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E. Etim[(x)]: TEMPERATURE-DEPENDENT GREEN'S FUNCTION APPROACH TO SCALING IN LEPTON NUCLEON SCATTERING.-

ABSTRACT.-

From the rather suggestive form of the weak and electromagnetic structure functions of the nucleon in the parton model and from field-current identity a new approach to deep scattering is proposed in which the structure functions are grand canonically averaged temperature-dependent Green's functions of currents. The aim is not to reproduce scaling but to show that if the structure functions do not scale trivially to zero in the Bjorken limit then the quasi periodic boundary condition between the retarded and advanced Green's functions imply an interesting proportionality relation between the neutrino and antineutrino nucleon structure functions referred to as form invariance. Form invariance follows if there exists a maximum temperature of hadronic matter above which thermal equilibrium is impossible and in the region of which the chemical potential diverges. From the relation of the Green's functions to the densities of energy and particle states we deduce from the boundary condition that the one must tend asymptotically to the other. Some applications are also discussed.

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## 1. - INTRODUCTION. -

All known models[1] of deep inelastic neutrino (-) and antineutrino (+) nucleon scattering are plagued by two kinds of arbitrariness. The first is in the ratio of $W_j^+(\nu, q^2)$ to $W_j^-(\nu, q^2)$ and the second in the ratio of $W_3^{\pm}(\nu, q^2)$ to $W_2^{\pm}(\nu, q^2)$, where $W_j^{\pm}(\nu, q^2)$ ($j = 1, 2, 3$) are the familiar weak structure functions of the nucleon. In this paper we propose a new approach to deep interaction dynamics and scale invariance based on the idea of temperature-dependent Green's functions of currents averaged over a grand canonical ensemble which eliminates this arbitrariness. This is possible because the structure functions $W_j^{\pm}(\nu, q^2)$ are invariants in the tensor decomposition of the retarded and advanced Green's functions of the weak hadronic currents and these latter are related through a periodic boundary condition. The idea of employing a grand canonical ensemble is suggested by the parton model[1] in which the structure functions are averages over assemblies consisting of different numbers of constituents of the nucleon. To obtain the periodic boundary condition in a compact form it is necessary to assume the asymptotic identity between fields and currents[2]. The structure functions $W_j^+(\nu, q^2)$ and $W_j^-(\nu, q^2)$ become asymptotically proportional, that is isotopically form invariant, in the scaling limit if there exists a limiting (maximum) temperature $T_c$ of hadronic matter in the region of which the average mass per particle or the chemical potential $\mu(T)$ of the hadronic system, together with its fluctuations diverge. Such divergences in the region of a critical temperature are the most fundamental aspect of scaling associated with a phase transition[3]. They are universal in that they always occur in one form or another in all systems, irrespective of their physical nature, which undergo phase changes. In the present case they admit of a critical index which is just the parameter that constrains the hadron mass spectrum to grow no faster than exponentially. In Veneziano type models[4] this parameter is fixed by the dimension of the space of certain vector operators whose eigenvalues are related to the mass of the hadrons. For the index so determined to give singularities of $\mu(T)$ as $T \to T_c$, if only logarithmically, the critical temperature $T_c$ would have to be much higher (circa 240 MeV) than the value of between 140 and 160 MeV used in most fits; to have a pole or branch point singularity, $T_c$ has to be even higher, ranging from 300 MeV to infinity. In section (2) we introduce the Green's functions and derive form invariance of the structure functions. In section (3) we discuss the relation of the retarded and advanced Green's functions to the density of energy levels and the density of particle states respectively and show thereby the equivalence of universal scaling and the familiar statistical bootstrap of the hadron mass spectrum. Section (4) is dedicated to applications.

## 2. - FINITE TEMPERATURE GREEN'S FUNCTIONS OF CURRENTS. -

In the parton model[1] the neutrino (-) and antineutrino (+) nucleon (B) structure functions $W_j^{\pm}(\nu, q^2;B)$ $(j = 1, 2, 3)$ are statistical averages, referred to as configuration mixing, over assemblies with different numbers of partons in the nucleon.

