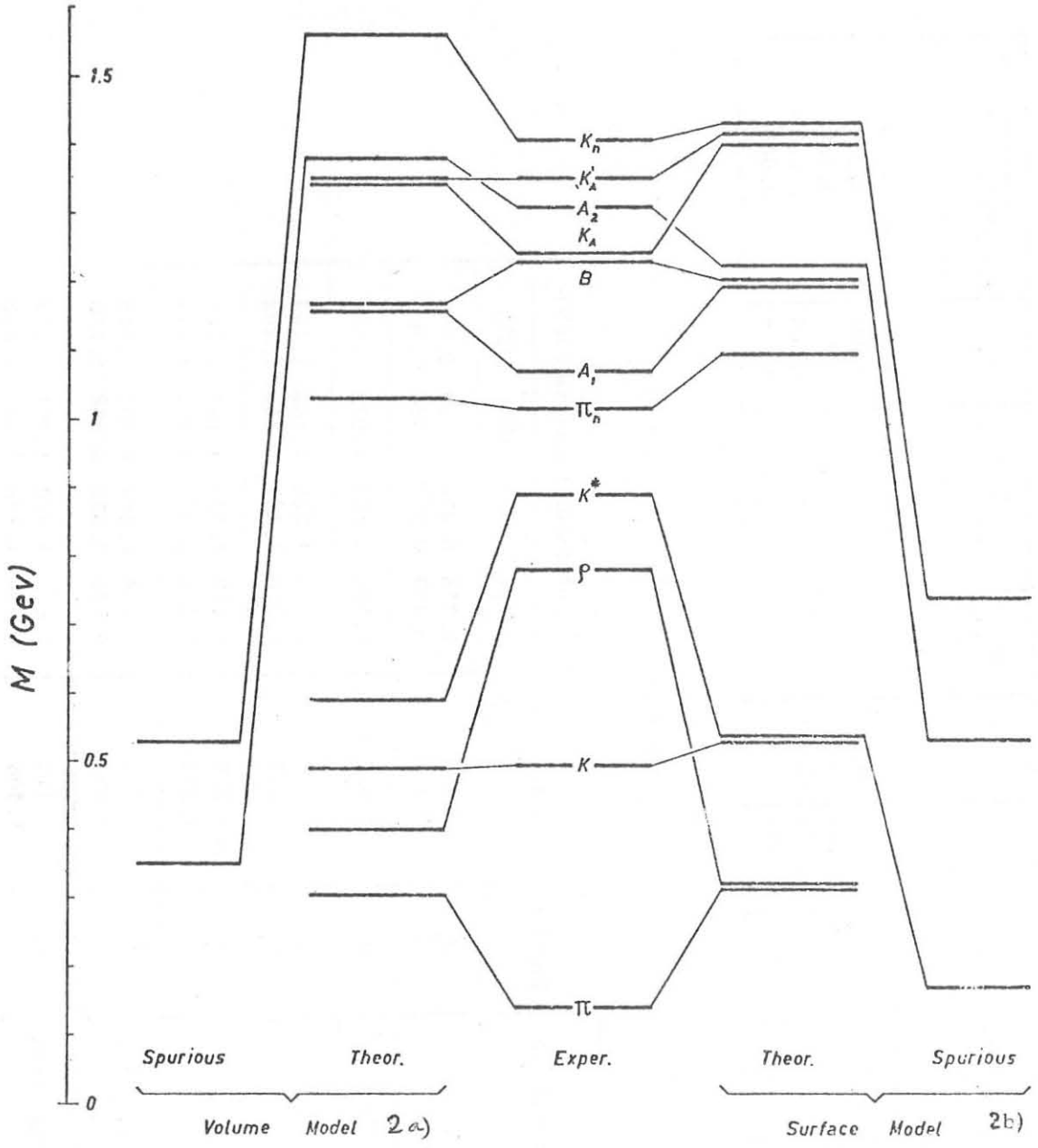

FIG. 3 - Comparison between the experimental spectrum, and the theoretical ones obtained by means of "large" quark masses (solutions 2 of Table II). Note the presence of spurious states in the low region of the spectrum.

4. - REGGE TRAJECTORIES. -

The equations for the bound states, already considered in section 2 and derived in detail in the Appendix, are written in such a fashion that they permit, in a straight-forward way, the interpolation to continuous complex values of the total angular momentum j. We thus obtain the so--called Regge trajectories j(s) associated with our spectrum of particles; more precisely we achieve the trajectories of the real part of j because we always neglect the decays of the bosons and consequently the imaginary part of j. Once the quark masses are fixed, the formalism gives four different types of bound state equations, leading to four trajectories. More precisely:

a) the coupled triplet equation (2.1) provides one trajectory with the choice $m_1=m_2=m_p=m_n$ (particles ϱ and A_2), and another one in the case $m_1=m_p=m_n$, $m_2=m_\lambda$ (particles K^* and K_n);

