Istituto Nazionale di Fisica Nucleare
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M. Baldo and E. Recami(x): RELATIVISTIC TRIANGLE-DIAGRAM CROSS-SECTIONS: APPLICATIONS TO ELEMENTARY PARTICLE AND NUCLEAR PHYSICS. -

"La filosofia è scritta in questo grandissimo libro che ci sta aperto innanzi agli occhi (io dico l'universo), ma non si può intendere se prima non si impara a intender la lingua, o conoscere i caratteri, ne' quali è scritto. Egli è scritto in lingua matematica, e i caratteri son triangoli, cerchi, ed altre figure geometriche, senza i quali mezzi è impossibile a intenderne umanamente parola".

Galileo Galilei

ABSTRACT.

In this work the authors evaluate explicitly the total cross-section for relativistic processes, with three final particles, proceeding through triangle-diagrams. It is suggested that the final state interactions here considered may contribute in explaining (partially, at least) some enhancements in the effective-mass distributions as due to kinematical effects. Our general formulas are for example applied - as proposed by Duimio and Recami(4) - to the pn π^+ system, and seem to allow reproducing the known peaking at about 2.2 GeV/c^2 in the dπ^+ distribution; our model, in particular, could give a possible justification for the appreciable deuteron survival in some collisions of mesons and deuterons, event at high projectile energies. The triangle-graph peaks appear highly asymmetric. Applications to nuclear physics are also considered, always with the aim of evidentiating the possible anomalous behaviours in some cross-sections produced esclusively by the pure kinematics.

(x) - On leave of absence from the Istituto di Fisica dell'Università di Milano.

I. - INTRODUCTION.

Recently, triangle graphs seem to have been successful in explaining some cross-section bumps, particularly, as kinematical effects due to final state interactions; both in elementary-particle[1] and in nuclear physics[2]. In the field of elementary particle physics it is for instance useful to identify the kinematical effects expecially when they contribute in resonance regions, in order to be able to "clean" better the experimental effective-mass distributions. In general, expecially in the high-energy cases, previous works were semi-quantitative or qualitatives in character.

In this paper we want, on one hand, carefully evaluate the effect of the triangular-graph Landau[3] singularity on the cross-section corresponding to processes (with three final particles) of the type

(1) $$A + B \longrightarrow C + D + E$$

supposed to proceed through the rather general diagram shown in Fig. 1. On the other hand, as an application, we want particularize our formulas to the reaction $pp \rightarrow pn\pi^+$ - as suggested by Duimio and Recami[4] - whose experimental interest has been growing up

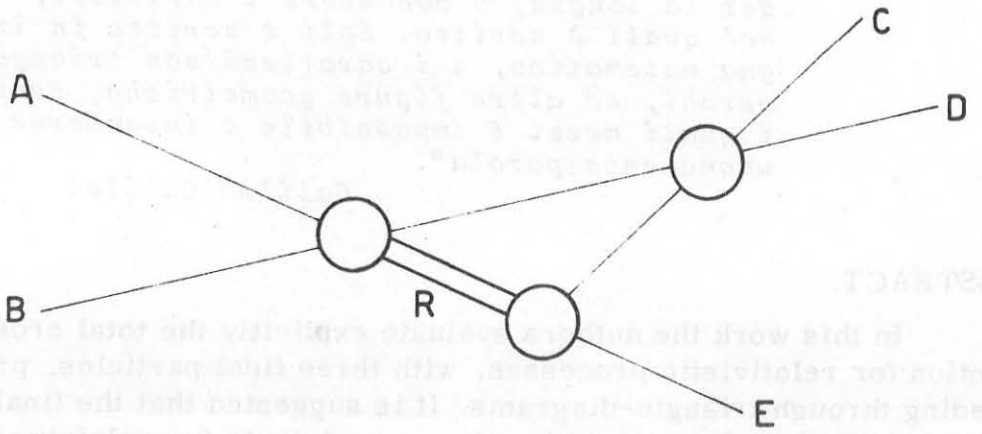

FIG. 1 - The general relativistic diagram here considered, for three-final-particle processes, with final state interaction of triangular type.

more and more in recent years. In that case, the triangle-graph (see Fig. 2) is interesting because its presence (as we shall see) might explain the not-negligible deuteron survival in some collisions e. g. of mesons and deuterons (see Fig. 3) even at high projectile-energies, notwithstanding the negligible deuteron binding energy. It is clear that our aim is not to explain all reaction (1) - the major contribution to which is coming from graphs without rescattering -, but to evaluate effects of the kinematical singularity connected with the diagram in Fig. 1.

