M. Ademollo, A. Barducci and J. Gomis: LAGRANGIAN FORMULATION AND CURRENT ALGE-
BRA IN THE NEVEU-SCHWARZ DUAL MODEL.

SUMMARY -

We study in some detail the field theory formulation in coordinate space of the Neveu-
-Schwarz model for the N pion amplitude and for the N pion emission from a single quark line. We
then consider the problem of current algebra for the fermion and for the boson case. In particu-
lar for the boson case we find a class of axial currents which satisfy the local SU(2)X SU(2) alge-
bra with the isospin currents and the PCAC relation between external real states.

1. - INTRODUCTION -

Recently a big deal of work has been devoted to the general features of the dual models,
both for what concerns the construction of models[1-6] with physical trajectories and a systema-
tic study of some more formal aspects. In particular people has been concerned with the pro-
blem of ghosts[7] in the case of unit intercept and with the study of conformal symmetry[8] rela-
ted to duality. A convenient approach to conformal symmetry is given by a field theoretical formu
lation[9] in two dimensions. The connection between dual models and conventional field theory has
also been investigated[10].

Our purpose here is to give a field theory formulation in ordinary space of the dual model
proposed by Neveu and Schwarz, both for the purely bosonic amplitudes[3] and for the emission
of pions from a single fermion (quark) line[5]. The relevant formalism of the model is summari-
zed in Section 2. In Section 3 we shall treat the wave equations for the infinite component boson
and fermion fields and their canonical quantization, and in Section 4 we shall find the local interac
tions for the fields from which the dual amplitudes can be obtained on the Born approximation.
Finally in Section 5 we shall study the vector and axial currents of the field theory model.

The Neveu-Schwarz model is in fact very appealing from this point of view, as it contains
axial currents[11] even in the boson case. Without considering the difficult problem of dual cur-
rents[12], which is related to the analytic structure of the form factors and to the duality cons-
traints, we shall only deal with local currents bilinear in the field operators. For the conserved
vector currents we take the canonical isospin currents and we look for axial currents which obey
current algebra and PCAC. While this problem can be easily solved for the fermion case, for the
boson case we shall recognize that is impossible. However we shall find a class of axial currents
which obey current algebra and PCAC in a weak sense, i.e. for the matrix elements between real
physical states.

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2. - THE NEVEU-SCHWARZ MODEL -

For our purpose it is useful to give a breif survey of the Neveu-Schwarz model[3,5].

We consider first the boson case[3]. The Fock space of the physical hadrons is generated by two sets of commuting operators $a_n^{\mu+}$ and $b_m^{\mu+}$. Instead of these standard operators we shall use

(2.1) $$A_n = \sqrt{n}\, a_n; \qquad A_{-n} = \sqrt{n}\, a_n^+; \qquad A_o = \sqrt{2}\, p; \qquad b_{-m} = b_m^+.$$

These obey the following commutation and anticommutation relations:

(2.2) $$\left[A_m^\mu, A_n^\nu \right] = - m\, g^{\mu\nu}\, \delta_{m,-n}, \qquad (m, n = \pm 1, \pm 2, \ldots\ldots),$$

(2.3) $$\left\{ b_m^\mu, b_n^\nu \right\} = - g^{\mu\nu}\, \delta_{m,-n}, \qquad (m, n = \pm \tfrac{1}{2}, \pm \tfrac{3}{2}, \ldots\ldots).$$

The amplitude for N pions as in Fig. 1 is written in the form

(2.4) $$A_N = \left\langle 0 \left| V(p_2) \frac{1}{L_o(\pi_1) - \tfrac{1}{2}} V(p_3) \ldots \frac{1}{L_o(\pi_{N-3}) - \tfrac{1}{2}} V(p_{N-1}) \right| 0 \right\rangle,$$
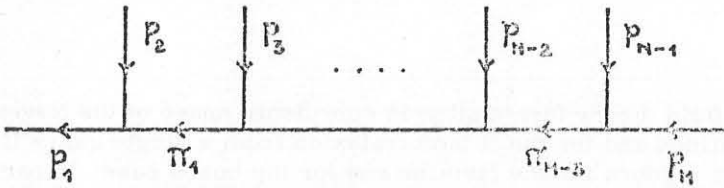

FIG. 1

where the pion is the ground state of the Fock space[13] and its squared mass is $p^2 = -1/2$. The vertex operator $V(p)$ is given by

(2.5) $$V(p) = p \cdot H\, V_o(p),$$

where

(2.6) $$H = \sum_{m=-\infty}^{\infty} b_m,$$

(2.7) $$V_o(p) = e^{-\sqrt{2}\, p \sum_{n=1}^{\infty} \frac{A_{-n}}{n}}\, e^{\sqrt{2}\, p \sum_{n=1}^{\infty} \frac{A_n}{n}}$$

$\left[L_o(\pi) - 1/2 \right]^{-1}$ is the meson propagator, where

(2.8) $$L_o(\pi) = -\pi^2 + R = -\pi^2 + R_a + R_b,$$

(2.9) $$R_a = -\sum_{n=1}^{\infty} A_{-n}\, A_n,$$

(2.10) $$R_b = -\sum_{m=\frac{1}{2}}^{\infty} m\, b_{-m} \cdot b_m.$$

We also define the operators

(2.11) $$L_n = L_{n_a} + L_{n_b},$$

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$$(2.12) \qquad L_{n_a} = -\frac{1}{2} : \sum_{k=-\infty}^{\infty} A_k \cdot A_{-n-k} :,$$

$$(2.13) \qquad L_{nb} = -\frac{1}{4} : \sum_{m=-\infty}^{\infty} (n+2m)b_m \cdot b_{-n-m} : .$$

which satisfy the Virasoro algebra

$$(2.14) \qquad \left[L_m, L_n \right] = (n-m)L_{m+n} + \frac{2}{3} n(n^2-1) \delta_{n,-m}.$$

