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C. Bernardini: ON THE PARAMETRIZATION OF THE ANGULAR DISTRIBUTIONS OF HADRONS PRODUCED VIA THE ONE-PHOTON ANNIHILATION CHANNEL IN e^+e^- REACTIONS. -

1. - Most of the peculiarities of the angular distribution of hadrons from e^+e^- annihilations in the center of mass are determined by the simple structure of the one photon channel. In an attempt to find model-independent parametrizations of the angular distribution of few observed charged tracks in events containing pseudoscalars with any multiplicity we got a simple answer for the single-track case, namely $a + b \cos^2\theta_p$ where a and b are energy-dependent parameters and θ_p is the track angle with the beam line. Also, the two-track distribution has been studied and some simple property shown; in particular, the possibility to describe final state dynamics in terms of only one angle variable. We will sketch in the following the physics and the mathematical techniques for the one - and two-track cases. We did not study cases with more than two tracks.

2. - Cabibbo and Gatto[1] have shown that, to lowest e. m. order the space part $\vec{J}(n)$ of the matrix element

$$\langle 0 | J_\mu | n \rangle \equiv J_\mu(n)$$

of the hadronic current operator in

$$e^+e^- \to n \qquad \text{(any hadronic state)}$$

is distributed according to a $\sin^2\theta$ law, θ being the angle of $\vec{J}(n)$ with the beam line (C. M. description is understood), because of it's coupling with the lepton current.

We will limit the analysis to the case in which n contains only spin-zero particles.

Also it would not be difficult to account for a possible polarization of the beam electrons, but we will study the no-polarization case.

A reference system using the beam line as polar axis will be called "scanner system" (S. S.); the "current system" (C. S.) will have $\vec{J}(n)$ as polar axis.

We do not know anything in general on how $\vec{J}(n)$ is formed out of the particle 3-momenta in n. But we can say that, given a state n (described by the nature and 3-momenta of each particle in n) the cross section for $e^+e^- \rightarrow n$ will be invariant under the following operations:

i - Rigid rotations around $\vec{J}(n)$
ii - Inversion $\vec{J}(n) \rightarrow -\vec{J}(n)$
iii - Rotations with $\vec{J}(n)$ at θ = constant, around the beam line.

The more, keeping the particle configuration fixed in C. S. , any change of the $\vec{J}(n)$ direction in the S. S. will be accounted for by the $\sin^2\theta$ law.

The single-track case corresponds to integration over all but one particle momenta. The two-track case concerns integrations over all but two particle momenta. Since usually the triggers require at least two tracks, the single-track analysis might be somewhat academic; nevertheless it illustrates well the power of the symmetries involved.

3. - We first consider the single-track case. Assume that a charged particle with momentum $\vec{p}$ is observed and no other particle is detected. Let us call θ_p, φ_p the polar angles of the track in the S. S. ; also, $p = |\vec{p}|$. Assume for a moment that the direction of $\vec{J}$ for that event is known. Then. call θ, φ the polar angles of J in the S. S. and Δ, χ the polar angles of the track in the C. S. (see Fig. 1). The symmetries will be exploited by saying