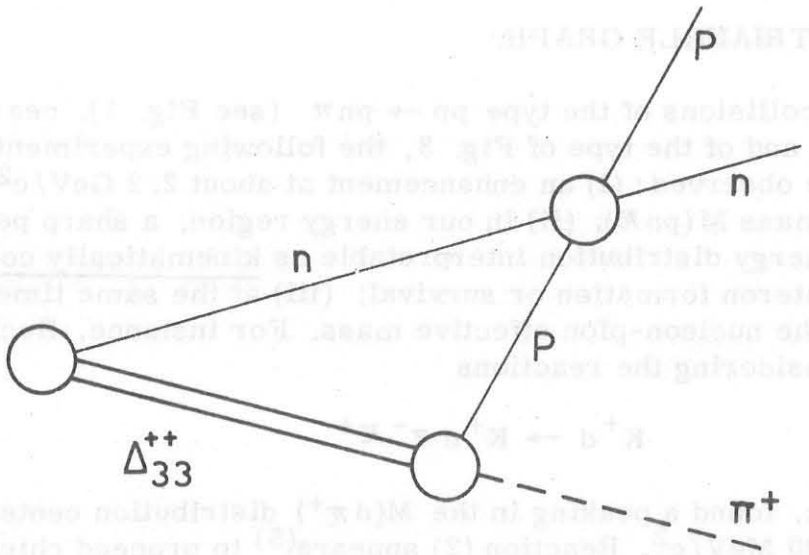

FIG. 2 - A particular triangle graph. It may account for many experimental features of reactions with a $pn\pi^+$ system in the final state (for instance it may interpret the experimental bump in the $np\pi$ effective-mass distribution at about 2.2 GeV/c^2 as a kinematical effect).

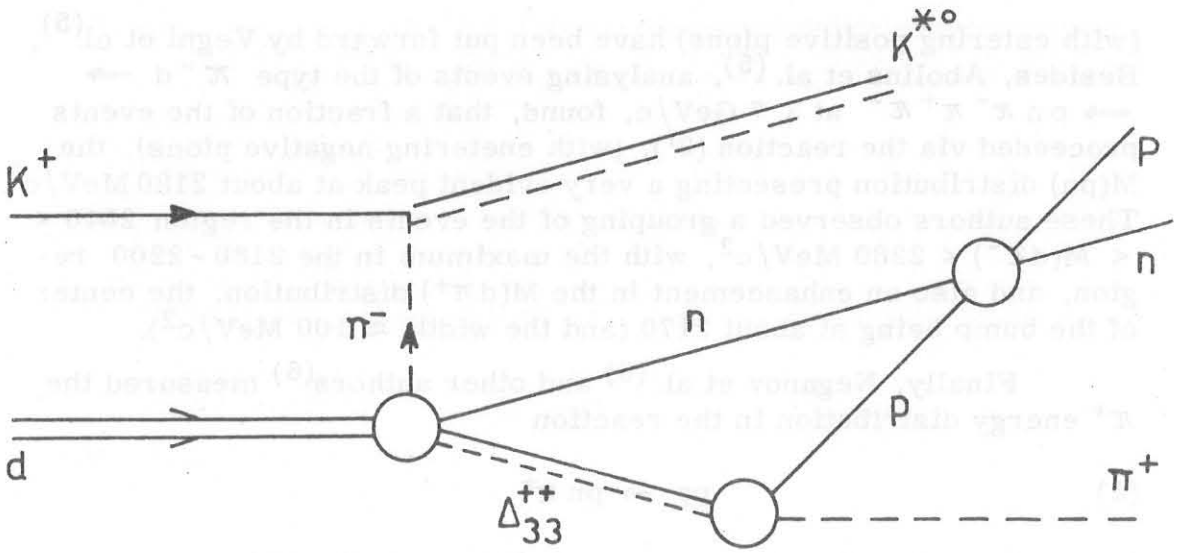

FIG. 3 - An example of triangle intervention in a more general (O. P. E.) diagram (See e. g. Buchner et al.[5]). The triangle (Landau) singularity can give a possible explanation of the appreciable deuteron survival.

II. - OUR TRIANGLE GRAPH.

In collisions of the type $pp \rightarrow pn\pi$ (see Fig. 1), near their threshold, and of the type of Fig. 3, the following experimental features were observed: (i) an enhancement at about 2.2 GeV/c^2 in the effective-mass $M(pn\pi)$; (ii) in our energy region, a sharp peak in the pion energy distribution interpretable as kinematically corresponding to deuteron formation or survival; (iii) at the same time Δ_{33} bumps in the nucleon-pion effective mass. For instance, Buchner et al.[5], considering the reactions

$$K^+ d \rightarrow K^+ d \pi^- \pi^+ \qquad (2)$$

at 3 GeV/c, found a peaking in the $M(d\pi^+)$ distribution centered around 2190 MeV/c^2. Reaction (2) appears[5] to proceed chiefly via pseudoscalar meson exchange between a K^* vertex and a $\pi^+ d$ vertex. Previously, Butterworth et al.[5] (while considering the same reaction at 2.3 GeV/c) had already explained qualitatively the peaking as a result of Δ_{33} production on the deuteron breakup, i. e. as the consequence of $\Delta(1236)$ formation between the π^+ and the proton of the deuteron, with subsequent recombination of the decay proton from the Δ with the spectator neutron. Analogue interesting considerations, but related to reaction:

$$\pi d \rightarrow \pi d \pi^- \pi^+ \qquad (2')$$

(with entering positive pions) have been put forward by Vegni et al.[5], Besides, Abolins et al.[5], analysing events of the type $\pi^- d \rightarrow$ $\rightarrow pn \pi^- \pi^+ \pi^-$ at 3.7 GeV/c, found, that a fraction of the events proceeded via the reaction (2'), (with enetering negative pions), the M(pn) distribution presenting a very evident peak at about 2180 MeV/c^2. These authors observed a grouping of the events in the region $2040 < M(d\pi^-) < 2280$ MeV/c^2, with the maximum in the 2180 - 2200 region, and also an enhancement in the $M(d\pi^+)$ distribution, the center of the bump being at about 2170 (and the width $\approx$ 100 MeV/c^2).

Finally, Neganov et al.[6] and other authors[6] measured the π^+ energy distribution in the reaction

$$pp \rightarrow pn\pi^+ \qquad (3)$$

(including also the deuteron-production cases) utilizing protons of 660 MeV, kinetic energy almost equivalent to a total energy of about 2200 MeV. They found a sharp peak at a pion kinetic c. m. energy of about 150 MeV, corresponding to production of couples p-n near their threshold (see Fig. 4). In reactions of the type (3), in our energy region, also Δ_{33} resonance intervention has been extensively evidenced (see for instance Fickinger et al.[8]). On the other hand,