$$W_j^{\pm}(\nu, q^2;p) = \sum_{N,\alpha} p_\alpha(N)\, W_{j\alpha N}^{\pm}(p) \tag{1.1}$$

$$W_j^{\pm}(\nu, q^2;n) = \sum_{N,\alpha} n_\alpha(N)\, W_{j\alpha N}^{\pm}(n) \tag{1.2}$$

where

$$W_{j\alpha N}^{\pm}(B) = \sum_k N_k(B) \left| I_k^{\pm} \right|^2 f_k^N\left(\frac{q^2}{2M\nu}\right) \varphi_{jk}(\nu, q^2) \tag{2}$$

with

$$\varphi_{1k}(\nu, q^2) = 1/M \tag{3.1}$$

$$\varphi_{2k}(\nu, q^2) = q^2/M\nu^2 \tag{3.2}$$

$$\varphi_{3k}(\nu, q^2) = 2\,\xi_k/\nu \tag{3.3}$$

The parameter $\xi_k$ is equal to +1 if the parton k is an antibaryon and -1 if a baryon; $N_k(B)$ is the total number of partons of type k in a state $(\alpha, N)$ of the nucleon with a given distribution $\alpha \equiv (N_k(B))$ of the number of partons of each type subject to the condition of having 3 valence quark partons and N pairs of quarks and antiquarks

$$\sum_k N_k(B) = 2N + 3 \tag{4}$$

The Cabibbo angle is set equal to zero so that $I_k^{\pm}$ in eq. (2) are the matrix elements of the isospin raising and lowering operators $I^{\pm}$ from the state of the parton k to others; $f_k^N(x)$ is the probability density normalised to unity

$$\int_0^1 f_k^N(x)\,dx = 1 \tag{5}$$

that a parton has a fraction $x = q^2/2M\nu$ of the nucleon's longitudinal momentum and $p_\alpha(N)$ ($n_\alpha(N)$) is the probability of finding a proton (neutron) in the state $(\alpha, N)$.

In the following we reinterprete the averaging in eqs. (1) as performed over a grand canonical ensemble representative of the nucleon as an interacting composite system. This enables us to carry out the averaging directly on matrix elements of the hadronic weak currents $J_\varsigma^{\pm}(x)$ without the need to introduce partons and the like. If $T = 1/\beta$ is the temperature of the composite system with chemical potential $\mu(T)$ we define the following Green's functions of currents[5]

$$W^{\pm}_{\varsigma\tau}(x,y;B) = \frac{\sum_\alpha \langle B(\alpha)|J_\varsigma^{\pm}(x)\,J_\tau^{\mp}(y)|B(\alpha)\rangle\, e^{-\beta(E_\alpha - \mu N_\alpha)}}{e^{-\beta(E_\alpha - \mu N_\alpha)}} =$$

$$= \frac{T_r(\exp(-\beta(H-\mu N))\,J_\varsigma^{\pm}(x)\,J_\tau^{\mp}(y))}{T_r(\exp(-\beta(H-\mu N)))} \tag{7}$$

$$= \langle J_\varsigma^{\mp}(x)\,J_\tau^{\mp}(y)\rangle$$

where $E_\alpha$, $N_\alpha$ are the eigenvalves of the Hamiltonian H and number operator N for a nucleon in the state $(\alpha, N)$. Using time translational invariance and the cyclic invariance of the trace we find

$$W^{\pm}_{\varsigma\tau}(x,y;B) = \langle e^{-\beta\mu N}\,J_\tau^{\mp}(\vec{y}, y_0 + i\beta)\,e^{\beta\mu N}\,J^{\pm}(x)\rangle \tag{7'}$$

If the $J_\varsigma^{\pm}(x)$ were particle field creaction (+) and annihilation (-) operators we would have from the commutation relations

(8) $$\left[N, J^{\pm}_{\varrho}(x)\right] = \pm J^{\pm}_{\varrho}(x)$$

(9) $$e^{-\beta\mu N} J^{\mp}_{\tau}(\vec{y}, y_o + i\beta)\, e^{\beta\mu N} = e^{\pm\beta\mu} J^{\mp}_{\tau}(\vec{y}, y_o + i\beta)$$

In the spirit of field-current identity[2] we assume that asymptotically the currents $J^{\pm}(x)$ behave like particle fields so that eq. (9) is valid. Substituting from it into (7') and setting $y_o = 0$ gives

(10) $$W^{\pm}_{\varrho\tau}(\vec{x}, x_o; \vec{y}, 0; B) = e^{\pm\beta\mu} \langle J^{\mp}_{\tau}(y, i\beta)\, J^{\pm}_{\varrho}(x) \rangle$$

Taking the Fourier transform of (10) and making use of the relations[1a]