It is convenient to introduce the gauge operators

$$(2.15) \qquad G_m = \sum_{n=-\infty}^{\infty} A_n \cdot b_{-m-n},$$

which satisfy the relations

$$(2.16) \qquad \left\{ G_m, G_n \right\} = 2 L_{m+n}.$$

The physical meson states obey the mass shell condition

$$(2.17) \qquad (L_o - \tfrac{1}{2})|\emptyset\rangle = 0$$

and the gauge conditions

$$(2.18) \qquad G_{-m}|\emptyset\rangle = 0, \quad (m = \tfrac{1}{2}, \tfrac{3}{2}, \ldots).$$

They have masses $m^2 = R_a + R_b - 1/2$, G parity and charge conjugation given by

$$(2.19) \qquad G = (-1)^{\sum_{m=1/2}^{\infty} b_{-m} \cdot b_m + 1} = (-1)^{2R_b+1},$$

$$(2.20) \qquad C = (-1)^{R_a + R_b + \frac{1}{2} - \frac{1-G}{4}}.$$

Therefore states with $m^2 = -(1/2)+2n$ or $m^2 = 2n$ must be isovector whereas states with $m^2 = (1/2) + 2n$ or $m^2 = 2n + 1$ must be isoscalar.

For the fermion case we define the harmonic oscillator operators d_n^μ, $d_{-n}^\mu = d_n^{\mu+}$, where n is an integer, and $d_0^\mu = -(i/\sqrt{2})\gamma_5\gamma^\mu$ satisfying the anticommutation relations

$$(2.21) \qquad \left\{ d_m^\mu, d_n^\nu \right\} = -g^{\mu\nu} \delta_{m,-n}, \quad (m, n = 0, \pm 1, \pm 2, \ldots).$$

We also define the operators

$$(2.22) \qquad L_{nd} = -\frac{1}{4} : \sum_{m=-\infty}^{\infty} (n+2m)d_m \cdot d_{-n-m} :$$

analogous to L_{nb} of Eq. (2.13) and satisfying the Virasoro algebra. The total generators for the fermion case are $L_n = L_{na} + L_{nd}$ and the gauge operators are

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(2.23)
$$F_n = i \gamma_5 \sum_{m=-\infty}^{\infty} A_m \cdot d_{-n-m},$$

with the anticommutation relations

(2.24)
$$\{ F_m, F_n \} = -2L_{m+n}.$$

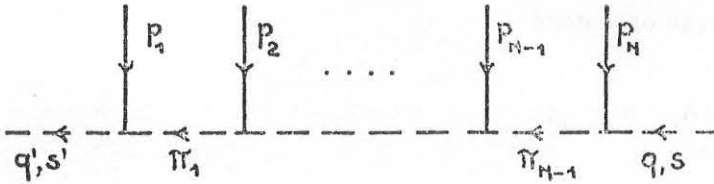The amplitude for the emission of N pions from the fermion (quark) line as in Fig. 2 is written in the form

FIG. 2

(2.25)
$$A_N = \bar{u}(q', s') \langle 0 | \gamma_5 \Lambda V_o(p_1) \frac{1}{F_o(\pi_1)-m} \cdots \gamma_5 \Lambda V_o(p_N) | 0 \rangle \cdot u(q, s),$$

where

(2.26)
$$\Lambda = (-1)^{\sum_{n=1}^{\infty} d_{-n} \cdot d_n}$$

and m is the mass of the quark ground state. The physical quark states satisfy the wave equation

(2.27)
$$(F_o - m) | \psi \rangle = 0$$

and the subsidiary conditions

(2.28)
$$F_{-n} | \psi \rangle = 0, \qquad (n = 1, 2, \ldots).$$

The amplitude (2.25) has the property that if we make a duality transformation leading to the quark-antiquark channel and we factorize on the first pole, we obtain for the meson sector the same amplitude of Eq. (2.4), provided that we have for the quark mass $m^2 = -1/2$.

3. - LAGRANGIAN FORMULATION -

We want to derive the N-pion amplitude (2.4) and the quark-pion amplitude (2.25) from a standard field theory model in Born approximation and we shall follow the procedure by Kleinert[10].

We first observe that the vertex operator V(p) in Eq. (2.4) has an exponential dependence on the external pion momentum and therefore is not suitable to be derived from a local interaction. By a heuristic procedure we can factorize each vertex as

(3.1)
$$V_o(p) \sim e^{-i \pi' \cdot A} e^{i \pi \cdot A},$$

(3.2)
$$A = i \sqrt{2} \sum_{n \neq 0} \frac{A_n}{n},$$

with $\pi' - \pi = p$, and then transform the propagators into $\left[\tilde{L}_o(\pi) - 1/2 \right]^{-1} = e^{i \pi \cdot A} \left[L_o(\pi) - \frac{1}{2} \right]^{-1} x$ x $e^{-i \pi \cdot A}$. The amplitude (2.4) would then take the form

(3.3) $A_N \propto \left\langle 0 \left| e^{-ip_1 \cdot A} P_2 \cdot H \dfrac{1}{\widetilde{L}_0(\pi_1) - \frac{1}{2}} P_3 \cdot H \cdots \dfrac{1}{\widetilde{L}_0(\pi_{N-3}) - \frac{1}{2}} P_{N-1} \cdot H e^{ip_N \cdot A} \right| 0 \right\rangle$

which is suitable for field theoretical treatment.