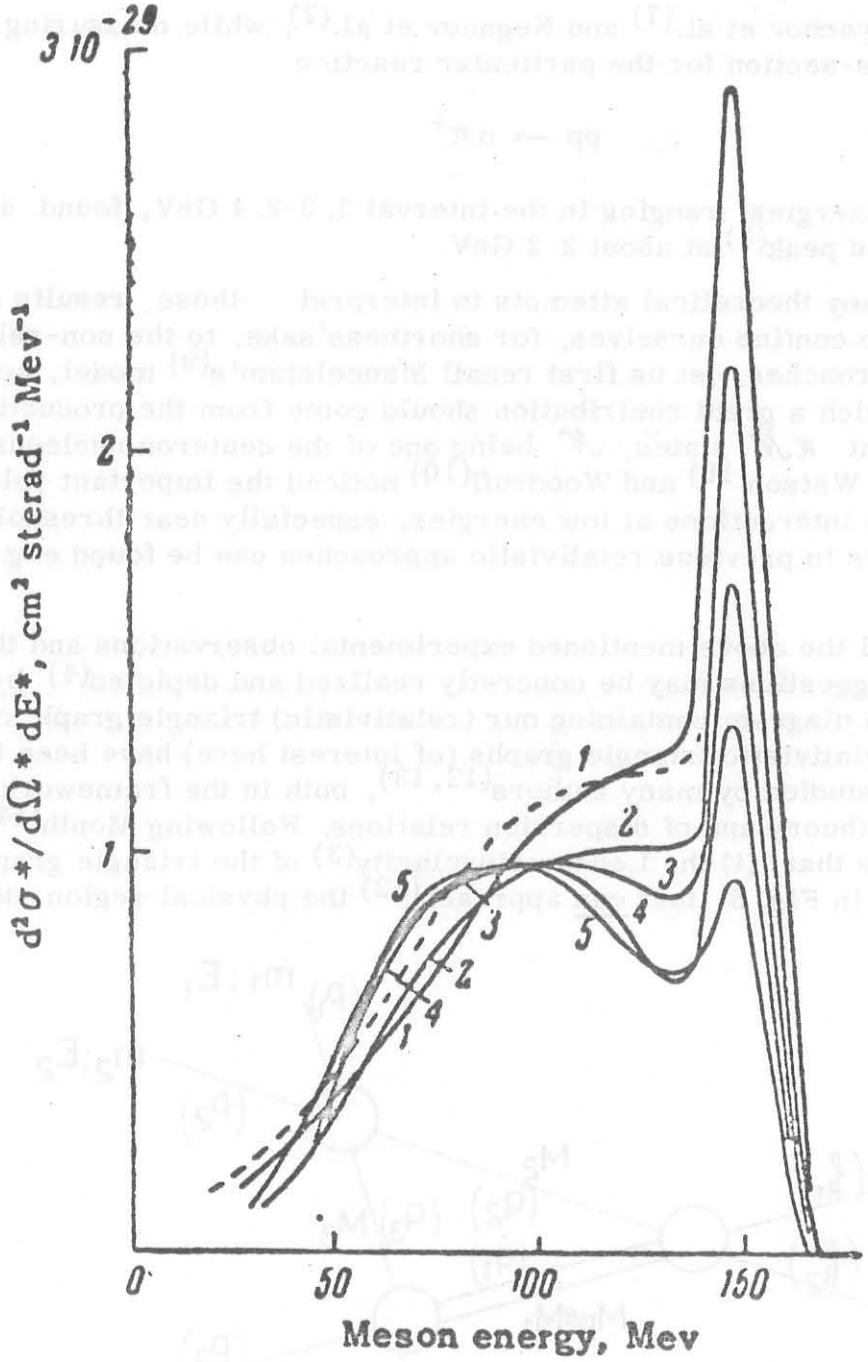

FIG. 4 - Spectrum, at various angles, of the c. m. s. kinetic energy for π^+ coming from $pp \rightarrow pn\pi$ reaction with proton beam-energy of 660 MeV (just corresponding to a total energy of about 2. 2 GeV). The peaks at 150 MeV correspond to production of couples p-n near their threshold. This plot is taken from Meshkovsky et al.[6].

Meshcheryachov et al.[7] and Neganov et al.[7], while measuring the total cross-section for the particular reaction

(4) $$pp \rightarrow d\pi^+$$

for total energies, ranging in the interval 2.0-2.4 GeV, found a pronounced peak[8] at about 2.2 GeV.

Many theoretical attempts to interpret those results are known. To confine ourselves, for shortness' sake, to the non-relativistic approaches, let us first recall Mandelstam's[9] model, according to which a great contribution should come from the production of resonant πN states, N being one of the deuteron nucleons. Secondly, Watson[10] and Woodruff[10] noticed the important role of final state interactions at low energies, especially near threshold. References to previous relativistic approaches can be found e. g. in ref. (11).

All the above mentioned experimental observations and theoretical suggestions may be concretely realized and depicted[4] by Feynmann diagram containing our (relativistic) triangle graph of Fig. 2. Relativistic triangle graphs (of interest here) have been theoretically studied by many authors[12,13], both in the framework of Landau's theory and of dispersion relations. Following Month[14] we emphasize that: (i) the Landau singularity[3] of the triangle graph(x) contained in Fig. 5, that can approach[12] the physical region, does

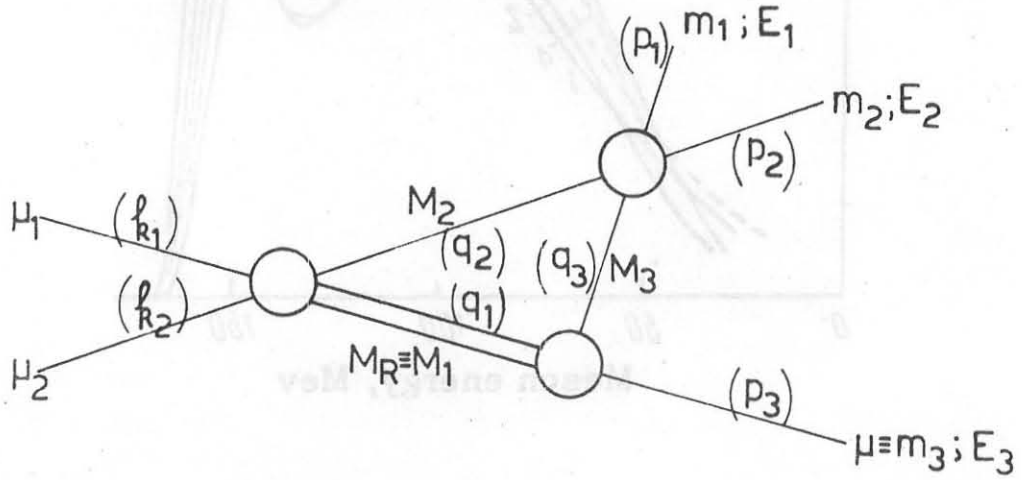

FIG. 5 - Our relativistic model (see the text). We use the same symbols to indicate particle masses and particle themselves. The M_R is a resonance decaying into M_3+m_3.

(x) - We use the same symbols to indicate particles and their rest-masses.

indeed approach close to it only if particles m_1 and m_2 are at threshold[14] (i. e. if their relative energy is negligible). This singularity is purely kinematical in origin and thus does not depend on the particular vertex structures; (ii) the "nearby singularity" of the triangle graph (see Fig. 5) is expected to yield a three particle peak - in the case of simple rescattering - at[14]:

$$(5) \qquad M^* \equiv \sqrt{(p_1+p_2+p_3)^2} \approx \sqrt{M_R^2+m_2^2+\frac{m_2}{m_1}(M_R^2+m_1^2-m_3^2)} \ ,$$

besides the threshold enhancement in $M(m_1 m_2)$. At the peak in our model the internal masses are on the mass-shell, as we shall see in the following; (iii) In the peak region particles m_1 and m_3 are expected to form again a "state" having the same mass of the resonance M_R. In general, formula (5) results to give the peak position within a few per cent (with respect to the incoming total kinetic energy).