(11.1) $$W^{\pm}_{\varrho\tau}(p,q) = \frac{1}{4\pi} \int d^4x\, e^{iqx} \langle p | J^{\pm}_{\varrho}(x)\, J^{\mp}_{\tau}(0) | p \rangle = \left[W^{\pm}_{\tau\varrho}(p,q)\right]^{x}$$

(11.2) $$W^{\pm}_{\varrho\tau}(p,q) = \left[W^{\pm}_{\varrho\tau}(-\vec{p}, p_o; -\vec{q}, q_o)\right]^{x}$$

where p is the 4-momentum of the nucleon and q the 4-momentum transfer, yields

(12.1) $$W^{+}_{\varrho\tau}(p,q) = e^{\beta(\mu - \nu)}\, W^{-}_{\varrho\tau}(p,q)$$

if $J^{+}_{\varrho}(x)$ is a creation operator and

(12.2) $$W^{+}_{\varrho\tau}(p,q) = e^{-\beta(\mu - \nu)}\, W^{-}_{\varrho\tau}(p,q)$$

if it is an annihilation operator. Eqs. (12) are the so-called quasi periodic boundary conditions familiar from statistical mechanics[5]. If, as $q^2$ and $\nu \to \infty$ with $q^2/2M\nu$ finite, the two sets of scale functions $F^{\pm}_{j}(\omega = q^2/2M\nu)$ defined by

(13.1) $$F_1^{\pm}(\omega) = M W_1^{\pm}(\nu, q^2)$$

(13.2) $$F_2^{\pm}(\omega) = \nu W_2^{\pm}(\nu, q^2)$$

(13.3) $$F_3^{\pm}(\omega) = \nu W_3^{\pm}(\nu, q^2)$$

do not both tend to zero, eqs. (12) will be valid if

(14.1) $$\beta = \frac{1}{T} \xrightarrow[\nu \to \infty]{} 0; \qquad \beta(\mu - \nu) = \text{const.}$$

(14.2) $$\beta = \frac{1}{T} \xrightarrow[\nu \to \infty]{} \beta_c = 0; \ \mu(T) \xrightarrow[T \to T_c]{} \mu_c < \infty$$

(14.3) $$\beta = \frac{1}{T} \xrightarrow[\nu \to \infty]{} \beta_c = 0; \qquad (\mu(T) - \nu) \xrightarrow[T \to T_c]{} \mu_o \neq 0$$

In the first case the temperature tends to infinity and there is little else to add. In the second case there exists a limiting (maximum) chemical potential $\mu_c$ and a limiting temperature $T_c$ above which thermal equilibrium is impossible. From eqs. (12) and (13) this means that either the $F_j^+(\omega)$ all tend to zero with the $F_j^-(\omega)$ non-vanishing or vice versa. For consistency Veneziano type models should reproduce this result; the first prediction of this kind was given by Landshoff[6] with $F_j^-(\omega;B) = 0$ for B equal to the proton and $F_j^+(\omega;B)$ non-zero. According to eq. (14.3) both sets of structure functions $F_j^{\pm}(\omega)$ are different from zero and for fixed j (j = 1, 2, 3) all four of them become proportional to each other

(15.1) $$F_j^+(\omega;p) = \chi F_j^-(\omega;p) = F_j^-(\omega;n)$$

(15.2) $$F_j^+(\omega;n) = \chi^{-1} F_j^-(\omega;n) = F_j^-(\omega;p)$$

where we have set

(16) $$\chi = e^{\beta_c \mu_o}$$

and made use of charge symmetry so that if eq. (12.1) is valid for the proton eq. (12.2) holds for the neutron. Eq. (14.3) is obviously the most interesting for besides a maximum temperature $T_c$ it predicts a divergence of the chemical potential $\mu(T)$ as $T \rightarrow T_c$. Only this case therefore corresponds to a universal asymptotic behaviour characteristic of scaling[3]. An important consequence of this universality is the form invariance of the structure functions as given in eqs. (15). The proportionality constant is fixed completely by thermodynamics while the interaction contained in the functions $F_j^-(\omega;p)$ are the same for neutrino and antineutrino scattering. The interaction is thus asymptotically independent of the charge of the exchanged intermediate vector bosons $W^\pm$ and of that of the nucleon target. The situation is as if the weak and electromagnetic interactions become asymptotically strong and consequently manifest a weak form of charge independence[7].