However the unitary transformation we have introduced

(3.4) $$T(p) = e^{ip \cdot A}$$

is singular and can only be defined through a suitable limiting procedure. A convenient definition is

(3.5) $T(p) = \lim\limits_{\varepsilon \to 1} e^{-\sqrt{2}p \sum\limits_{n \neq 0} \frac{\varepsilon^{|n|}}{n} A_n} = \lim\limits_{\varepsilon \to 1} (1 - \varepsilon^2)^{-p^2} e^{\sqrt{2}p \sum\limits_{n=1}^{\infty} \frac{\varepsilon^n}{n} A_{-n}} e^{-\sqrt{2}p \sum\limits_{n=1}^{\infty} \frac{\varepsilon^n}{n} A_n}$

the limit $\varepsilon \to 1$ to be taken at the end of calculations.

The transformed hamiltonian operator $\widetilde{L}_0$ is

(3.6) $$\widetilde{L}_0(p) = T(p) L_0(p) T^{-1}(p) = L_0(p) - p \cdot \Gamma - (c-1)p^2,$$

where

(3.7) $$\Gamma = \sqrt{2} \sum\limits_{n \neq 0} A_n,$$

and c is an infinite constant which in terms of the parameter ε is $c = (1+\varepsilon^2)/(1-\varepsilon^2)$.

A different limiting procedure would be to consider only a finite number N of oscillator mo̱des and letting $N \to \infty$ at the end. In this case we have $c = 1+2N$.

We have to observe, however, that the T transformation, in the way we want to use it, is not always free of ambiguities. For instance we may have problems when we transform operators like L_n and we want to use them in an amplitude like (3.3). The reason of the ambiguity is that in passing from (2.4) to (3.3) we want to identify $V_0(p)$ with $T(-p)$, apart from a multiplicative diveṟgent constant, according to (3.5). Now this identification is in general not allowed since $V_0(p)$ and $T(-p)$ belong to different representations of the conformal group, $V_0(p)$ having spin $J_a = -p^2$ [8] and $T(p)$ having $J_a = 0$, apart from the contribution of the zero mode. Thus, for example, for the ope̱rators $W_n = L_0 - L_n$ we have

(3.8) $$T(p)W_n(p) T^{-1}(p) = W_n(0)$$

while, as is well known

(3.9) $$V_0(-p)W_n(p)V_0^{-1}(-p) = W_n(0) - np^2.$$

Therefore we see that we have ambiguities in the limit $\varepsilon \to 1$ of (3.5) when we transform operators which are sensitive to the representation to which the transformation belongs. This was not the case of L_0 and therefore no such problem arises for the amplitudes. Concerning the case of operators like W_n, it may be useful to consider the combinations

(3.10) $$M_n = W_n - nW_1; \qquad M_{-n} = M_n^+, \qquad (n > 1)$$

which commute with a tensor operator of any spin.

The excited meson states in the rest frame are given by the basic vectors $| \alpha \beta \rangle$ of the Fock space, where α and β stand for the occupation numbers of the a_n and b_n oscillators. The states with momentum p will be obtained from the states at rest by a boost transformation and are indicated by $| p, \alpha \beta \rangle$. Of course on the mass shell the energy is given by $p_0 = (\vec{p}^{\,2} + R - (1/2))^{1/2}$. Furthermore the meson states undergo the T transformation and become

(3.11)
$$|\overline{p, \alpha\beta}\rangle = T(p)|p, \alpha\beta\rangle .$$

The whole boson system can be described by an infinite component local field, whose components $\Phi_{\alpha\beta}(x)$ can be labelled by the quantum numbers of the basic Fock space.

The lagrangian for the free field is taken of the form

(3.12)
$$\mathcal{L}(x) = \Phi^{+}(x)(c\overleftrightarrow{\partial}_{\mu}\overrightarrow{\partial}^{\mu} + \frac{i}{2}\Gamma_{\mu}\overleftrightarrow{\partial}^{\mu} - R + \frac{1}{2})\Phi(x),$$

from which we obtain the wave equation

(3.13)
$$\left[\tilde{L}_{o}(i\partial) - \frac{1}{2}\right]\Phi(x) = 0.$$

The subsiduary conditions of Eq. (2.18) will be interpreted as conditions on the state vectors rather than conditions for the field, in analogy with the Gupta-Beuler procedure for the electromagnetic field. We then have for any physical state $|\psi\rangle$

(3.14)
$$\tilde{G}_{-m}(i\partial)\overset{(+)}{\Phi}(x)|\psi\rangle = 0,$$

(3.15)
$$\tilde{G}_{-m}(i\partial) = G_{-m}(0) + \sqrt{2}i\,H .$$

Therefore the components of the field are linearly independent and, on the other hand, the vectors of the Hilbert space can be divided into real and spurious states according to whether they satisfy or not Eq. (3.14).

From (3.12) the canonical momentum of the field is

(3.16)
$$\pi(x) = \Phi^{+}(x)(c\overleftarrow{\partial}_{o} + \frac{i}{2}\Gamma_{o}),$$

and we postulate the canonical commutation relations

(3.17)
$$\left[\Phi_{\alpha\beta}(x), \pi_{\alpha'\beta'}(y)\right]_{x_{o}=y_{o}} = i\delta(\vec{x} - \vec{y})\delta_{\alpha\alpha'}\delta_{\beta\beta'}$$

(3.18)
$$\left[\Phi_{\alpha\beta}(x), \Phi_{\alpha'\beta'}(y)\right]_{x_{o}=y_{o}} = \left[\pi_{\alpha\beta}(x), \pi_{\alpha\beta}(y)\right]_{x_{o}=y_{o}} = 0.$$