b) the pure triplet case (eq. (2.2) with s=1) gives two trajectories (particles A_1 and K_A) with the two choices for the masses;

c) the particles π and B from one hand and KK'_A from the other, lie on the trajectories corresponding to the singlet equation (2.2) with s=0;

d) the trajectory related to the particle π_n is deduced from the coupled triplet equation, for the peculiar case j=0, where the selection rules allow one orbital momentum only (ℓ =s=1). Such an equation reads

$$(4.1)\qquad \omega^0_{1\,1} J_{1/2}(\omega^0_{1\,1})/J_{3/2}(\omega^0_{1\,1})+\Lambda K_{1/2}(\Lambda)/K_{3/2}(\Lambda)+V^0_{1\,1,\,1\,1}=0\,.$$

It is an immediate consequence of our model to conclude that the A_2 and the ϱ particles have overlapping trajectories, in agreement with the degeneracy observed experimentally.

Fig. 4 shows the comparison between the experimental masses and the trajectories calculated by means of the set of parameters 1a of Table II.

The general behaviours are satisfactory both for the intercepts at s=0 and the slopes, for the volume model with small masses, even if a certain little curvature is present. In Fig. 5 the same trajectories are shown for s ranging between 2 and 10 GeV^2. The trajectories are still rising at 10 GeV^2 where they however deviate remarkably from the straight line. The solution 1b (surface model) provides completely absurd trajectories (see Fig. 6).

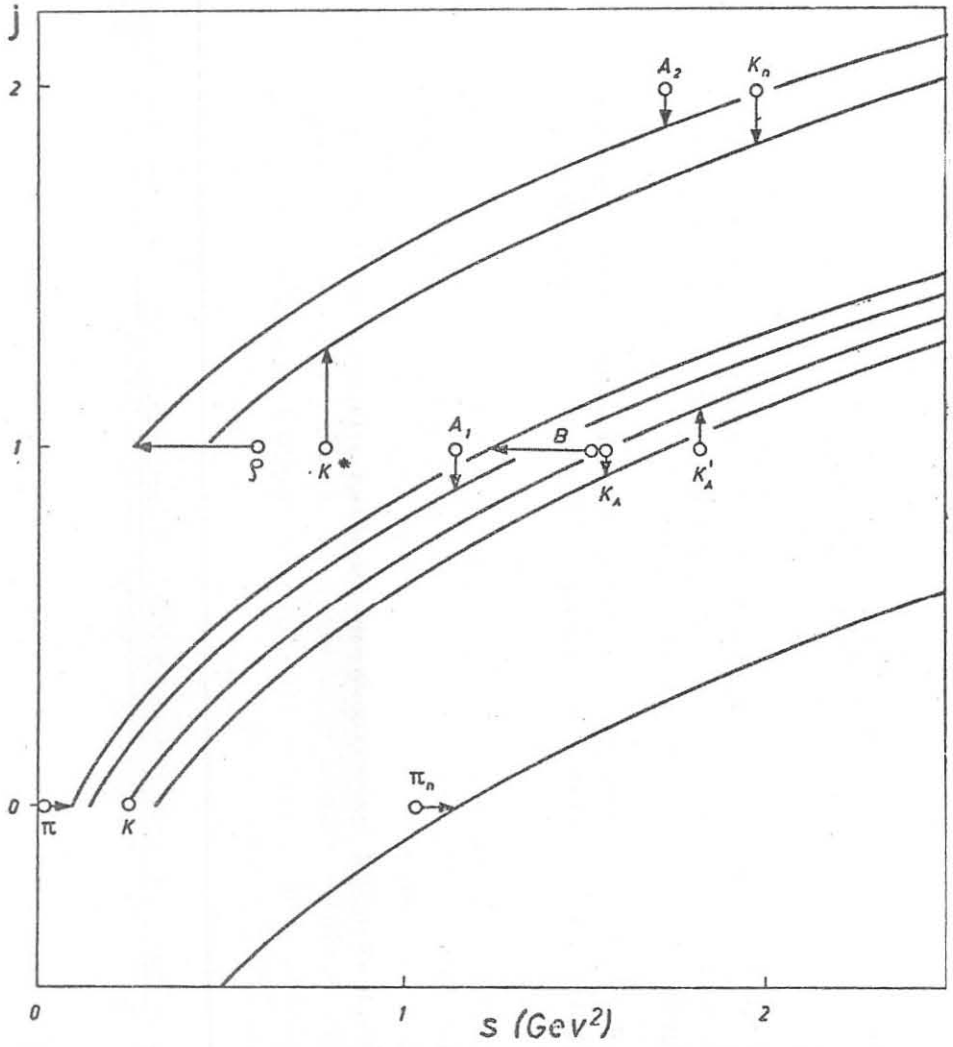

FIG. 4 - Plot of the Regge trajectories calculated by the parameters (1a). The experimental masses are shown for comparison, and correlated by arrows with the corresponding trajectories.