## 3. - SCALING AND THE HADRONIC MASS SPECTRUM. -

The Green's function method introduced in the last section is, by the nature of the averaging involved, a bootstrap theory of hadrons in which any one hadron is a composite of others. Scale invariance is a constraint in this theory. Two important consequences of this constraint are the existence of a maximum temperature $T_c$ of hadronic matter and the divergence of the chemical potential $\mu(T)$ as $T \rightarrow T_c$. But there are others and we shall investigate these now.

If $\varepsilon_\rho^{(\lambda)}$ is the spin vector of helicity $\lambda$ ($\lambda = 0, \pm 1$) of the intermediate vector bosons $W^\pm$, then by definition $\varepsilon_\rho^{(\lambda)} \varepsilon_\tau^{(\lambda)} W^-_{\rho\tau}(\nu, q^2)$ is proportional to the transition probability for a process in which a particle (current) of spin vector $\varepsilon_\rho^{(\lambda)}$ and 3-momentum $\vec{q}$ when added to the composite nucleon system increases its energy by $\nu$. This transition probability in turn is proportional to the density of particle states $\rho(E)$ available for the added particle. In the language of nuclear physics this corresponds to particle capture. Similarly $\varepsilon_\rho^{(\lambda)} \varepsilon_\tau^{(\lambda)} W^+_{\rho\tau}$ x x$(\nu, q^2)$ is proportional to the probability for a process in which the same particle is removed (emitted) leading to a decrease of energy by $\nu$. This corresponds to a nuclear particle decay and the associated probability is proportional to the density of energy states $\sigma(E)$ into which the compound system can decay. From the condition of detailed balancing which is a direct consequence of our use of an equilibrium ensemble we find

$$\frac{\varepsilon_\rho^{(\lambda)} \varepsilon_\tau^{(\lambda)} W^+_{\rho\tau}(\nu, q^2)}{\varepsilon_\rho^{(\lambda)} \varepsilon_\tau^{(\lambda)} W^-_{\rho\tau}(\nu, q^2)} = \frac{\sigma(E)}{\rho(E)} = \chi \tag{17}$$

so that asymptotically the hadron mass spectrum $\varrho(m)$ tends to the density of energy states $\sigma(m)$. This result is generally referred to as statistical bootstrap and was first postulated by Hagedorn in the form[8]

$$\frac{\log \varrho(m)}{\log \sigma(m)} \xrightarrow[m \to \infty]{} \text{const} \simeq 0(1) \tag{18}$$

The necessity of having both $W^{\pm}_{\varrho\tau}(\nu, q^2)$ different from zero in the Bjorken limit follows immediately from eq. (17). It is well known that this bootstrap can only be satisfied by an exponentially growing mass spectrum[8]

$$\varrho(m) = C\, m^{-b} \exp(\beta_c m) \tag{19}$$

where b is a positive real number and $T_c = 1/\beta_c$ the maximum temperature of hadronic matter. The parameter b is unknown: however the condition of universal scaling, that is the divergence of $\mu(T)$ in the Bjorken limit constrains it to lie in the interval $0 \leq b \leq 2$. To see this consider the grand partition function $Z(\beta) = Z(\beta, \mu = \mu(\beta))$

$$\begin{aligned} Z(\beta) &= T_r(\exp(-\beta(H-\mu N)) \\ &= \exp(\beta\mu\bar{N})\, Z'_c(\beta) \end{aligned} \tag{20}$$

where $\bar{N}$ is the average number of hadrons (partons) in the nucleon and $Z'_c(\beta)$ is defined by

$$Z'_c(\beta) = T_r(\exp(-\beta(H-\mu(N-\bar{N})))) \tag{21}$$

Let $\bar{E}(T)$ be the average energy with respect to the partition function $Z(\beta)$ at the temperature T and $\bar{E}_c(T)$ the average energy with respect to $Z'_c(\beta)$. Taking the derivative of (20) with respect to $\beta$ yields

$$\begin{aligned} -\frac{\partial(\log Z)}{\partial\beta} &= \bar{E}(T) - \mu\bar{N} - \beta\bar{N}\frac{\partial\mu}{\partial\beta} \\ &= \bar{E}_c(T) - \mu\bar{N} - \beta\bar{N}\frac{\partial\mu}{\partial\beta} - \beta\mu\frac{\partial\bar{N}}{\partial\beta} \end{aligned} \tag{22}$$

whence

$$\bar{N}(T) = \bar{N}_c - \int_T^{T_c} \frac{\bar{E}(t) - \bar{E}_c(t)}{t\mu(t)}\, dt \tag{23}$$