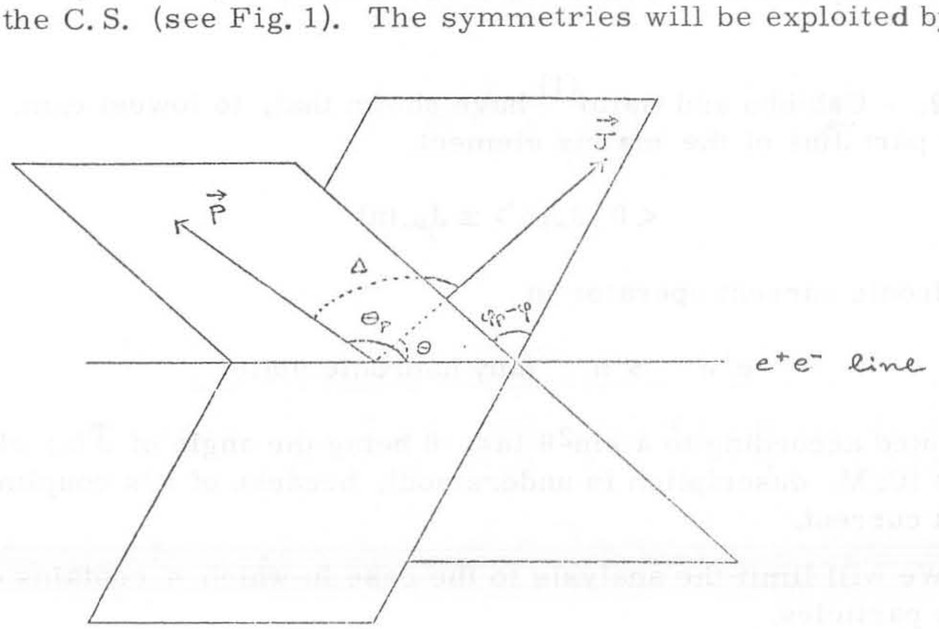

FIG. 1

that, should we know $\vec{J}$, the probability of the event would be of the form

$$\frac{3}{8\pi} W(\Delta, p) \sin\Delta \, d\Delta \, d\chi \, \sin^3\theta \, d\theta \, d\varphi$$

where W is a function of Δ normalized over the sphere. Because of ii, § 2:

$$W(\Delta) = W(\pi - \Delta)$$

Therefore $W(\Delta, p)$ can be expanded in the form

$$W(\Delta, p) = w_o(p) + w_2(p) P_2(\cos\Delta) + \ldots.$$

where P_n is a Legendre polynomial (even indexes only). Because of the normalization, $w_o(p)$ is known and equals $1/4\pi$.

Actually, we do measure θ_p, φ_p so that integration over the remaining angles must be performed. By using

$$\cos\Delta = \cos\theta \cos\theta_p + \sin\theta \sin\theta_p \cos(\varphi - \varphi_p)$$

and the composition formula for Legendre polynomials, the following distribution is found after integration over θ, φ for the track-angles in the S. S. :

(1) $$\frac{d\Omega_p}{4\pi} \left\{ 1 - \frac{4\pi}{5} w_2(p) P_2(\cos\theta_p) \right\}$$

Therefore, the following theorem holds:

Single-track theorem. - Any state produced via the one-photon channel in $e^+e^- \rightarrow$ (spin zero hadrons) will give a single track distribution in scanner-space of the form (1).

The known cases of $e^+e^- \rightarrow \pi^+\pi^-$ (K^+K^-) and $e^+e^- \rightarrow \pi^+\pi^-\pi^o$ correspond to $W \sim \delta(1 - \cos^2\Delta)$ and $W \sim \delta(\cos\Delta)$ respectively. For them, $w_2 = 5/4\pi$ and $w_2 = -5/8\pi$. In general

$$-2 \leq \frac{4\pi}{5} w_2(p) \leq 1$$

because of positivity requirement on (1).

When no information on the multiplicity is known, $w_2(p)$ must be understood as the sum over all possible hadronic states (we can safely assume that such states contain only π and K).

The cross section for the single track will be written in general

$$\frac{d\sigma}{dp\,d\Omega_p} = A(p)\left\{1 - \frac{4\pi}{5}\, w_2(p)\, P_2(\cos\theta_p)\right\}$$

Integration over p gives

$$4\pi \int dp\, A(p) = \sigma_{Total}, \quad \text{the total cross section. Define}$$

$$\overline{w}_2 = 4\pi \int dp\, w_2(p)\, A(p)/\sigma_{Total}$$

then, when the momentum p is not measured

$$\frac{d\sigma}{d\Omega_p} = \frac{\sigma_{Total}}{4\pi}\left\{1 - \frac{4\pi}{5}\, \overline{w}_2\, P_2(\cos\theta_p)\right\}$$

Note that, performing these integrations over the kinematical limits for p, the lower limit could be less than the threshold for an actual apparatus and, the worst, the threshold could be angle-dependent. Therefore, a word of warning must be said concerning the comparison with data.

4. - Let us consider now the two-track case. Labeling the parameters of the two particles by the index 1 and 2, we will call θ_i, φ_i the polar angles in the S. S. and Δ_i, χ_i those in the C. S., shown in Fig. 2. Aslo, θ_{12} is the angle between the two tracks. To lighten the notations, put

$$z_i = \cos\Delta_i \quad , \quad z = \cos\theta \quad , \quad x_i = \cos\theta_i$$