In our case (see Fig. 2), for instance, we can first deduce and observe, on a qualitative or semi-quantitative ground: (i) The contribution of the triangle graph of Fig. 2 to high energy reaction scattering of the type πd and K^+d (see Fig. 3) could explain why, in a not negligible part of these collisions, the final proton and neutron appear to form again a deuteron (deuteron survival) despite its negligible binding energy. Our model can explain as well the (sharp) enhancement (at the peak energy, see eq. (6)) in the π^+ energy distribution observed[6] in reaction (3) and reproduced in Fig. 4, which kinematically corresponds (as already mentioned) to deuteron production or to production of couples p-n near their threshold[x]; (ii) The expected three particle peak will be at:

$$(6) \qquad M(pn\,\pi^+) \approx \sqrt{2(\mathcal{N}^2+\Delta_{33}^2)-\pi^2} \approx 2191 \text{ MeV}/c^2,$$

in agreement with the experimental data[5-8] for reaction $pp \rightarrow d\pi^+$, as already partially reported. Here $\mathcal{N}$ is the nucleon mass and Δ_{33} the mass of the wellknown $\Delta(1236)$ resonance[o]. In particular this fact can account for the well-observed bump in the total cross-section for reaction (4) at an entering total energy[7, 11, 15] of about 2.2 GeV, and for the inverse reaction[15] $d\pi^+ \rightarrow pp$; (iii) Our model explains why, in the reactions and regions here considered, the pion of the $(pn)\pi^+$ final system seems to "resonate" with each nucleon, without "deuteron breaking".

(x) - In any case, we can neglect the small deuteron binding energy, and not distinguish between a deuteron and a p-n couple at threshold.

(o) - A similar peak has been, obviously, observed in the $pp\pi$ system. See e. g. Reay et al.[8].

III. - GENERAL RELATIVISTIC FORMULATION.

On the basis of our previous considerations, it was tempting to perform detailed relativistic(x) calculations for the general case of reactions, with two initial and three final (spin-zero) particles:

$$\mu_1 + \mu_2 \rightarrow m_1 + m_2 + m_3 \tag{1'}$$

proceeding through the diagram depicted in Fig. 5. For simplicity, we neglect for the moment spin considerations. Afterwards, we will apply our formulas to the case (3)-(4). Subsequently, we will list other possible applications, which will be considered in further papers. The three internal masses are named $M_1 = M_R$, M_2, M_3 (in a clockwise sense). Particle M_R is a resonance, with halfwidth Γ, decaying into $M_3 + \mu$. Then M_3 rescatters with M_2. Quantities

$$E_1, E_2, E_3$$

are the final particle total energies.

In the overall c.m.s, and in natural units, the differential cross-section for the global process of Fig. 5 may be written

$$d\sigma = \frac{\pi^2 (g_1 g_2 g_3)^2}{4 E_1 E_2 E_3} \left| (s - \mu_1^2 - \mu_2^2)^2 - 4 \mu_1^2 \mu_2^2 \right|^{-1/2} \times$$

$$\times \delta^4 (P_i - P_f) \left| \mathcal{M}(s,z) \right|^2 d\vec{p}_1 \, d\vec{p}_2 \, d\vec{p}_3 \tag{7}$$

where: s is the square total energy, quantities P_i, P_f are the total initial and final four-momenta respectively, $\vec{P}_j$ $(j=1,2,3)$ are the final three-momenta, and $z \equiv (p_1 + p_2)^2$. To write (7) we factorized the triangle and vertex contribution, by supposing the vertex invariant amplitudes g_j, $(j=1,2,3)$, to be almost constant in the energy region of interest[16]. Quantity $\mathcal{M}$ is esentially the triangle contribution(o):

$$\delta^4 (P_i - P_f) \mathcal{M}(s,z) = \int \frac{d^4 q_1 \, d^4 q_2 \, d^4 q_3}{(q_1^2 - M_R^2)(q_2^2 - M_2^2)(q_3^2 - M_3^2)} \times$$

$$\times \delta^4 (k_1 + k_2 - q_1 + q_2) \, \delta^4 (p_3 + q_1 - q_3) \, \delta^4 (p_1 + p_2 + q_3 - q_2) \, , \tag{8}$$

(x) - Also in our energy regions ("near threshold") a non-relativistic evaluation would be lacking in correct meaning: e. g. in reaction (4) the pion c. m. kinetic energy equale about its rest-energy.