with $\bar{N}_c$ an integration constant. $\bar{N}(T)$ is thus an increasing function of T which becomes sharp and maximum (equal to $\bar{N}_c$) at $T = T_c$. The fluctuation $\overline{\Delta N}^2 = \bar{N}^2 - \bar{N}^2$ in $\bar{N}(T)$ is therefore small. This result could have been established directly by observing that $\bar{N}(T)$ is canonically conjugate to $\mu(T)$ and since the latter diverges in the scaling limit $\bar{N}(T)$ becomes sharp and hence has vanishing fluctuation. This result allows us to approximate $Z_c'(\beta)$ by a canonical partition function $Z_c(\beta)$ and write

$$Z_c'(\beta) \simeq Z_c(\beta) + O(\overline{\Delta N}^2) \tag{21'}$$

$$\begin{aligned} Z_c(\beta) &= \mathrm{Tr}(\exp(-\beta H)) \\ &= \int_{E_o}^{\infty} \sigma(\mathcal{E}) \exp(-\beta\mathcal{E})\, d\mathcal{E} = \int_{m_o}^{\infty} \rho(m) \exp(-\beta m) dm \end{aligned} \tag{24}$$

where the second equality follows from (17) with the constant $\chi$ suppressed and kinetic energy terms neglected. In simple forms of the parton model[1] $\bar{N}_c$ is related to the average value of the Bjorken variable through the normalisation (cf. eqs. (1) and (2))

$$\int_o^1 \omega\, f_k^N(\omega)\, d\omega = \bar{\omega}_\alpha(N) = \frac{1}{N} \tag{25.1}$$

$$\sum_{N,\alpha} p_\alpha(N)\, \bar{\omega}_\alpha(N) = \bar{\omega} = \frac{1}{\bar{N}(T)} \xrightarrow[T \to T_c]{} \frac{1}{\bar{N}_c} \tag{25.2}$$

where N stands for the total number of partons in the configuration $\alpha$. Any way one looks at it $\bar{N}_c$ is finite so that from the relation[9]

(26) $$\bar{E}(T) = \mu(T)\bar{N}(T) \simeq \mu(T)\bar{N}_c \simeq \bar{E}_c(T)$$

it follows that the average energy $\bar{E}(T)$ diverges as $T \rightarrow T_c$ since $\mu(T)$ does so.

The average $\bar{E}_c(T)$ is easily computed from eqs. (19), (21') and (24) with the results[8b]

(27.1) $$\bar{E}_c(T0 = E_o + \frac{T T_c}{T_c - T}, \qquad b = 0$$

(27.2) $$\lim_{T \rightarrow T_c} \bar{E}_c(T) = (T_c - T)^{-1+\varepsilon}, \qquad 0 \leq \varepsilon < 1, \quad 0 \leq B < 2,$$

(27.3) $$\lim_{T \rightarrow T_c} \bar{E}_c(T) = -E_o \log\left(\frac{T_c - T}{T_o}\right), \qquad b = 2$$

(27.4) $$\bar{E}_c(T = T_c) = E_o\left(\frac{b-1}{b-2}\right), \qquad b > 2$$

In all cases $\bar{E}_c(T)$ is an increasing function of T with a finite maximum at $T_c$ for $b > 2$ and a singularity for $b \leq 2$. Because of the dependence of the singularity structure of $\mu(T)$, from eqs. (26) and (27), on the parameter b we refer to this latter simply as the critical index in conformity with the use in statistical mechanics[3]. With this thermodynamical meaning for b one can also characterise the Bjorken limit dynamically as that in which the temperature becomes critical, i.e. $T \rightarrow T_c$.

Eq. (27.4) corresponds to (14.2) and hence to the case in which either $W_j^+(\nu, q^2)$ or $W_j^-(\nu, q^2)$ tends to zero[6]. Values of the critical index for which this happens are given by Veneziano-type models. In these models the number of particle states with mass $m_h = (h/\alpha')^{1/2}$, $\alpha' = 1\,GeV^{-2}$, is equal to the degeneracy of the eigenvalue h of the operator

(28) $$L = \sum_{\tau=1}^{d} \sum_{l=1}^{\infty} l\, a^{+}_{\tau l}\, a_{\tau l}$$