The field $\Phi(x)$ can be expanded in plane waves as

(3.19)
$$\Phi_{\alpha\beta}(x) = \sum_{\alpha'\beta'}\int\frac{d^{3}p}{(2\pi)^{3/2}}\sqrt{\frac{m_{\alpha'\beta'}}{2p_{o}}}\left[u_{\alpha\beta}(p, \alpha'\beta')e^{-ipx}a_{+}(p, \alpha'\beta') + u_{\alpha\beta}(-p, \alpha'\beta')\times\right.$$
$$\left.\times e^{ipx}a_{-}^{+}(p, \alpha'\beta')\right],$$

where the "spinors" $u_{\alpha\beta}(p, \alpha'\beta')$ are defined by

(3.20)
$$u_{\alpha\beta}(p, \alpha'\beta') = \langle\alpha\beta|\overline{p, \alpha'\beta'}\rangle,$$

and satisfy the equation

(3.21)
$$\left[\tilde{L}_{o}(p) - \frac{1}{2}\right]u(p, \alpha\beta) = (-cp^{2} - p\cdot\Gamma + R - \frac{1}{2})u(p, \alpha\beta) = 0.$$

We see from Eq. (3.22) that the negative energy solutions $u(-p, \alpha\beta)$ can be related to the positive energy solutions through

(3.22) $$u(-p, \alpha\beta) = (-1)^{\sum_{n=1}^{\infty} \frac{1}{n} A_{-n} \cdot A_n} u(p, \alpha\beta),$$

corresponding to a PT transformation.

They also satisfy the orthogonality and completeness relations (14)

(3.23) $$u^{+}(\vec{p}, p_o, \alpha\beta)\left[c(p_o + p_o') + \Gamma_o\right]u(\vec{p}, p_o', \alpha'\beta') = \frac{2p_o}{m_{\alpha\beta}}\delta_{\alpha\alpha'}\delta_{\beta\beta'}$$

(3.24) $$\sum_{\alpha\beta, \varepsilon = \pm} \frac{1}{2} cm_{\alpha\beta}\, u(\varepsilon p, \alpha\beta)u^{+}(\varepsilon p, \alpha\beta) = I,$$

(3.24') $$\sum_{\alpha\beta, \varepsilon = \pm} \frac{cm_{\alpha\beta}}{2\varepsilon p_o}\, u(\varepsilon p, \alpha\beta)u^{+}(\varepsilon p, \alpha\beta) = 0.$$

For the creation and destruction operators we then obtain the standard commutation relations

(3.25) $$\left[a_{\pm}(p, \alpha\beta),\ a_{\pm}^{+}(p', \alpha'\beta')\right] = \delta(\vec{p} - \vec{p}')\delta_{\alpha\alpha'}\delta_{\beta\beta'}$$

(3.26) $$\left[a_{\pm}(p, \alpha\beta),\ a_{\pm}(p', \alpha'\beta')\right] = \left[a_{\pm}^{+}(p, \alpha\beta),\ a_{\pm}^{+}(p', \alpha'\beta')\right] = 0,$$

(3.27) $$\left[a_{\pm}(p, \alpha\beta), a_{\mp}^{+}(p', \alpha'\beta')\right] = \left[a_{\pm}(p, \alpha\beta),\ a_{\mp}(p', \alpha'\beta')\right] = \left[a_{\pm}^{+}(p, \alpha\beta), a_{\mp}^{+}(p', \alpha'\beta')\right] = 0.$$

The Feynman propagator results to be

(3.28) $$S_F(x-y) = \frac{1}{(2\pi)^4}\int d^4p\ e^{-ip(x-y)}\frac{1}{\tilde{L}_o(p) - \frac{1}{2}}$$

A similar procedure can be used to quantize the quark field $\psi(x)$. We start from the lagrangian density

(3.29) $$\mathcal{L}(x) = \bar{\psi}(x)(i\gamma_\mu \overleftrightarrow{\partial}^\mu - \gamma_5 \Delta_\mu \overleftrightarrow{\partial}^\mu + i\gamma_5 N - m)\psi(x),$$

where

(3.30) $$\Delta = \sqrt{2} \sum_{n \neq 0} d_n,$$

(3.31) $$N = \sum_{n \neq 0} A_n \cdot d_{-n},$$

from which we obtain the wave equation

(3.32) $$\left[\tilde{F}_o(i\partial) - m\right]\psi(x) = 0,$$

(3.33) $$\tilde{F}_o(i\partial) = F_o(i\partial) - \gamma_5 \Delta \cdot \partial.$$

The canonical momentum is

(3.34) $$\pi(x) = \psi^{+}(x)(i - \gamma_o\gamma_5\Delta_o),$$

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with the standard anticummutation relations with the field. The field $\psi_{\tau,\alpha\delta}(x)$, where τ labels the Dirac components, has the momentum expansion

$$\psi_{\tau,\alpha\delta}(x) = \sum_{\pm s} \sum_{\alpha'\beta'} \int \frac{d^3p}{(2\pi)^{3/2}} \sqrt{\frac{m_{\alpha'\delta'}}{P_o}} \left[U_{\tau,\alpha\delta}(ps,\alpha'\delta') e^{-ipx} \right. \times$$

(3.35)

$$\left. \times\ b(ps,\alpha'\delta') + V_{\tau,\alpha\delta}(ps,\alpha'\delta') e^{ipx} d^+(ps,\alpha'\delta') \right] ,$$

where b and d^+ are the quark destruction and creation operators satisfying the standard anticommutation relations and the spinors U and V are solutions of the wave equations

(3.36) $\quad \left[\tilde{F}_o(p) - m \right] U(ps,\alpha\delta) = (\not{p} - m + i\,\gamma_5 N + i\,\gamma_5 p\cdot\Delta)U(ps,\alpha\delta) = 0,$