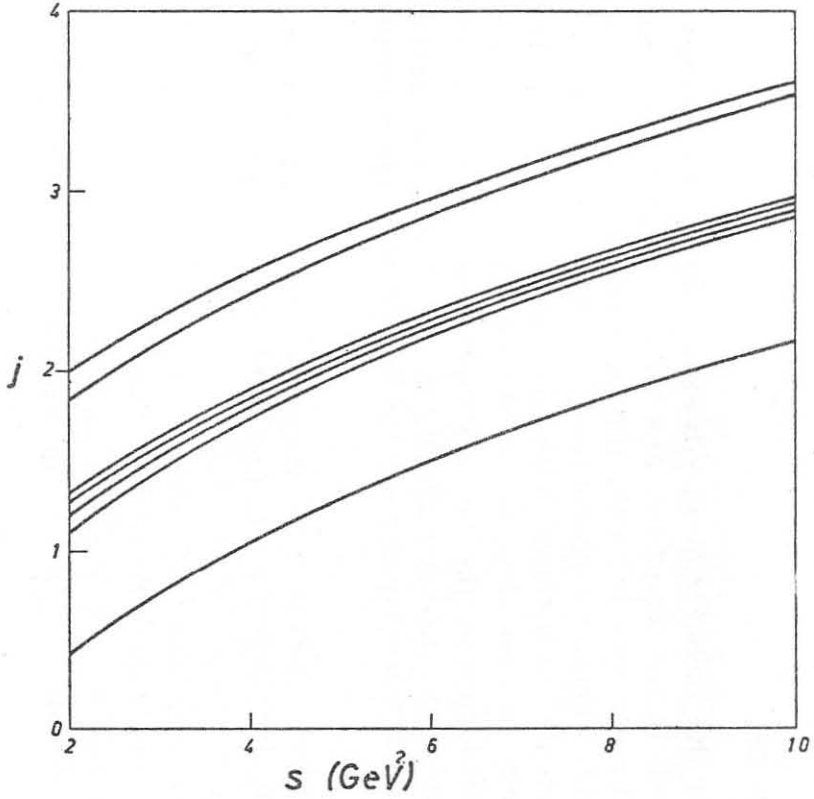

FIG. 5 - Continuation of the Regge trajectories of Fig. 4, between 4 and 10 GeV^2. Note the different scale for s and j.

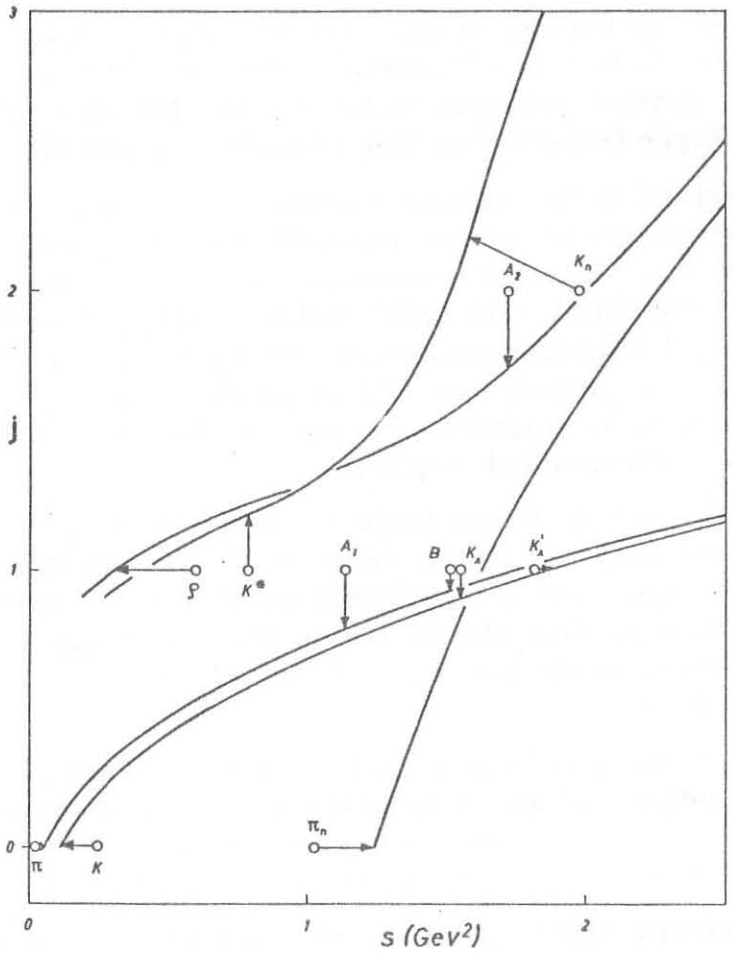

FIG. 6 - Plot of the Regge trajectories calculated by parameters (1b).