(o) - We use the metric (+ - - -).

k_1, k_2 being the initial four-momenta. So that:

$$(9)\qquad \mathcal{M}(s,z) = -\frac{i\pi^2}{2}\int_0^1 \frac{d\alpha_1 d\alpha_2 d\alpha_3\,\delta(1-\sum_{j=1}^{3}\alpha_j)}{(\sum_{j=1}^{3}\alpha_j M_j^2)-\alpha_1\alpha_3\mu^2-\alpha_1\alpha_2 s-\alpha_2\alpha_3 z}\,.$$

The total cross-section Lorentz-invariant expression, as function of the total energy s, is:

$$\sigma(s) = N(s)\int_{-1}^{1} d\cos\theta \int_{m_1}^{\infty}\sqrt{E_1^2-m_1^2}\,dE_1\int_{\mu}^{\infty}\delta(\sqrt{s}-\sum_{j=1}^{3}E_j)\quad x$$

$$(10)\qquad x\sqrt{E_3^2-\mu^2}\,dE_3\,\frac{|\mathcal{M}(s,z)|^2}{\sqrt{s}-E_1-E_3}\Bigg|_{\vec{p}_2=-\vec{p}_1-\vec{p}_3}\,,$$

where $N(s)\equiv 2\pi^4(g_1g_2g_3)^2\left|(s-\mu_1^2-\mu_2^2)^2-4\mu_1^2\mu_2^2\right|^{-1/2}$, and where, in performing angular integrations, we related $\vec{p}_1$ to a frame joint with $\vec{p}_3$, for every fixed $\vec{p}_3$-direction (while $\vec{p}_3$ refers to an arbitrary fixed frame). Precisely: $\cos\theta \equiv \hat{\vec{p}}_1\cdot\hat{\vec{p}}_3$, and: $z = z(E_1, E_3, \cos\theta)$.

Some kinematical details are given in Appendix A. By taking in due account the phase-space boundaries, we get:

$$\sigma(s) = N(s)\left\{\int_{-1}^{1} d\cos\theta\int_{m_1}^{a}\sqrt{E_1^2-m_1^2}\,dE_1\sqrt{E_3^{(-)2}-\mu^2}\quad x\right.$$

$$x\,\frac{|\mathcal{M}(s,E_1,E_3^{(-)},\cos\theta)|^2}{\sqrt{s}-E_1-E_3^{(-)}} + \int_{-1}^{0} d\cos\theta\int_{a}^{b(\cos\theta)}\sqrt{E_1^2-m_1^2}\,dE_1\quad x$$

$$(11)\qquad x\left[\frac{\sqrt{E_3^{(-)2}-\mu^2}\,|\mathcal{M}(s,E_1,E_3^{(-)},\cos\theta|^2}{\sqrt{s}-E_1-E_3^{(-)}} + \right.$$

$$\left.\left. + \frac{\sqrt{E_3^{(+)2}-\mu^2}\,|\mathcal{M}(s,E_1,E_3^{(+)},\cos\theta)|^2}{\sqrt{s}-E_1-E_3^{(+)}}\right]\right\},$$

where(x):

(x) - Sometimes the dependence on s is understood.

$$(12)\quad \begin{cases} a \equiv \dfrac{(\sqrt{s}-\mu)^2 + m_1^2 - m_2^2}{2(\sqrt{s}-\mu)} ; \\ b = b(\cos\theta) \equiv \\ \quad \equiv \dfrac{\sqrt{s}(c-2\mu^2) - \left\{ s(c-2\mu^2)^2 - (s-\mu^2\sin^2\theta)\left[c^2 - 4\mu^2(s+m_1^2\cos^2\theta)\right]\right\}^{1/2}}{2(s-\mu^2\sin^2\theta)} ; \\ c \equiv s + m_1^2 - m_2^2 + \mu^2 . \end{cases}$$

and where:

$$(13)\quad \begin{cases} E_3^{(\mp)} = \dfrac{AC \mp B\sqrt{D}}{A^2 - B^2} ; \\ A \equiv 2(\sqrt{s} - E_1); \quad B \equiv 2\sqrt{E_1^2 - m_1^2}\cos\theta ; \quad C \equiv s + R - 2E_1\sqrt{s} ; \\ D \equiv C^2 - \mu^2(A^2 - B^2) . \end{cases}$$

The explicit expression of $\mathcal{M}$ is given in Appendix B, following Valuev[16]. In Appendix C we forward also a formula yielding the μ-particle energy distribution in the particular case $m_1 = m_2$ (i. e., for example, the pion energy distribution in the reaction (3)). The Landau singularity associated with the triangle diagram here considered can appear on the "physical" sheet when[16] quantities y_j^2 (see B(2)) are either all greater or all smaller than +1. Our triangle graph (Fig. 5) satisfies that condition, provided that the intermediate particles M_R, M_2 can go on the mass-shell. In fact, its structure requires necessarily: $y_1 < -1$; $y_2 > +1$; $y_3 < -1$(x), the former two inequalities being always fulfilled and the latter one only implying $\sqrt{s} > M_R + M_2$. Therefore our formulas are intended to be considered for $\sqrt{s} > M_R + M_2$, and the expression of $\mathcal{M}$ has been evaluated in that region.