(3.37) $\quad \left[\tilde{F}_o(-p) - m \right] V(ps,\alpha\delta) = (-\not{p} - m + i\,\gamma_5 N - i\,\gamma_5 p\cdot\Delta)V(ps,\alpha\delta) = 0.$

The spinors U and V can be factorized in the form

(3.38) $\quad\quad U_{\tau,\alpha\delta}(ps,\alpha'\delta') = u_\tau(ps, m_{\alpha'\delta'})u_{\alpha\delta}(p,\alpha'\delta'),$

(3.39) $\quad\quad V_{\tau,\alpha\delta}(ps,\alpha'\delta') = v_\tau(ps, m_{\alpha'\delta'})u_{\alpha\delta}(-p,\alpha'\delta'),$

where $u_\tau(ps, m_{\alpha'\delta'})$ and $v_\tau(ps, m_{\alpha'\delta'})$ are the usual Dirac spinors for a particle of mass $m_{\alpha'\delta'}$ and the quantities $u_{\alpha\delta}(p,\alpha'\delta')$ are, like for the boson case:

(3.40) $\quad\quad\quad u_{\alpha\delta}(p,\alpha'\delta') = \langle \alpha\delta | \overline{p,\alpha'\delta'} \rangle .$

The spinors U and V satisfy the orthogonality and completeness conditions

(3.41) $\quad U^+(\vec{p}s,\alpha\delta)(1+i\gamma_o\gamma_5\Delta_o)U(ps';\alpha'\delta') = \dfrac{P_o}{m_{\alpha\delta}}\,\delta_{ss'}\delta_{\alpha\alpha'}\delta_{\delta\delta'}$

(3.42) $\quad V^+(\vec{p}s,\alpha\delta)(1+i\gamma_o\gamma_5\Delta_o)V(ps',\alpha'\delta') = \dfrac{P_o}{m_{\alpha\delta}}\,\delta_{ss'}\delta_{\alpha\alpha'}\delta_{\delta\delta'}$

$$\sum_{\pm s,\alpha\delta} \frac{m_{\alpha\delta}}{P_o} \left[U(ps,\alpha\delta)U^+(ps,\alpha\delta) + V(ps,\alpha\delta)V^+(ps,\alpha\delta) \right] \times$$

(3.43)

$$\times\ (1 + i\gamma_o\gamma_5\Delta_o) = I.$$

Finally the quark propagator is

(3.44) $\quad\quad S_F(x-y) = \dfrac{1}{(2\pi)^4} \int d^4p\ e^{-ip(x-y)} \dfrac{1}{\tilde{F}_o(p) - m}$

4. - INTERACTIONS -

In order to derive the Neveu-Schwarz amplitude for N pions, we consider the interaction lagrangian

(4.1)
$$\mathcal{L}_I(x) = g \Phi^+(x) H_\mu \Phi(x) \partial^\mu \varphi(x),$$

where $\Phi(x)$ is the infinite component boson field and $\varphi(x)$ is the pseudoscalar field corresponding to the external pions. The asymmetry[15] existing in Eq. (4.1) between internal and external particles reflects an analogous asymmetry which is already present in the dual models, where the internal line states behave like the states of a bound system while the external states behave like elementary particles. On the other hand, if we want to describe the symmetric three-reggeon vertex by means of a lagrangian trilinear and symmetric in the field Φ, we will have a non local interaction, as visualized by the Harari-Rosner duality diagrams.

Considering now the amplitude corresponding to the Feynman graph of Fig. 1 we have essentially

(4.2)
$$A_N = g^{N-2} \langle \bar{0} | P_2 \cdot H \frac{1}{\tilde{L}_o(\pi_1) - \frac{1}{2}} P_3 \cdot H \dots \frac{1}{\tilde{L}_o(\pi_{N-3}) - \frac{1}{2}} P_{N-1} \cdot H | \bar{0} \rangle .$$

Performing now the unitary transformation $T(p)$ we obtain, by use of (3.5), (2.7) and (2.5):

(4.3)
$$A_N = \left[(\frac{c}{2})^{\mu^2} g \right]^{N-2} \langle 0 | V(p_2) \frac{1}{L_o(\pi_1) - \frac{1}{2}} V(p_3) \dots \frac{1}{L_o(\pi_{N-3}) - \frac{1}{2}} V(p_{N-1}) | 0 \rangle$$

where $\mu^2 = p_i^2$ is the external pion mass. In order to identify (4.3) with the Neveu-Schwarz amplitude (2.4) we must have $g = (c/2)^{-\mu^2}$, where c is the divergent constant introduced in Section 3. Therefore we would have $g \approx 0$ for $\mu^2 > 0$, while we have $g \approx \infty$ for the actual model where duality requires $\mu^2 = -1/2$.

For the fermion case we consider the interaction lagrangian

(4.4)
$$\mathcal{L}_I(x) = i g \bar{\psi}(x) \gamma_5 \Lambda \psi(x) \varphi(x)$$

For the amplitude of Fig. 2 we now have

(4.5)
$$A_N = g^N \bar{u}(q's') \langle \bar{0} | \gamma_5 \Lambda \frac{1}{\tilde{F}_o(\pi_1) - m} \gamma_5 \Lambda \dots \frac{1}{\tilde{F}_o(\pi_{N-1}) - m} \cdot \gamma_5 \Lambda | \bar{0} \rangle u(qs),$$

and after operating with the transformation T we obtain

(4.6)
$$A_N = \left[(\frac{c}{2})^{\mu^2} g \right]^N \bar{u}(q's') \langle 0 | \gamma_5 \Lambda V_o(p_1) \frac{1}{F_o(\pi_1) - m} \dots \gamma_5 \Lambda V_o(p_N) | 0 \rangle \cdot u(qs).$$

Again this amplitude coincides with (2.25) provided that $g = (c/2)^{-\mu^2}$ like for the boson case.