5. - CONCLUSIONS. -

SU(3) breaking. The mechanism of breaking the SU(3) symmetry with the difference of the mass values between the quark and the n-p doublet works nicely in most bosonic states such as π, π_n, A_1, B, ϱ, A_2, K, K_A, K'_A, K^*, K_N, but exhibits its limits in not providing the experimental masses of some resonance such as ω, ϕ, η, η', where one probably has to introduce an SU(3) breaking term directly in the potential.

This fact however seems to be a minor point in comparison with what we have achieved from our analysis.

The volume and the surface models. Our phenomenological analysis dealt essentially with two different choices of non-relativistic potentials, called the volume and the surface models.

The former, by looking at the best fit, appears much more realistic than the latter. In fact our "surface" solutions, both with small and high masses, predict spurious states in the spectrum and absurd behaviour of the Regge trajectories (see Figures 2, 3 and 6).

Coming instead to the volume models we obtain an adequate interpretation of the spectrum by means of small effective quark masses ($m_n=m_p=1.57$ GeV, $\Delta m\simeq 0.22$ GeV), strong central attractive forces, less strong but equally essential spin-orbit and spin-spin forces ($U_{ss}=-0.201$; $U_{ls}=-0.131$ GeV) and almost zero tensor forces ($V_t=-0.004$ GeV). The last result justifies completely our initial guess on the tensor force and predicts physically no mixture between the different orbital angular momentum states in the coupled triplets.

The evaluation of the Regge trajectories provides rising trajectories j(s) with a behaviour that in the range of s between 0 and $3\,\mathrm{GeV}^2$ is quite satisfactory from the experimental point of view, and gives the correct intercept at s=0. One should notice that the Regge trajectories are computed in terms of the parameters determined from the spectrum, without any adjustment.

The range of the forces is R=0.47 f. This number verifies well the kinematical condition of non-relativistic motion (see section 1). As far as the possible relation between our phenomenological range and a gluon particle is concerned, we obtain for this a mass around 0.420 GeV.

This elementary object can be made coincide with an SU(3) singlet, scalar, pseudoscalar or also vector. We must consider the value obtained for its mass just as an indicative order of magnitude and not at all definite. In order to complete the predictions that our volume model gives, we notice that if the reduced quark mass obtained for the quark antiquark system, is verified also in the three quark system, it is consistent with the anomalous magnetic moment of the nucleon, where an effective quark mass of order of the third of the proton's gives the correct order of magnitude[5].

A second type of solution obtained from the "volume" model has higher masses ($m_p=m_n=18.538$ GeV) but creates serious troubles in the spectrum because it predicts spurious states in the range of masses below 1.5 GeV (see Fig. 3), which are not present experimentally.

The basic coupling. The picture that comes out from the "volume" model after determining its parameters makes one formulate, as we discussed in our introduction, conjectures on more fundamental descriptions necessarily based on the relativistic local field theory.

We therefore continue here the discussion started in our first section when we proposed three possible relativistic renormalizable

models with the introduction of a boson responsible for the quark-quark and antiquark-antiquark forces, the so-called gluon field.

We must attribute it a mass around 0.5 GeV and owing to the fact that the effective quark mass comes out smaller than the expected free quark mass, we favour the scalar or the pseudoscalar couplings (spin or gluon). From these interaction terms one easily understands in a simple manner how the most relevant effect might be the drastic reduction of the mass of the quark, and the residual force is treated by a still strong non-relativistic potential. This does not mean that a vector basic interaction, with a vector gluon, is not possible at all. In fact the strong coupling theory allows the possibility that higher order contributions dominate above the lowest order ones, thus permitting the creation of a scalar force which needs the exchange of at least two vector gluons. In this case however, one has to understand the mechanism that enhances the scalar term from the many gluon exchange and reduces the others.

We finally would like to point out that the square well potential, regular at r=0, is not necessarily contrasting the Yukawa trilinear couplings such as those proposed in section 1, because of the strong coupling constant. Indeed we may simply imagine a mechanism of cancellation at the point r=0 between the Yukawa potential $g^2 \exp(-\mu r)/r$ due to the Born term (gluonic exchange) and the superposition of Yukawa terms $\int_{a_0}^{\infty} \sigma(a) \exp(-ar) da / r$ coming from higher order contributions, that eliminates the singularity at r=0 and regularizes the potential at the origin.

The Thomas term. Our spin-orbit potential is not very strong in comparison with the central one but it is such that, in combination with the spin-spin term, determines one type of coupling (volume model) instead of the other (surface). We therefore have here an argument against the validity of the vector coupling that, in analogy with quantum electrodynamics, should favour a spin-orbit potential of the Thomas type[3].

In our case such a term would provide a delta function on the spherical surface of radius R (R being the range of the central force) that is essentially what one obtains in our surface model which seems refused by the experimental fit.