IV. - REACTION $pp \longrightarrow pn\pi$.

As an application, let us consider reaction (3), in which case: $\mu_1 = \mu_2 = m_1 = m_2$ are the nucleon mass and $m_3 = \mu$ is the pion one. The intermediate resonance is the $\Delta(1236)$. The mass values are taken from ref. (18). The numerical computations - somewhat lengthy in the "programming" phase - were performed by using a

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(x) - This third inequality is necessary only when singularity appearance is required to be possible.

UNIVAC-1108 computer.

Our model, if it is active, contributes - in the singularity region - by producing essencially p-n couples at threshold (deuteron production). So that in our phylosophy the model(o) should reproduce (in the 2.2 GeV/c^2 region) the observed peaking in the pp $\rightarrow$ dπ^+ reaction total cross-section. In Fig. 6 we forward our theoretical curve as function of the total energy (together with the Lorentz-invariant pn π "phase-space" behaviour). The existing data have uncertainties too large for our purposes, and are a little obsolete : no recent data are available for a satisfactory comparison in our anomalous singularity region. The normalization constant $c^{\alpha}\hbar^{\beta}(g_1 g_2 g_3)^2$ has the physical dimensions $[M^4 L^9 T^{-10}]$; its value resulted about : $(g_1 g_2 g_3)^2 \approx 3 \times 10^{-7}$, in MeV and natural units.

Our formulas really satisfy the conditions pointed out in Sec. III in the infinite range : $\sqrt{s} > M_\Delta + m$. But, as we considered the g_1, g_2, g_3 to be approximately constant, we must limit any comparison with experience in small energy ranges(x). And obviously, our mechanism will not be the only one contributing to the complete reaction (3)! In Fig. 7 we show only the (Lorentz-invariant) "phase-space" behaviour for reaction pp $\rightarrow$ pnπ^+ in a larger total-energy range.

The actual peak position does not result where expected from (5), but at an energy less than 1 % higher. The theoretical enhancement appears to be highly asymmetric, as it seems usual in the high energy field, at least according to other analogous triangle-diagram calculations in progress. Spin consideration seems to have no damping effect, but it could (slightly) enlarge the shape of the theoretical peak(1)

To obtain angular distributions, one had to take into account the correct dependences of the g_j, (j=1, 2, 3), on the proper kinematical variables.

Our model appears to make the hypothesis of existence of a di-proton resonance(15) an unnecessary one.

V. - OTHER POSSIBLE APPLICATIONS.

Other possible applications have been already proposed, e. g., by Month(14), who suggested to explain the "mesonic resonance"(18) E(1420) by means of the graph of Fig. 1 with R = $K^*(890)$; C = K; D = $\bar{K}$; E = π, and the "mesonic" peak(18, 19) $K_{3/2}(1175)$, with

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(o) - As our model is purely kinematical, the aim was not to obtain a theoretical curve perfectly comparable with the experimental data.
(x) - In fact, e. g., the quantity g_1 (of course related to the elastic p-n scattering amplitude) drops down rapidly as one gets far from the (p-n) zero relative-energy region.

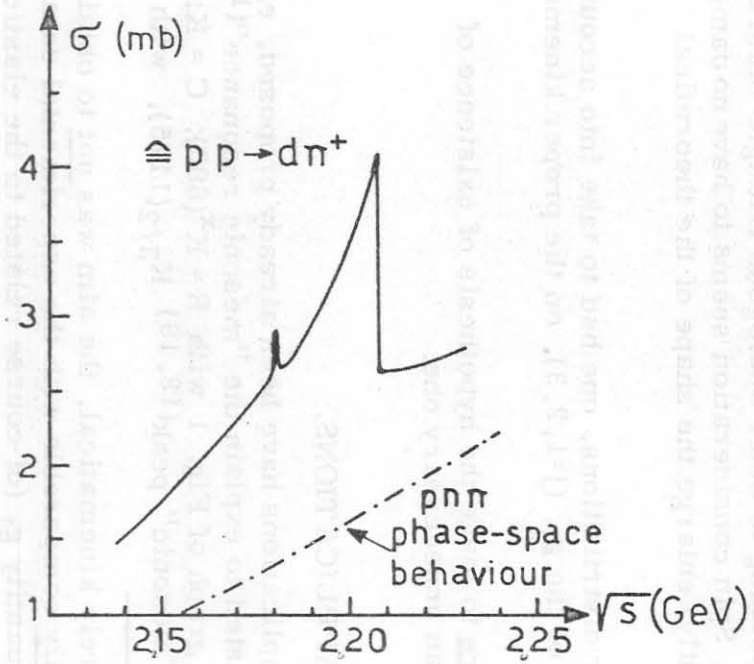

FIG. 6 - Our theoretical curve for reaction (3) cross-section, in the singularity region, as function of the total energy. Our model sems to explain the peak for deuteron production, ruling out the "di-proton" resonance hypothesis (see the text).