5. - CURRENTS AND CURRENT ALGEBRA -

In this Section we want to study the vector and axial currents of the field theory model we have consider so far, and we shall be particularly interested in current algebra and PCAC.

We consider first the simpler case of fermions. We shall assume isospin symmetry, each quark state belonging to an isospin doublet. From (3.29) we have for the isospin canonical vector currents

(5.1)
$$V^\mu_{a_i}(x) = \bar{\psi}(x) F^\mu \frac{\tau_a}{2} \psi(x),$$

where

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$$(5.2) \qquad F^{\mu} = \gamma^{\mu} + i\gamma_5 \Delta^{\mu}$$

satisfy the anticommutation relations

$$(5.3) \qquad \left\{ F^{\mu}, F^{\nu} \right\} = 2cg^{\mu\nu}.$$

We can easily calculate the equal time commutators (ETC) of the current components by making use of the anticommutation relations

$$(5.4) \qquad \left\{ \psi(x), \bar{\psi}(x') \right\}_{x_0 = x'_0} = \delta(\vec{x} - \vec{x}') \frac{1}{c} F^0,$$

and we obtain results quite analogous to the ordinary Dirac theory. Specifically we get

$$(5.5) \qquad \left[V_a^0(x), V_b^{\mu}(x') \right]_{x_0 = x'_0} = i\,\varepsilon_{abc} V_c^{\mu}(x)\,\delta(\vec{x} - \vec{x}'),$$

$$(5.6) \qquad \left[V_a^i(x), V_b^j(x') \right]_{x_0 = x'_0} = \delta(\vec{x} - \vec{x}') \left[i\,\varepsilon_{abc}\,\delta_{ij} V_c^0(x) + \right.$$
$$\left. + \frac{1}{2}\,\delta_{ab}\,\varepsilon_{ijk}\,\bar{\psi}(x) i F^k F^5 \psi(x) \right],$$

where we have set

$$(5.7) \qquad F^5 = \frac{i}{c^2} F^0 F^1 F^2 F^3.$$

There are different kinds of axial currents in this model. Writing in general

$$(5.8) \qquad A_a^{\mu}(x) = \bar{\psi}(x) F_5^{\mu} \frac{\tau_a}{2} \psi(x),$$

we can classify the covariants F_5^{μ} in two classes according to the behaviour of the current under the charge conjugation operator

$$(5.9) \qquad C = C_D (-1)^{\sum_{n=1}^{\infty}(\frac{1}{n} A_{-n}A_n + d_{-n}d_n)},$$

where C_D is the usual Dirac charge conjugation matrix. We have for example the following types of covariants F_5^{μ}, apart from terms containing derivatives:

$$(5.10) \qquad C = +1: \quad \gamma^{\mu}\gamma_5, F^{\mu}F^5, \sigma^{\mu\nu}d_{\nu}, \Delta\,d^{\mu}, \ldots$$
$$C = -1: \quad d^{\mu}, \gamma^{\mu}A\cdot d, \gamma_5 A^{\mu}, \ldots$$

Current algebra for the time components requires

$$(5.11) \qquad \frac{1}{c} F_5^0 F^0 F_5^0 = F^0,$$

and we see that none of the covariants of the second class (C = -1) listed in (5.10) can satisfy (5.11), while all the covariants of the first class, with appropriate combinations of the d's and normalization constants, can satisfy.

In order to restrict the choice we further require PCAC in the form

$$(5.12) \qquad \partial_{\mu} A_a^{\mu}(x) \propto i\bar{\psi}(x)\,\gamma_5 \Delta \frac{\tau_a}{2} \psi(x),$$

where the right-hand side is the pseudoscalar current which couples to the pion, according to (4.4).
We then remain with the covariant

$$(5.13) \qquad F_5^\mu = F^\mu \gamma_5 \Lambda,$$

which satisfies both (5.11) and (5.12). In fact using the wave equation (3.32) we obtain

$$(5.14) \qquad \partial_\mu A_a^\mu(x) = 2im\, \bar\psi(x)\, \gamma_5\, \Lambda\, \frac{\tau_a}{2}\, \psi(x).$$

We can easily calculate the ETC of current algebra and we find

$$(5.15) \qquad \left[V_a^0(x), A_b^\mu(x')\right]_{x_0=x'_0} = \left[V_a^\mu(x), A_b^0(x')\right]_{x_0=x'_0} = i\,\varepsilon_{abc}\, A_c^\mu(x)\, \delta(\vec x - \vec x'),$$

$$(5.16) \qquad \left[V_a^i(x), A_b^j(x')\right]_{x_0=x'_0} = \delta(\vec x - \vec x')\left[i\,\varepsilon_{abc}\,\delta_{ij}A_c^0(x) + \frac{1}{2}\delta_{ab}\,\varepsilon_{ijk}\,\bar\psi(x)iF^k F^5 \gamma_5\,\Lambda\,\psi(x)\right],$$

$$(5.17) \qquad \left[A_a^\mu(x), A_b^\nu(x')\right]_{x_0=x'_0} = \left[V_a^\mu(x), V_b^\nu(x')\right]_{x_0=x'_0}$$