$a^+_{\tau 1}$ and $a_{\tau 1}$ are an infinite set of creation and annihilation operators in a vector space of dimension d. For $h \to \infty$ the degeneracy is given by[4]

$$(29) \qquad \rho(m) \xrightarrow[m \to \infty]{} \frac{1}{\sqrt{2}} \left(\frac{d}{24}\right)^{(1+d)/4} m^{-(1+d)/2} \exp\left(2\pi \sqrt{\frac{d\alpha'}{6}}\, m\right)$$

which on comparison with (19) gives

$$(30.1) \qquad T_c = 1/\beta_c = \frac{1}{2\pi} \sqrt{\frac{6}{d\alpha'}}$$

$$(30.2) \qquad b = \frac{1}{2}(1+d)$$

If $a_{\tau 1}$ and $a^+_{\tau 1}$ are Lorentz vector operators $d=4$ and $b=5/2$ but if d is computed from (30.1) using the experimental value of $T_c$ between 140 and 160 MeV d lies between 6 and 8 and b between 7/2 and 9/2, all of which values give non-singular behaviour of $\mu(T)$. To produce a singularity of $\mu(T)$ at most three modes of the vector operators $a_{\tau 1}$ and $a^+_{\tau 1}$ can contribute to the energy operator in eq. (28). From eq. (30.1) this will raise the critical temperature to at least 220 MeV. Stringent conditions are therefore imposed on any form of Ward-like identities in the dual resonance model which must not only produce the critical index but must also ensure a renormalization of the critical temperature.

In this section the intrinsically dynamical nature of the observed scaling of the lepton-nucleon structure functions in the Bjorken limit has been emphasized through some of its important consequences. This is to be contrasted with the purely abstract approach through conformal invariance[10]. It is however known that Bjorken scaling does not follow from conformal invariance alone but that it is a dynamical property[11]. As discussed previously scaling is a bootstrap constraint; when applied to conformal symmetry it is a requirement on the spectrum of allowed representations of the conformal algebra on the light cone[11]. Rather than delve into these matters we give in the next section some simple applications of the ideas developed up till now.

## 4. - APPLICATIONS. -

To compute neutrino and antineutrino nucleon cross sections one has to know , besides eq. (15), the ratio of $W_3^{\pm}(\nu, q^2)$ to $W_2^{\pm}(\nu, q^2)$.

Using the Callan-Gross relation[12]

$$(31) \qquad \omega F_1^{\pm}(\omega) = \frac{1}{2} F_2^{\pm}(\omega)$$

this ratio is related to how the inequalities

$$(32) \qquad 0 \leq \frac{\omega}{2} \left| F_3^{\pm}(\omega) \right| \leq \omega F_1^{\pm}(\omega) \leq \frac{1}{2} F_2^{\pm}(\omega)$$

are satisfied, in otherwords it depends on the model one has for the vector-axial vector interference terms in the matrix elements of equation (7). Quite generally one can therefore write

$$(33) \qquad \omega \left| F_3^{\pm}(\omega) \right| = \Lambda(\omega) F_2^{\pm}(\omega)$$

where

$$(34) \qquad 0 \leq \Lambda(\omega) \leq 1$$

and use has been made of eqs. (15) which imply

$$(35) \qquad \Lambda^{+}(\omega) = \Lambda^{-}(\omega) = \Lambda(\omega)$$

From the parton model formulae, eqs. (1) and (2), one easily derives the Adler[13], Bjorken[14] and Gross-Llewellyn sum rules[1a] respectively

$$(36.1) \qquad \int_0^1 (F_{2p}^{+}(\omega) - F_{2p}^{-}(\omega)) \frac{d\omega}{\omega} = 2$$

$$(36.2) \qquad \int_0^1 (F_{3p}^{+}(\omega) + F_{3p}^{-}(\omega)) d\omega = -6$$

where the first two sum rules summarised in (36.1) are the same in the Bjorken limit as a result of (31). The Llewellyn Smith equality[15], which also follows from eqs. (1) and (2), becomes in integral form

(37)
$$\frac{1}{12}\int_0^1 (F^-_{3p}(\omega) - F^-_{3n}(\omega)) = \int_0^1 (F_{1p}(\omega) - F_{1n}(\omega)))\,d\omega =$$
$$= \frac{1}{6} + \frac{1}{3}\sum_{N,\alpha} p_\alpha(N)(N_1(p) - N_2(p))$$

where $F_1(\omega)$ and $F_2(\omega)$ are the electroproduction structure functions and because of charge symmetry written out as