Relations with previous work. Some results we obtain were already derived by several authors[6] with even simpler assumptions on the shape of the potential. In particular we like to mention the dominance of the central forces in relation with the non central ones, the very little mixture of the orbital angular momentum for the ϱ and the A_2 particles and finally some features on the bosonic Regge trajectories. These trajectories, however, were related with a peculiar central potential, i.e.

the harmonic oscillator and, according to our knowledge, they were not connected with a more complete type of interaction that includes non central forces as well.

Being aware of the impossibility of drawing drastic conclusions, we point out however that many aspects of the whole analysis we made seem to be more consistent with a scalar (or the pseudoscalar) SU(3) invariant field theory. Our results do not deny however the fact that more fundamental theories and wider symmetries (such as chiral SU(3)x xSU(3)) might be possible with the vector interaction.

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APPENDIX. -

It is worthwhile to show with some details the formal treatment of the model on which our calculations are based. In fact, although similar exactly soluble models are known in the literature[7], the case of the tensor delta coupling has not been stressed so far, as in our knowledge.

We start from the system of coupled equations which are derived in the general case of a scattering problem with non-central forces, in the non-relativistic limit[8]:

$$(A.1)\quad \sum_{\ell's'}\left[\left(-\frac{d^2}{dr^2}+\frac{\ell''(\ell''+1)}{r^2}+k^2\right)\delta_{\ell'\ell''}\,\delta_{s's''}+2m\,v^{j}_{\ell''s'',\,\ell's'}(r)\right]\psi_{\ell's',\ell s}(r)=0^{(x)},$$

where

$$(A.2)\qquad v^{j}_{\ell''s'',\ell's'}(r)=\langle \ell''s''j|\,v(r)\,|\ell's'j\rangle,$$

are the matrix elements of the interaction, in the representation where the orbital angular momentum of relative motion $\bar{\ell}$ is coupled with the channel-spin $\bar{s}\,(=\bar{s}_1+\bar{s}_2)$ to give the total angular momentum $\bar{j}^{(++)}$. The radial wave-function $\psi(r)$ has the following asymptotic behaviour:

$$(A.3)\quad \lim_{r\to\infty}\psi^{j}_{\ell's',\ell s}(r)\propto\delta_{\ell\ell'}\,\delta_{ss'}\exp\left[-i(kr-\ell\pi/2)\right]-S^{j}_{\ell's',\ell s}\exp\left[-i(kr-\ell'\pi/2)\right].$$

Other symbols in eqs. (A.1)...(A.3) are quite obvious, being only necessary to remember that the center-of-mass frame of reference is employed.

We assume the following potential:

$$(A.4)\qquad v^{j}_{\ell''s'',\ell's'}(r)=U^{j}_{\ell''s'',\ell's'}\theta(r,R)+RV^{j}_{\ell''s'',\ell's'}\,\delta(r-R),$$

where

$$(A.5a)\qquad \theta(r,R)=1 \qquad \text{for } r\leq R,$$

(x) - Units $\hbar=c=1$ are employed.

(++) - Parity conservation is also assumed.

(A. 5b) $$\theta(r, R) = 0 \qquad \text{for } r > R,$$

and

(A.6a) $$U^{j}_{\ell''s'', \ell's'} = \langle \ell''s''j | U_c + U_{ss}\, \bar{s}_1 \cdot \bar{s}_2 + U_{\ell s}\, \bar{\ell} \cdot \bar{s} | \ell's'j \rangle ,$$

(A.6b) $$V^{j}_{\ell''s'', \ell's'} = \langle \ell''s''j | V_{ss}\, \bar{s}_1 \cdot \bar{s}_2 + V_{\ell s}\, \bar{\ell} \cdot \bar{s} + V_t Q | \ell's'j \rangle ,$$

with

(A. 7) $$Q = 4 \left[3(\bar{s}_1 \cdot \bar{r})(\bar{s}_2 \cdot \bar{r})/r^2 - \bar{s}_1 \cdot \bar{s}_2 \right] \text{(x)}.$$

In other words we introduce a volume (square well) interaction with central, spin-spin and spin-orbit components, and a surface (Dyrac delta function) interaction with spin-spin, spin-orbit and tensor components. Throughout the paper, the cases $V_{ss}=V_{\ell s}=0$, and $U_{ss}=U_{\ell s}=0$ are referred to conventionally as "volume" and "surface" model respectively.