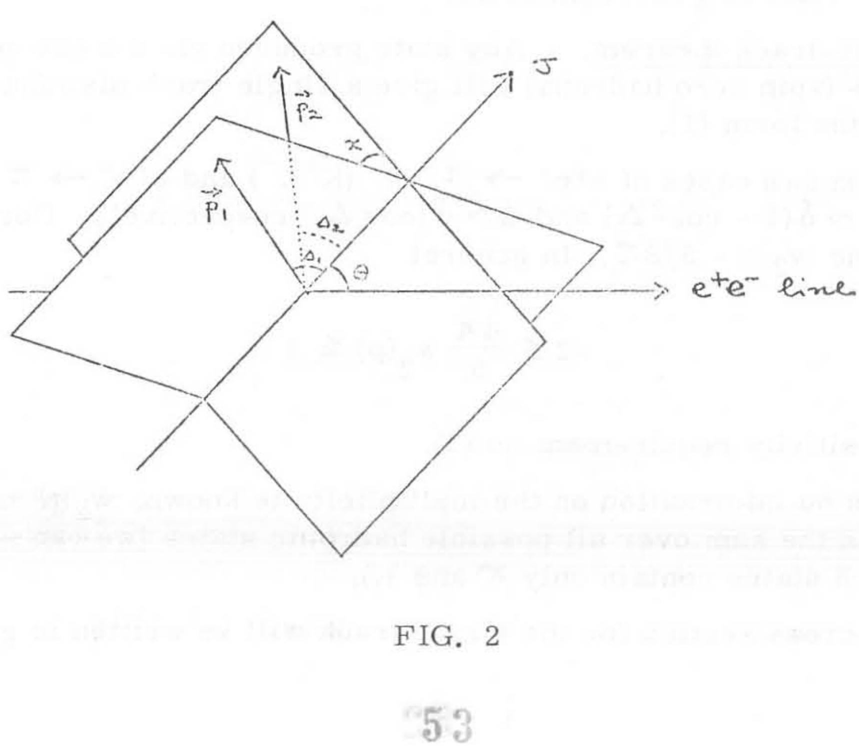

FIG. 2

and $\chi = \chi_1 - \chi_2$, $z_{12} = x_1 x_2 - \sqrt{1 - x_1^2}\sqrt{1 - x_2^2}\cos(\varphi_1 - \varphi_2)$

Assuming again the direction of $\vec{J}$ to be known (θ, φ) then, becau se of the symmetries we would write for the probability of the $\vec{p}_1$, $\vec{p}_2$, $\vec{J}$ configuration

$$\frac{3}{8\pi} W(z_1, z_2, \chi; p_1 p_2)\, dz_1\, d\chi_1\, dz_2\, d\chi_2 (1 - z^2)\, dz d\varphi$$

where W is a normalized function having the property ii), § 2

$$W(-z_1, -z_2, \chi; p_1 p_2) = W(z_1, z_2, \chi; p_1 p_2)$$

The analogous procedure to the single-track case would be to integrate over z, φ keeping θ_i, φ_i fixed for both particles:

$$\frac{3}{8\pi} d\Omega_1\, d\Omega_2 \int_0^{2\pi} d\varphi \int_{-1}^{+1} (1 - z^2)\, dz\, W(z_1, z_2, \chi; p_1 p_2)$$

$d\Omega_i$ being the solid angle for track i in the S. S.

That this would hardly produce simple results is seen in the cir cumstance that, while

$$z_i = z x_i + \sqrt{1 - z^2}\sqrt{1 - x_i^2}\cos(\varphi - \varphi_i)$$

is a simple relation, the formula

$$\cos\chi = \frac{z_{12} - z_1 z_2}{\sqrt{1 - z_1^2}\sqrt{1 - z_2^2}}$$

is exceedingly complicate. Therefore, the most natural expansion

$$W(z_1, z_2, \chi; p_1 p_2) = \sum W_{rs}^m (p_1 p_2) P_r(z_1) P_s(z_2) \cos m\chi$$

cannot be exploited in a simple way[2].

We shall therefore consider a different parametrization in which the particle having momentum $\vec{p}_1$ is analyzed in the S. S. whereas particle 2 $(\vec{p}_2)$ is analyzed with respect to the polar axis $\vec{p}_1$, as shown in Fig. 3. We introduce now the angles:

- α , between the $(\vec{p}_1, \vec{J})$ plane and the plane containing $\vec{p}_1$ and the beam line

α_2, between the ($\vec{p}_1$, $\vec{p}_2$) plane and the plane of $\vec{p}_1$ and the beam-line.