We now consider the boson case. Here the Fock space is the sum of two subspaces corresponding to the I = 0 and I = 1 states; correspondingly the field $\Phi_\sigma(x)$ will be labelled with an isospin index σ with $\sigma = 0$ for the I = 0 component and $\sigma = 1, 2, 3$ for the I = 1 components. The isovector field has an expansion like (3.19) where the sum is restricted to the states α', β' with $R_{\alpha'\beta'} = 2n$ (odd G states) and $R_{\alpha'\beta'} = 2n + 1/2$ (even G states) and the isoscalar field has a similar expansion over the other states. The commutation relation (3.17) will be replaced by

$$(5.18) \qquad \left[\Phi_\sigma(x), \pi_\tau(x')\right]_{x_0 = x'_0} = i\,\delta(\vec x - \vec x')\,\delta_{\sigma\tau}\, P_I,$$

where $P_I = 1/2\left[1+(-1)^I \tilde C G\right]$ is the projector on the subspace of isospin I and is explicitly given by

$$(5.19) \qquad \sum_{(\alpha\beta)_I,\ \varepsilon=\pm} \frac{1}{2}cm_{\alpha\beta}\, u(\varepsilon p, \alpha\beta) u^+(\varepsilon p, \alpha\beta) = P_I,$$

which replaces the completeness relation (3.24). The canonical vector current of isospin obtained from (3.12) is

$$(5.20) \qquad V_a^\mu(x) = \varepsilon_{abc}\, \Phi_c^+(x)\, (-c\overleftrightarrow\partial^\mu + i\Gamma^\mu)\Phi_b(x).$$

From this we can easily obtain the following ETC:

$$(5.21) \qquad \left[V_a^0(x), V_b^0(x')\right]_{x_0=x'_0} = i\,\varepsilon_{abc}\, V_c^0(x)\, \delta(\vec x - \vec x'),$$

$$(5.22) \qquad \left[V_a^0(x), V_b^i(x')\right]_{x_0=x'_0} = i\,\varepsilon_{abc}\, V_c^i(x)\, \delta(\vec x-\vec x') + ic(2\,\delta_{ab}\,\delta_{cd} - \delta_{ac}\,\delta_{bd} - \delta_{ad}\,\delta_{bc})\times$$
$$\times \Phi_c^+(x)\,\Phi_d(x)\,\partial^i\,\delta(\vec x - \vec x'),$$

$$(5.23) \qquad \left[V_a^i(x), V_b^j(x')\right]_{x_0=x'_0} = 0.$$

The vector current (5.20) when inserted in a dual graph gives rise, after the T transformation, to the current vertex

$$(5.24) \qquad V^\mu(q) = (\tfrac{c}{2})^{q^2} e^{-\sqrt{2}\,q \sum\limits_{n=1}^{\infty} \frac{A_{-n}}{n}} (\pi^\mu + \pi'^\mu + \Gamma^\mu) e^{\sqrt{2}\,q \sum\limits_{n=1}^{\infty} \frac{A_n}{n}},$$

where $q = \pi' - \pi$ is the momentum of the current and π and π' are the momenta of the adjacent meson lines. The factor $(c/2)q^2$ is a divergent form factor and represents the obvious extension of the factor $(c/2)\mu^2$ for the pion vertex in (4.3). Once again the presence of this divergence is a consequence[16] of the infinite number of oscillator modes in the dual models. To eliminate such a divergence one has the unpleasant choice of inserting a factor $(c/2)-q^2$ in the interaction.

For the axial current the situation is much more complicated. PCAC would suggest, from the pion interaction (4.1), an axial current of the form $\Phi^+ H^\mu \Phi$, but this would give vanishing ETC. Since we are interested in current algebra we shall take for the time component

$$(5.25) \qquad A_a^o(x) = -iT_a^{\sigma\tau} \left[\pi_\sigma(x) G \Phi_\tau(x) - \Phi_\sigma^+(x) G^+ \pi_\tau^+(x) \right],$$

where T_a are hermitian 4×4 matrices and G is a pseudoscalar, odd G parity operator. The only operators having these properties are of the type A.b, apart from terms containing derivatives which would give Schwinger terms in the ETC. It then follows that G is non diagonal in the energy representation and will in general connect $I = 0$ to $I = 1$ states.

The current algebra relations for the time components

$$(5.26) \qquad \left[V_a^o(x), A_b^o(x') \right]_{x_o = x'_o} = i\, \varepsilon_{abc} A_c^o(x)\, \delta(\vec{x} - \vec{x}'),$$

$$(5.27) \qquad \left[A_a^o(x), A_b^o(x') \right]_{x_o = x'_o} = i\, \varepsilon_{abc} V_c^o(x)\, \delta(\vec{x} - \vec{x}'),$$

require the following conditions:

$$(5.28) \qquad \left[P_1 I_a, GT_b \right] = i\, \varepsilon_{abc} GT_c,$$

$$(5.29) \qquad G \left[(T_a^{\varrho s} T_b^{s\tau} - T_b^{\varrho s} T_a^{s\tau}) P_1 + (T_a^{\varrho o} T_b^{o\tau} - T_b^{\varrho o} T_a^{o\tau}) P_o \right] G =$$
$$= i\, \varepsilon_{abc} P_1 I_c^{\varrho\tau},$$

where I_a are the isospin matrices for the reducible representation with $I = 0, 1$. We can immediately see that (5.28) and (5.29) cannot be identically satisfied in operator form. In fact the left-hand side of (5.29) has non diagonal matrix elements, while the right-hand side is diagonal.

In spite of this difficulty we shall find that the current algebra relations (5.26) and (5.27) and also PCAC can be satisfied between physical states, by making use of the gauge conditions.