In the case $s_1=s_2=1/2$, in which we are mainly interested, we have for each j two coupled triplet states $|j-1, 1, j\rangle$, $|j+1, 1, j\rangle$, and two pure (singlet and triplet) states $|0j0\rangle$, $|1j1\rangle$ (o), the matrix elements of the operators of eq. (A.6) being in this case:

(A.8a) $$\langle \ell''s''j | \bar{s}_1 \cdot \bar{s}_2 | \ell's'j \rangle = \frac{1}{2} \delta_{\ell'\ell''} \delta_{s's''} \left[s'(s'+1) - 3/2 \right],$$

(A.8b) $$\langle \ell''s''j | \bar{\ell} \cdot \bar{s} | \ell's'j \rangle = \frac{1}{2} \delta_{\ell'\ell''} \delta_{s's''} \left[j(j+1) - \ell'(\ell'+1) - s'(s'+1) \right],$$

(A.8c) $$\langle \ell''s''j | Q | \ell's'j \rangle = 2 \sqrt{30\,(2\ell'+1)(2\ell''+1)}\; (-1)^{j+1}\, \delta_{s'1}\, \delta_{s''1} \begin{pmatrix} \ell' & 2 & \ell'' \\ 0 & 0 & 0 \end{pmatrix} \begin{Bmatrix} \ell' & 1 & j \\ 1 & \ell'' & 2 \end{Bmatrix}$$

(x) - This is the usual tensor operator of the literature, the factor 4 coming from the fact that we employ the spin operator $\bar{s}_i = (1/2)\,\bar{\sigma}_i$ $(i=1, 2)$.

(o) - This is obviously not true in the particular case j=0, where only two states $|000\rangle$ and $|110\rangle$, both pure, are allowed. However, for the sake of simplicity, all the forthcoming discussion is referred to the general $(j \neq 0)$ case.

For the coupled states eq. (A.1) leads to the following coupled equations (ℓ =j-1, j+1):

$$\left[\frac{d^2}{dr^2} - \frac{j(j-1)}{r^2} + k^2 - 2m\, v^j_{j-1,1;j-1,1}(r)\right]\psi^j_{j-1,1;\ell,1}(r) - \tag{A.9a}$$

$$- 2m\, v^j_{j-1,1;j+1,1}(r)\,\psi_{j+1,1;\ell,1}(r) = 0$$

$$-2m\, v^j_{j+1,1;j-1,1}(r)\,\psi^j_{j-1,1;\ell,1}(r) + \tag{A.9b}$$

$$+\left[\frac{d^2}{dr^2} - \frac{(j+1)(j+2)}{r^2} + k^2 - 2m\, v^j_{j+1,1;j+1,1}(r)\right]\psi^j_{j+1,1;\ell,1}(r) = 0,$$

While the two pure states equations assume the form (s=0, 1):

$$\left[\frac{d^2}{dr^2} - \frac{j(j+1)}{r^2} + k^2 - 2m\, v^j_{js,js}(r)\right]\psi^j_{js,js}(r) = 0\,. \tag{A.10}$$

Eqs. (A.9) and (A.10) are decoupled everywhere but on the interaction radius R, owing to the fundamental assumption of a δ tensor interaction, so that the internal and external solutions may be immediately drawn. For the coupled case (A.9) we have (ℓ, ℓ'=j-1, j+1):

$$\psi^j_{\ell'1,\ell 1}(r) = A^j_{\ell'1,\ell 1} F_{\ell'}(r\sqrt{2m\, U^j_{\ell 1,\ell 1} + k^2}); \qquad r \leq R, \tag{A.11a}$$

$$\psi^j_{\ell'1,\ell 1}(r) = B^j_1\left[I_{\ell'}(kr)\delta_{\ell\ell'} - S^j_{\ell'1,\ell 1} 0_{\ell'}(kr)\right]; \qquad r > R \tag{A.11b}$$

(x)

In other words the wavefunctions $\psi^j_{j-1,1;j-1,1'}$, $\psi^j_{j+1,1;j-1,1}$ from one hand, and $\psi^j_{j-1,1;j+1,1'}$, $\psi^j_{j+1,1;j+1,1}$ from the other, satisfy the same Schrodinger equations (A.9), but are subjected to different asymptotic

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(x) - For the definition of the regular (F_ℓ) and irregular (G_ℓ), ingoing ($I_\ell = G_\ell - iF_\ell$) and outgoing ($0_\ell = I^*_\ell$) solutions of the free Schodinger equation, as well as of other special functions which will be introduced later, see ref. (9).

conditions (A.3) and show therefore different external behaviour (A.11b). Then, by matching the internal and external logarithmic derivatives (taking carefully into account the discontinuity introduced by the δ function in the wavefunction derivative), with elimination of the coefficients A, B, one derives four equations for the S matrix elements. We use the following compact form:

$$S^j_{\ell 1,\ell 1}\left[Z^j_{\ell 1}(\omega^j_{\ell 1})-L_\ell(\lambda)\right]+S^j_{\ell 1,\ell' 1}V^j_{\ell 1,\ell' 1}0_{\ell'}(\lambda)/0_\ell(\lambda)- \tag{A.12a}$$
$$-\exp\left[-2i\phi_\ell(\lambda)\right]\left[Z^j_{\ell 1}(\omega^j_{\ell 1})-L^x_\ell(\lambda)\right]=0,$$

$$S^j_{\ell 1,\ell 1}V^j_{\ell 1,\ell' 1}+S^j_{\ell 1,\ell' 1}\left[Z^j_{\ell' 1}(\omega^j_{\ell' 1})-L_{\ell'}(\lambda)\right]0_{\ell'}(\lambda)/0_\ell(\lambda)- \tag{A.12b}$$
$$-\exp\left[-2i\phi_\ell(\lambda)\right]V^j_{\ell 1,\ell' 1}=0,$$

where the choice $\ell=j-1$, $\ell'=j+1$ corresponds to the first couple of equations, and the opposite choice to the second one.