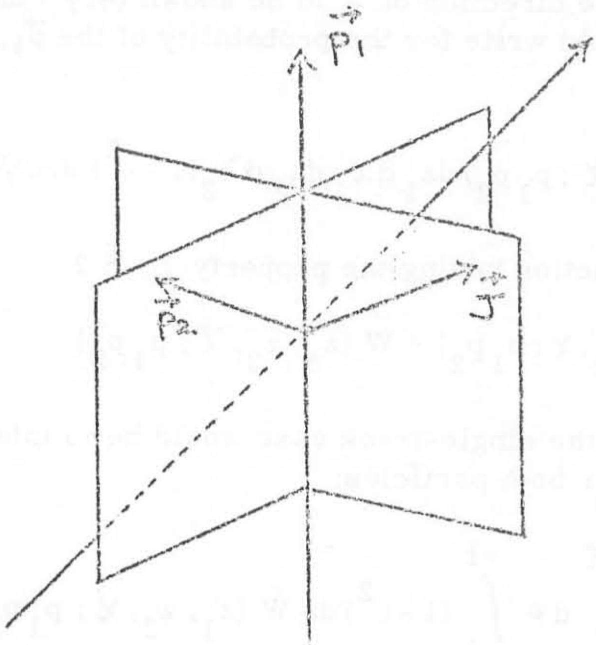

FIG. 3

Thus, α and α_2 describe rotations around the $\vec{p}_1$ axis. A configuration is fully determined, for this two-track case, by giving z_1, z_{12}, $\alpha - \alpha_2$; that is, the dynamics will be contained in a function W (z_1, z_{12}, $\alpha - \alpha_2$; $p_1 p_2$) and the average over the directions of $\vec{J}$ will be performed by integrating over z_1 and α .

It follows that the probability to find p_1 in the solid angle $d\Omega_1$ and p_2 in the solid angle $d\Omega_{12}(= dz_{12} d\alpha_2)$ from track 1 is

$$\frac{3}{8\pi} d\Omega_1 d\Omega_{12} \int_0^{2\pi} d\alpha \int_{-1}^{+1} dz_1 \, (1 - z^2) \, W \, (z_1, z_{12}, \alpha - \alpha_2; \, p_1 p_2) \equiv$$

$$\equiv d^4Q \, (x_1, \varphi_1; \, z_{12}, \alpha_2)$$

Since

$$z = z_1 x_1 + \sqrt{1 - x_1^2} \sqrt{1 - z_1^2} \cos\alpha$$

we proceed again to expand W:

$$W \, (z_1, z_{12}, \alpha - \alpha_2; p_1 p_2) = \sum W_r^m \, (z_{12} \, p_1 p_2) \, P_r \, (z_1) \cos m(\alpha - \alpha_2)$$

Then, by integration

$$d^4Q = d\Omega_1 \, d\Omega_{12} \Big\{ W_0^0 (z_{12}; p_1 p_2) + \frac{1}{10} W_2^0 (z_{12}; p_1 p_2)(1 - 3x_1^2) -$$

$$- \frac{3}{8} \Big[\Sigma g_r W_r^1 (z_{12}; p_1 p_2) \Big] \sqrt{1 - x_1^2} \cos \alpha_2 - \frac{1}{2} \Big[W_0^2 (z_{12}; p_1 p_2) -$$

$$- \frac{1}{5} W_2^2 (z_{12}; p_1 p_2) \Big] \cos 2\alpha_2 (1 - x_1^2) \Big\}$$

where $g_r = \int_{-1}^{+1} \sqrt{1 - z_1^2} \, P_r (z_1) \, dz_1$.

The explicit form of g_r is:

$$g_r = 2 \frac{\cos^2(\frac{\pi r}{2})}{(1 - r^2)(3 + r)}$$

with $g_1 = 0$ (by the definition formula). Therefore, $g_r = 0$ for r odd and

$$g_0 = \frac{2}{3} \quad , \quad g_2 = - \frac{2}{15} \quad , \quad g_4 = - \frac{2}{105}$$

showing that only the first few terms in the sum (W_0^1 and W_2^1) contribute appreciably.