(38)
$$\int_0^1 (F^+_{1p}(\omega) - F^+_{1n}(\omega))\,d\omega = \int_0^1 (F^-_{1n}(\omega) - F^-_{1p}(\omega))\,d\omega$$

one gets $p_\alpha(N) = n_\alpha(N)$; $N_1(p)$, $N_2(p)$ are respectively the total number of antiquarks $\bar{q}_1$ and $\bar{q}_2$ in the configuration $(\alpha, N)$. Making use of CVC, the asymptotic equality of the vector and axial vector form factors and the electroproduction sum rules(16)

(39.1)
$$I_p = \int_0^1 F_{2p}(\omega)\frac{d\omega}{\omega} = \frac{1}{3}\int_0^1 F^-_{2p}(\omega)\frac{d\omega}{\omega} = 0.58$$

(39.2)
$$\Delta I = \int_0^1 (F_{2p}(\omega) - F_{2n}(\omega))\frac{d\omega}{\omega} = 0.19$$

one finds from eqs. (15.1), (33), (36) and the definition

(40)
$$\lambda \cdot 3 I_p = \int_0^1 \Lambda(\omega) F^-_{2p}(\omega)\frac{d\omega}{\omega}, \qquad \lambda \leq 1$$

(41.1)
$$\chi = 1 + 2/3 I_p = 2.15$$

(41.2)
$$\lambda = 3(\chi - 1)/(\chi + 1) = 1.09$$

Experimentally the above solutions for $\chi$ and $\lambda$ are unacceptable since from eq. (37) they imply

$$\Delta I = 0.33 \quad \lambda = 0.36 \tag{42}$$

against the experimental value in (39.2). Besides, taking the maximum temperature in eq. (16) to be $T_c = 160\,\text{MeV}$ one finds from (41.1) and (16) that $\mu_o = 120\,\text{MeV}$, a value much smaller than the mass of any known hadron. Considering the uncertainties in the experimental numbers one could make short work of eqs. (41) and (42) and set

$$\begin{aligned} \chi &= 2 \\ \lambda &= 1 \\ \Delta I &= 1/3 \end{aligned} \tag{43}$$

as predictions of the parton model so that the sum rules (36) and (37) are compatible with each other if the inequalities (32) are saturated. In this case the Adler and Bjorken sum rules are both contained in the sum rule

$$\int_o^1 (F^-_{3p}(\omega) - F^-_{3n}(\omega))\, d\omega = 2 \tag{44}$$

in the Bjorken limit. The simplest way to make progress from here is to substitute the experimental value of $\Delta I$ directly into (37) to get

$$\int_o^1 (F^-_{3p}(\omega) - F^-_{3n}(\omega))\, d\omega = 6\,\Delta I = 1.14 \tag{45}$$

which is the modified form of (44). Eq. (45) fixes the value of the constant $\lambda$, for taking $T_c = 160\,\text{MeV}$ and $\mu_o = 140\,\text{MeV}$ in eq. (16) and introducing (15.1) into (45) gives

$$\chi = 2.4 \tag{46.1}$$

$$\lambda = 2\,\Delta I/(\chi - 1) \cdot I_p = 0.47 \tag{46.2}$$

Since $\mu_o$ cannot be less than the pion mass (cf. eqs. (26) and (27.1) it follows from the above equations that the saturation solution $\lambda = 1$ is excluded. This is a non-trivial constraint on the structure functions

$F_3^{\pm}(\omega)$ which could not have been guessed from considerations of scaling kinematics only. The arbitrariness of the assumption $\lambda = 1$ can now be seen from eq. (45). In the field-theoretical model of Drell, Levy and Yan[17] and in the diffractive model of Harari[18] $F_{2p}(\omega) = F_{2n}(\omega)$ so that $\Delta I$ is identically zero. From eqs. (15) and (45) this means that either $\chi = 1$ or $\lambda = 0$ or both. Drell et al. take $\chi = \lambda = 1$ and Harari $\chi = 1$, $\lambda = 0$. In any case $\Delta I = 0$ with $\chi \geq 1$ leaves $\lambda$ completely arbitrary. This is the arbitrariness referred to in the introduction and it subsists even in those models where $\Delta I \neq 0$. An additional input is necessary to eliminate this arbitrariness and this is what eqs. (15) and (16) have accomplished.