A simple solution of Equations (5.28) and (5.29) is to take the T matrices satisfying

$$(5.30) \qquad \left[I_a, T_b \right] = i\, \varepsilon_{abc} T_c,$$

$$(5.31) \qquad T_a^{rs} T_b^{st} - T_b^{rs} T_a^{st} = T_a^{ro} T_b^{ot} - T_b^{ro} T_a^{ot} = i\, \varepsilon_{abc} I_c^{rt},$$

and the operator G such that

$$(5.32) \qquad \pi_\varrho(x) G^2 \Phi_\tau(x) \approx \pi_\varrho(x)\, \Phi_\tau(x),$$

where $\approx$ means that the two sides are equal between physical states. A solution for (5.30) and (5.31) is the following

$$(5.33) \qquad T_a^{rs} = I_a^{rs} = i\, \varepsilon_{ras};$$

$$(5.34) \qquad T_a^{or} = T_a^{ro} = \delta_{ra}; \quad T_a^{oo} = 0.$$

For G we shall take an expression of the form

$$(5.35) \qquad G = \sum_{m=-M}^{M} a_m \widetilde{G}_m,$$

where M is an arbitrary half integer and $\widetilde{G}_m = \widetilde{G}_m(i\partial)$ are the gauge operators. The independence of G on the derivatives requires for (3.15)

$$(5.36) \qquad \sum_{m=-M}^{M} a_m = 0.$$

For G^2 we obtain, from (2.16):

$$(5.37) \qquad G^2 = \sum_{n=-2M}^{2M} c_n \widetilde{L}_n,$$

where

$$(5.38) \qquad c_n = \sum_m a_m a_{n-m}.$$

Since $\sum_n c_n = (\sum_m a_m)^2 = 0$, we have:

$$(5.39) \qquad G^2 = -\sum_n c_n \widetilde{W}_n = -\sum_{n=2}^{2M} (c_n \widetilde{M}_n + c_{-n} \widetilde{M}_{-n}) - \widetilde{W}_1 \sum_{n=1}^{2M} n c_n - \widetilde{W}_{-1} \sum_{n=1}^{2M} n c_{-n},$$

where $\widetilde{M}_n = M_n(0)$ are the operators defined in (3.10). Since $\Gamma = i[L_o, A]$ transforms under the Virasoro algebra like a $J = -1$ representation (apart from the zero mode contribution) it commutes with $\widetilde{M}_n$ and we have for physical states

$$(5.40) \qquad \left[\widetilde{M}_{-n} + \frac{1}{2}(n-1)\right] \Phi^{(+)}(x) |\psi\rangle = 0; \quad \langle\psi| \pi^{(-)}(x) \left[\widetilde{M}_n + \frac{1}{2}(n-1)\right] = 0.$$

Therefore we can satisfy (5.32) provided that

$$(5.41) \qquad \sum_{n=1}^{2M} n c_n = 0; \quad c_o = 2.$$

From (5.36) and (5.41) it also follows $\sum_{n=1}^{2M} n c_{-n} = 0$.

We can easily realize that G cannot be taken hermitian, i.e. with $a_{-m} = a_m^*$. In fact since we have

$$(5.42) \qquad \sum_{n=1}^{2M} n c_n = -\sum_{m=-M}^{M} (\sum_{k=0}^{2M} a_{m-k})(\sum_{k=0}^{2M} a_{k-m}),$$

this expression could not vanish except for all $a_m = 0$.

The axial current which results from the preceding discussion is of the form

14.

(5.43) $A_a^\mu(x) = -iT_a^{\sigma\tau} \Phi_\sigma^+(x)\left[(c\overleftarrow{\partial}^\mu + \frac{i}{2}\Gamma^\mu)G - G^+(c\overrightarrow{\partial}^\mu - \frac{i}{2}\Gamma^\mu)\right]\Phi_\tau(x),$

to which corresponds the dual vertex

(5.44) $A^\mu(q) = (\frac{c}{2})q^2\left[(\pi'^\mu + \frac{1}{2}\Gamma^\mu)V_o(q)G(\pi) + G^+(\pi')V_o(q)(\pi^\mu + \frac{1}{2}\Gamma^\mu)\right],$

where $G(\pi) = \sum_m a_m G_m(\pi)$ and the other notations are the same as in Eq. (5.24).

    Expression (5.43) simplifies greatly between physical states in virtue of (3.14) and reduces to the form required by PCAC, provided that we have for the a_m the further condition

(5.45) $$\sum_{m=\frac{1}{2}}^{M} a_m = 0.$$

In fact in this case we obtain

(5.46) $$A_a^\mu(x) \approx -\frac{1}{\sqrt{2}} \sum_{m=\frac{1}{2}}^{M} m\,(a_m + a_m^*)\,T_a^{\sigma\tau}\Phi_\sigma^+(x)H^\mu\Phi_\tau(x).$$

    Since (5.33) and (5.34) can also be expressed in the form

(5.47) $$T_a^{\sigma\tau} = \frac{1}{2}T_r(\tau_\sigma\tau_a\tau_\tau),$$

with $\tau_o = 1$, (5.46) gives the axial current suggested by PCAC with the correct isospin factor of the Chan-Paton[17] form.

    In conclusion, we have found a class of axial currents for bosons of the form (5.43), which satisfy the local SU(2)xSU(2) current algebra without Schwinger terms, but only in a weak sense, i.e. between external real states. PCAC is also satisfied in the weak sense. To this purpose we want to remark that since the reduced form (5.46) for the axial current would give zero ETC, the spurious states play a crucial role in the saturation of the current algebra commutators.

    We finally observe that the operator G, resulting from (5.35), (5.36), (5.41) and (5.45), depends for any given M, on 2M-3 arbitrary complex parameters. It turns out that a consistent solution for G, leading also to a non zero PCAC constant in (5.46), exists only for $M \geqslant 5/2$.

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