Also, by using the spherical harmonics

$$Y_{mn} (\theta, \varphi) = \cos m\varphi \; P_n^m (\cos \theta)$$

the general formula can be rewritten:

$$d^4Q = d\Omega_1 \, d\Omega_{12} \Big\{ W_0^0 \, Y_{00} (\theta_1, \alpha_2) -$$

(2)
$$- \frac{1}{5} W_2^0 \, Y_{02} (\theta_1, \alpha_2) - \frac{3}{8} (\Sigma g_r W_r^1) \, Y_{11} (\theta_1, \alpha_2) -$$

$$- \frac{1}{6} (W_0^2 - \frac{1}{5} W_2^2) \, Y_{22} (\theta_1, \alpha_2) \Big\}$$

From this general formula we can deduce simpler formulas for special cases:

a) Which is the probability that, given a track (track 1, say, pointing in the θ_1, φ_1 direction) we find another (track 2) making an angle θ_{12} with $\vec{p}_1$, irrespective of the orientation of the plane ($\vec{p}_1$, $\vec{p}_2$)? (remember $z_{12} = \cos\theta_{12}$).

This is simply obtained by integrating over α_2:

(3)
$$d^3Q\,(x_1, \varphi_1, z_{12}) = 2\pi\, d\Omega_1\, dz_{12} \left\{ W_0^0\,(z_{12}; p_1 p_2) + \right.$$
$$\left. + \frac{1}{10} W_2^0\,(z_{12}; p_1 p_2)(1 - 3x_1^2) \right\} = 2\pi\, d\Omega_1\, dz_{12} \left\{ W_0^0 - \frac{1}{5} W_2^0\, P_2\,(x_1^2) \right\}$$

a formula that reminds of the single track theorem.

It is also evident that

(4)
$$8\pi^2\, W_0^0\,(z_{12}; p_1 p_2)\, dz_{12}$$

is the probability to find a second track at an angle θ_{12} with the first.

b) The general formula shows that the probability to find a track-pair whose ($\vec{p}_1$, beam) and ($\vec{p}_1$, $\vec{p}_2$) planes form an angle α_2 (integrate over x_1, φ_1, z_{12}) is:

(5)
$$\frac{d\alpha_2}{2} \left[1 + A\cos\alpha_2 + B\cos 2\alpha_2 \right]$$

where

$$A = -\frac{3\pi}{4} \sum g_r \int_{-1}^{1} dz_{12}\, W_r^1$$

$$B = -\frac{8}{3}\pi^2 \int_{-1}^{+1} dz_{12} \left[W_0^2 - \frac{1}{5} W_2^2 \right]$$

a) and b) are just examples of what can be done to analyze events; here we want to emphasize that the main result expressed by the general formula for d^4Q is:

Two-track theorem. - The distribution in space of two-tracks from any $e^+e^- \to$ (spin zero hadrons) annihilation process depends on four unknown functions of the single variable θ_{12}, the angle between the two tracks. The dependence on the other 3 angle variables is completely determined and given by the general formula (2).

Eventually as a check and example, it is easily verified that for $e^+e^- \longrightarrow 2\pi$ (2K), since

$$W = \frac{1}{8\pi^2} \delta(1+z_{12}) \left[\delta(1+z_1) + \delta(1-z_1) \right]$$

one has $W_r^m = 0$ if $m > 0$; $W_r^o = 0$ if r odd. For even r

$$W_r^o = \frac{2r+1}{8\pi^2} \delta(1+z_{12})$$

whence $d^4Q \sim d\Omega \, d\Omega_{12} \sin^2\theta_1 \, \delta(1+z_{12})$.

Also, for $e^+e^+ \rightarrow 3\pi\,(2K+\pi)$, write

$$W = \frac{1}{4\pi^2} \delta(z_1) H(z_{12}, p_1 p_2)$$

whence:

$$W_r^m = 0 \quad \text{for} \quad m > 0$$

$$W_r^o = 0 \quad \text{for odd } r$$

$$W_r^o = \frac{2r+1}{8\pi^2} H(z_{12}; p_1 p_2) P_r(0)$$

so that

$$d^4 Q \sim d\Omega_1 \, d\Omega_{12} H(z_{12}; p_1 p_2)(1+\cos^2\theta_1)$$

5. - A full exploitation of the two-track theorem is made difficult by the presence of the momentum variables p_1 and p_2 and by the fact that one has to reconstruct unknown functions (the W_r^m) rather than to determine numerical values of parameters.