In eqs. (A.12) $\phi_\ell(\lambda)=tg^{-1}\left[F_\ell(\lambda)/G_\ell(\lambda)\right]$ is the hard sphere phase shift and

$$L_\ell(\lambda)\equiv\mathcal{L}_\ell(\lambda)+iP_\ell(\lambda)=R\left[\frac{d0_\ell}{dr}\right]_{r=R}/0_\ell(\lambda)$$

the external logarithmic derivative, while the internal logarithmic derivative Z reads:

$$Z^j_{\ell s}(\omega^j_{\ell s})=R\left[\frac{dF_\ell}{dr}\right]_{r=R}/F_\ell(\omega^j_{\ell s})+V^j_{\ell s,\ell s}= \tag{A.13}$$
$$=\omega^j_{\ell s}F_{\ell-1}(\omega^j_{\ell s})/F_\ell(\omega^j_{\ell s})-\ell+V^j_{\ell s,\ell s}$$

All functions in eqs. (A.12) are evaluated on the interaction radious, namely:

$$\lambda=kR, \tag{A.14a}$$

$$\omega^j_{\ell s}=R\sqrt{2mU^j_{\ell s,\ell s}+k^2}. \tag{A.14b}$$

From eqs. (A.12) one gets (being as before $\ell, \ell' = j-1, j+1$):

$$S^j_{\ell 1, \ell 1} = \exp\left[-2i\phi_\ell(\lambda)\right] \times$$

(A.15a)

$$\times \frac{\left[Z^j_{\ell 1}(\omega^j_{\ell 1}) - L^x_\ell(\lambda)\right]\left[Z^j_{\ell' 1}(\omega^j_{\ell' 1}) - L_{\ell'}(\lambda)\right] - \left[V^j_{\ell 1, \ell 1}\right]^2}{\left[Z^j_{\ell 1}(\omega^j_{\ell 1}) - L_\ell(\lambda)\right]\left[Z^j_{\ell' 1}(\omega^j_{\ell' 1}) - L_{\ell'}(\lambda)\right] - \left[V^j_{\ell 1, \ell' 1}\right]^2} .$$

$$S^j_{\ell 1, \ell' 1} = \pm \exp\left[-i\{\phi_\ell(\lambda) + \phi_{\ell'}(\lambda)\}\right] .$$

(A.15b)

$$\cdot \frac{2i\sqrt{P_\ell(\lambda) P_{\ell'}(\lambda)}\, V^j_{\ell 1, \ell' 1}}{\left[Z^j_{\ell 1}(\omega^j_{\ell 1}) - L_\ell(\lambda)\right]\left[Z^j_{\ell' 1}(\omega^j_{\ell' 1}) - L_{\ell'}(\lambda)\right] - \left[V^j_{\ell 1, \ell' 1}\right]^2} .$$

Finally, by matching the (internal and external) wavefunctions, one derives four normalization constants, and is able to write down the wavefunction. The last constant is free for an overall ψ normalization.

For the pure states, the S matrix elements may be derived either from eq. (A.10) by a similar procedure, or directly from the expressions (A.15) with the formal substitutions $\ell, \ell' \to j$; $1 \to s\,(=0, 1)$.

Since $S^j_{jo, j1} = V^j_{jo, j1} = 0$ (see eqs. (A.8)), one gets:

(A.16)

$$S^j_{js, js} = \exp\left[-2i\phi_j(\lambda)\right] \frac{Z^j_{js}(\omega^j_{js}) - L^x_j(\lambda)}{Z^j_{js}(\omega^j_{js}) - L_j(\lambda)} ; \qquad s = 0, 1.$$

The unitary may be easily checked in eqs. (A. 15) and is transparent in eq. (A.16) since Z is real.