In terms of a perturbative series expansion of the Green's functions it is clear that the parton model is that approximation in which only first order graphs are summed over. This is obviously a weak coupling approximation in which the contributions of higher order and multiple scattering graphs are considered negligible. It is not at all surprising that this approximation is inadeguate for as seen in section (2) the weak and electromagnetic interactions become effectively strong in the scaling limit. This can also be understood by considering how radiative corrections behave at very high energy. The complete series expansion of the Green's functions contains the parton model approximation as the Born term. The intention in this section is not to device an approximation scheme to carry out the perturbative summation in the strong coupling regime but to show how from eqs. (46) one can compute both differential and total neutrino and antineutrino nucleon cross sections from electroproduction data. For this purpose we shall, for simplicity, take the function $\Lambda(\omega)$ independent of $\omega$ and equal to the constant $\lambda$. The cross sections then become[1a]

(47.1) $$\frac{d\sigma^{\pm}(p)}{d\omega} = \frac{G^2 ME}{\pi} \frac{1}{3} (2 \mp \lambda) F_2^{\pm}(p)$$

(47.2) $$\frac{d\sigma^{\pm}(n)}{d\omega} = \frac{G^2 ME}{\pi} \frac{1}{3} (2 \mp \lambda) F_2^{\mp}(p)$$

(48.1) $$\sigma^{+}(p) = \frac{G^2 ME}{\pi} \frac{\chi}{3} (2 - \lambda)\, 3\, J_p = 0.55 \frac{G^2 ME}{\pi}$$

(48.2) $$\sigma^{-}(p) = \frac{G^2 ME}{\pi} \frac{1}{3} (2 + \lambda)\, 3\, J_p = 0.33 \frac{G^2 ME}{\pi}$$

(48.3) $$\sigma^{+}(n) = \frac{G^2 ME}{\pi} \frac{1}{3} (2-\lambda) 3 J_p = 0.23 \frac{G^2 ME}{\pi}$$

(48.4) $$\sigma^{-}(n) = \frac{G^2 ME}{\pi} \frac{\chi}{3} (2+\lambda) 3 J_p = 0.79 \frac{G^2 ME}{\pi}$$

where G is the weak coupling constant, E the incident LAB energy of the lepton and $J_p$ is given by the electroproduction sum rule[16]

(49) $$J_p = \int_0^1 F_{2p}(\omega) d\omega = \frac{1}{3} \int_0^1 F^{-}_{2p}(\omega) d\omega = 0.14$$

From eqs. (48) we have

(50.1) $$\frac{1}{2}(\sigma^{+}(p) + \sigma^{+}(n)) = 0.39 \frac{G^2 ME}{\pi}$$

(50.2) $$\frac{1}{2}(\sigma^{-}(p) + \sigma^{-}(n)) = 0.56 \frac{G^2 ME}{\pi}$$

(50.3) $$\frac{\sigma^{+}(p) + \sigma^{+}(n)}{\sigma^{-}(p) + \sigma^{-}(n)} = 0.7$$

The ratio in (50.3) is more than twice that predicted by the field-theoretical model of Drell et al.[17]. The only experimental number available on the cross sections is[19]

(51) $$\frac{1}{2}(\sigma^{-}(p) + \sigma^{-}(n)) = (0.51 \pm 0.13) \frac{G^2 ME}{\pi}$$

For the sum rules in eqs. (36) we now have

(36.1') $$\int_0^1 (F^{+}_{2p}(\omega) - F^{-}_{2p}(\omega)) \frac{d\omega}{\omega} = 2.44$$

(36.2') $$\int_0^1 (F^{+}_{3p}(\omega) + F^{-}_{3p}(\omega)) d\omega = -2.78$$

This last equation is very different from its original version ; tests of it are doubtless of great importance.

In conclusion we can say that the statistical slant given to Bjorken scaling has confirmed its intrinsically dynamical nature and the genuine identity of properties between it and Kadanoff-Widom scaling. Both forms of scaling are manifestations characteristic of the critical region. This fact suggests a theoretical tool, the method of the renormalisation group, for the investigation of the Bjorken limit. This method has already proved useful in giving a microscopic basis to scaling associated with phase transitions[3c] and has been re-proposed by Wilson[20] for investigating scattering in the deep region. Because in the high energy case one is saddled with a highly relativistic problem, the formulae applicable to this regime are not simple transcripts from many-body theory. Moreover it is still very unuclear how to exploit the relationship between scale invariance and the validity of Ward-like identities in the dual resonance model. These problems are presently under study.

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