When momentum analysis is not done, the W's must be integrated over the momentum spectrum for the two particles. This integration will generally require knowledge of the z_{12} dependence of the kinematical limits for p_2, given p_1. Nevertheless, integration over the momentum spectrum will give average functions $\overline{W}_r^m(z_{12})$ of the single variable z_{12}

such that $d^4\bar{Q}$, constructed by replacing $W_r^m(z_{12}; p_1 p_2)$ with $\bar{W}_r^m(z_{12})$ in (2), is related to the two track cross section by:

$$\frac{d^4\sigma}{d\Omega_1 \, d\Omega_{12}} = \sigma_{Total} \frac{d^4\bar{Q}}{d\Omega_1 \, d\Omega_{12}}$$

Now, it is a better procedure than to reconstruct unknown functions in $d^4\bar{Q}$ to introduce empirical functions depending on few parameters in order to get fits.

We choose the following kind of empirical functions:

$$\bar{W}_r^m(z_{12}) = \frac{1}{16\pi^2}(a_r^m + b_r^m z_{12}^2)$$

Note that when $r = m = 0$, because of (4) and the meaning of W_o^o one has

$$b_o^o = 3(1 - a_o^o)$$

Also, from (3) we get after integration over z_{12}

$$a_2^o + \frac{1}{3} b_2^o = 4\pi \bar{w}_2$$

that is a relation with the parameter appearing in the single-track formula (1).

Since the coefficients $\bar{W}_r^1(z_{12})$ appear in the general formula (2) as a linear combination:

$$\sum_o^\infty g_r \bar{W}_r^1(z_{12})$$

we only need the parameters

$$a_1 = \sum g_r a_r^1$$

$$b_1 = \sum g_r b_r^1$$

Also, the sum $W_o^2 - 1/5\ W_2^2$ appears in the last term of 2, so that only

$$a_2 = a_o^2 - \frac{1}{5} a_2^2, \qquad b_2^o = b_o^2 - \frac{1}{5} b_2^2$$

have a role in the fit.

In conclusion, we propose to represent the two-track data by a formula containing 7 parameters:

$$a_o^o,\ a_2^o,\ b_2^o\ (\text{or}\ \bar{w}_2),\ a_1,\ b_1,\ a_2,\ b_2$$

In particular, formula (5) contains the parameters

$$A = -\frac{3\pi}{32}(a_1 + \frac{1}{3} b_1)$$

$$B = -\frac{1}{3}(a_2 + \frac{1}{3} b_2)$$

and formula (3) shows that, when integrating over α_2, the distribution $d^3\bar{Q}$ is given by a 3 parameter (a_o^o, a_2^o, b_2^o) formula.

Positivity requirements impose some inequalities whose model independent form is of the following type:

$$\bar{W}_o^o(z_{12}) \geq 0, \quad \text{any } z_{12}$$

$$\bar{W}_o^o(z_{12}) - \frac{1}{5}\bar{W}_2^o(z_{12}) P_2(x_1) \geq 0 \quad \text{any } z_{12}, x_1$$

$$B \leq 1, \quad A \leq 1 + B \qquad (\text{from (5)})$$

and so on.

When we use the empirical formula, these inequalities become

$$0 \leq a_o^o \leq \frac{3}{2}$$

$$a_o^o \geq -\frac{1}{10} a_2^o \qquad \text{if } a_2^o \text{ is negative}$$

$$a_o^o \geq \frac{1}{5} a_2^o \qquad \text{if } a_2^o \text{ is positive}$$

$$a_2 + \frac{1}{3} b_2 > -3$$

$$a_2 + \frac{1}{3} b_2 - \frac{9\pi}{32}(a_1 + \frac{1}{3} b_1) < 3$$

and so on.

In conclusion, because of solid angle limitations in the experimental set-ups the reconstruction of the total cross section for hadronic annihilation events will require a knowledge of the geometrical efficiency and of the angular distribution of the produced particles. We have shown here how to proceed when one or two tracks are detected; the aim of the present work is therefore to help to judge of the sensitivity of an apparatus and, perhaps, to optimize it's performances.

I want to thank Bruno Bartoli, Giorgio Capon and Mario Greco for comments.

REFERENCES AND FOOTNOTES. -

(1) - N. Cabibbo and R. Gatto, Phys. Rev. 124, 1577 (1961).

(2) - Notice that, if $S(p_1 p_2)$ is the momentum spectrum for p_2 at a fixed p_1

$$4\pi \int dp_2\, S(p_1 p_2)\, W_{ro}^o(p_1 p_2) = w_r(p_1)$$

the single-track parameters. Since by ii) § 2 r+s must be even, r even follows when s = 0.