Now from eqs. (A.15) and (A.16) one immediately finds the S matrix poles in the complex k plane, for the coupled and pure case respectively, namely:

(A.17)

$$\left[Z^j_{j-1,1}(\omega^j_{j-1,1}) - L_{j-1}(\lambda)\right]\left[Z^j_{j+1,1}(\omega^j_{j+1,1}) - L_{j+1}(\lambda)\right] - \left[V^j_{j-1,\,1;\,j+1,\,1}\right]^2 = 0,$$

(A.18) $$Z^j_{js}(\omega^j_{js})-L_j(\lambda)=0; \quad , s=0,1.$$

In particular, for pure imaginary momentum $k=i\sqrt{2m\varepsilon}$, being the (positive) binding energy of the system, eqs. (A.17) and (A.18) are the bound state equations. Nevertheless, in this particular case, they are better written in function of the real argument

(A.19) $$\Lambda = R\sqrt{2m\varepsilon},$$

namely:

(A.20) $$\left[Z^j_{j-1,1}(\omega^j_{j-1,1})-L^{(-)}_{j-1}(\lambda)\right]\left[Z^j_{j+1,1}(\omega^j_{j+1,1})-L^{(-)}_{j+1}(\lambda)\right]-\left[V^j_{j-1,1;j+1,1}\right]^2=0,$$

(A.21) $$Z^j_{js}(\omega^j_{js})-L^{(-)}_j(\Lambda)=0; \qquad s=0,1,$$

where

(A.22) $$L^{(-)}_\ell(\Lambda)=R\left[\frac{d0^{(-)}_\ell}{dr}\right]_{r=R}/0^{(-)}_\ell(\Lambda)=-\Lambda 0^{(-)}_{\ell-1}(\Lambda)/0^{(-)}_\ell(\Lambda)-\ell,$$

$0^{(-)}_\ell$ being the (real) outgoing solution of the free Schrödinger equation for negative energy.

Since F_ℓ and $0^{(-)}_\ell$ are simply related to the Bessel functions $J_{\ell+1/2}$ and $K_{\ell+1/2}$ respectively[9], one can use, in eqs. (A.20) and (A.21), in place of eqs. (A.13) and (A.22), the following expression which is more suitable for extension to complex j values:

(A.23) $$Z^j_{\ell s}(\omega^j_{\ell s})-L^{(-)}_\ell(\Lambda)=\omega^j_{\ell s}\,J_{\ell-1/2}(\omega^j_{\ell s})/J_{\ell+1/2}(\omega^j_{\ell s})+$$
$$+\,K_{\ell-1/2}(\Lambda)/K_{\ell+1/2}(\Lambda)+V^j_{\ell s,\ell s}.$$

By means of the matrix elements (A.8), and making use of the properties of the functions J and K, the Regge trajectories are immediately drawn from eqs. (A.20), (A.21) (A.23).

Finally, although the equations developed so far through the S matrix formalism are all we need for our calculations, it is worthwhile

to outilne breafly the direct solution of the bound state problem, which must lead obviously to the same equations (A.20), (A.21), but gives in addition the bound state wavefunctions.

We start as before with the coupled states: one immediately recognizes that, in the bound state case, not only the Schrodinger equations (A.9), but also the asymptotic conditions (A.3), are the same for functions $\psi^j_{\ell 1;j-1,1}$ and $\psi^j_{\ell 1;j+1,1}$, so that two labels may be dropped, by writing $\psi^j_{\ell s,\ell' s'} \equiv \psi^j_{\ell s}$. Then the internal and external solutions (A.11) become:

$$\psi^j_{\ell 1}(r) = A^j_{\ell 1} F_\ell \left(r\sqrt{2m(U^j_{\ell 1,\ell 1} - \varepsilon)}\right); \qquad r \leq R, \tag{A.24a}$$

$$\psi^j_{\ell 1}(r) = B^j_{\ell 1} 0^{(-)}_\ell \left(r\sqrt{2m\varepsilon}\right), \tag{A.24b}$$

and the logarithmic derivatives matching (A.12) read now:

$$A^j_{j-1,1}\left[Z^j_{j-1,1}(\omega^j_{j-1,1}) - L^{(-)}_{j-1}(\Lambda)\right] + A^j_{j+1,1} V^j_{j-1,1;j+1,1} 0^{(-)}_{j-1}(\Lambda)/0^{(-)}_{j+1}(\Lambda) = 0\,, \tag{A.25a}$$

$$A^j_{j-1,1} V^j_{j-1,1;j+1,1} 0^{(-)}_{j+1}(\Lambda)/0^{(-)}_{j-1}(\Lambda) + A^j_{j+1,1}\left[Z^j_{j+1,1}(\omega^j_{j+1,1}) - L^{(-)}_{j+1}(\Lambda)\right] = 0\,. \tag{A.25b}$$

The homogeneous system (A.25) admits non-trivial solutions $A^j_{j-1,1}$, $A^j_{j+1,1}$ if and only if eq. (A.20) is satisfied. Then, from one of the equations (A.25), and from wavefunctions continuity on the interaction radious one gets three of the four unknown coefficients (the latter being as usual an overall normalization constant) and is able to write down the wavefunction.

The pure states wavefunctions are obtained in a similar way, together with the eigenvalue equations (A.21).

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