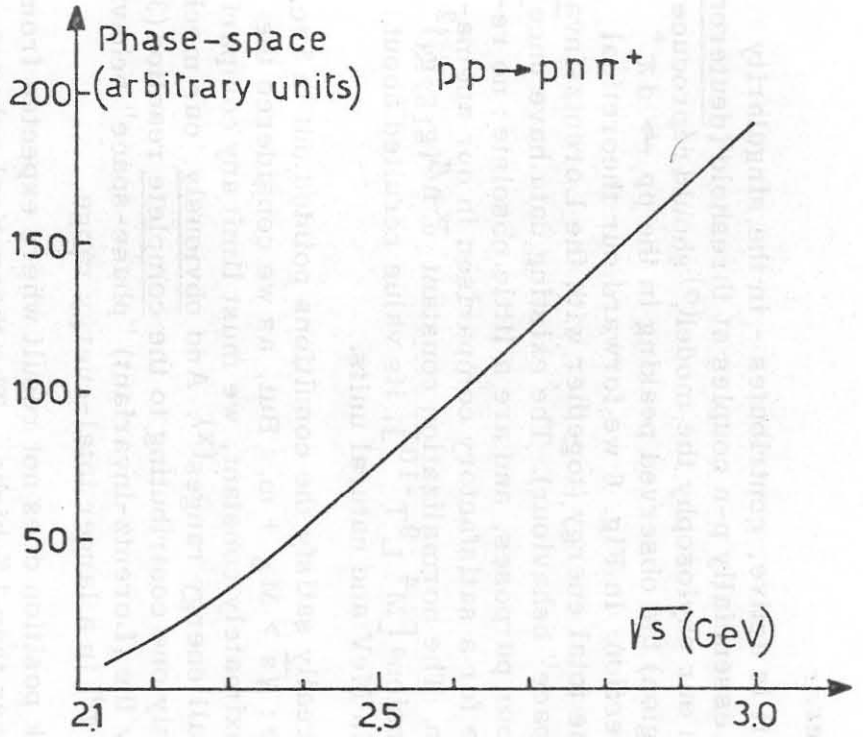

FIG.7 - Lorentz-invariant "phase-space" behaviour for the (complete) reaction pp → pn π^+ as function of the total energy.

R = $K^*(890)$; C = D = π; E = K. The same mechanism (with R = ϱ and C = D = E = π) has been involved[14] also for the A_1 enhancement[18,19]; in that case one expects a grouping of the events at the three kinematical "extremes" of a Dalitz plot[14]. Really, those three clusters where observed[20], but at a different effective mass M($\pi\pi\pi$), intermediate between the A_1 and A_2. We report that fact, without being able at present to clarify this situation.

As already said, we consider useful - in any case - to identify possible kinematical effects, e. g. , of the type here studied, expecially when they contribute in resonance regions, in order to explain some partial apparent "resonance decays", as well as to compare the experimental distributions with a curve more sophisticated than the phase-space one. We list here some other possible applications (see Fig. 8), that will be analysed in detail in a further paper:

a) We suggest that the triangle graph in Fig. 8a can contribute with a bump in the region of the resonances N(1680) and N(1688), which - at least in the inelastic processes - dominantly decay into N$\pi\pi$ with large widths. The intermediate N(1550) is known to decay into Nπ for the 35 %[18];

b) According to us, the graphs in Fig. 8b, with N(1550) and N'(1470) as intermediate resonances(x), may account for the humps D_{35}(1950) and P_{13}(1855), reported e. g. by Lovelace[22], and quoted by Rosenfeld et al.[18] as possible "threshold effects". Those humps have have been claimed e. g. by the CERN group[22] as a result of their Argand diagram analysis. Besides, it is interesting the graph[14,21] having $M_R = \Delta(1236)$, which could explain the fraction (50%) of N(1518) decaying into N$\pi\pi$ (or better the events with that decay and which fall kinematically under the N(1518) peak). The Δ^{++}(1236) just dominates the N(1518) decays into N$\pi\pi$, and itself is known to decay almost totally into Nπ;

c) Another graph[14,21] that would be of interest to study is the one depicted in Fig. 8c (with Λ(1520) as intermediate resonance), which might contribute to the Σ(1765) enhancement. The 15% of the events grouped under that peak correspond to decays just into Λ(1520)π;

d) We propose also that the complete diagram shown in Fig. 8d might give account for the "D^{+++}(2520)" enhancement, in the ppπ^+ effective-mass-distribution, observed in pp $\rightarrow$ pp$\pi^+\pi^-$ reactions at 4 GeV/c by Kidd et al.[23]. Successively, Alexander et al.[23] did not observe that peak but at a different (higher) entering momentum: 5. 5 GeV/c. On the contrary, Reay et al.[8], at other (lower) momenta: 2. 8, 3. 2 and 3. 65 GeV/c, did not found amy evidence for a true resonance D^{+++}, but noticed (at different angular directions) humps corresponding to M(ppπ) $\approx$ 2520 MeV/c^2 which ones were just inter

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(x) - These resonances where observed to decay abundantly into Nπ.

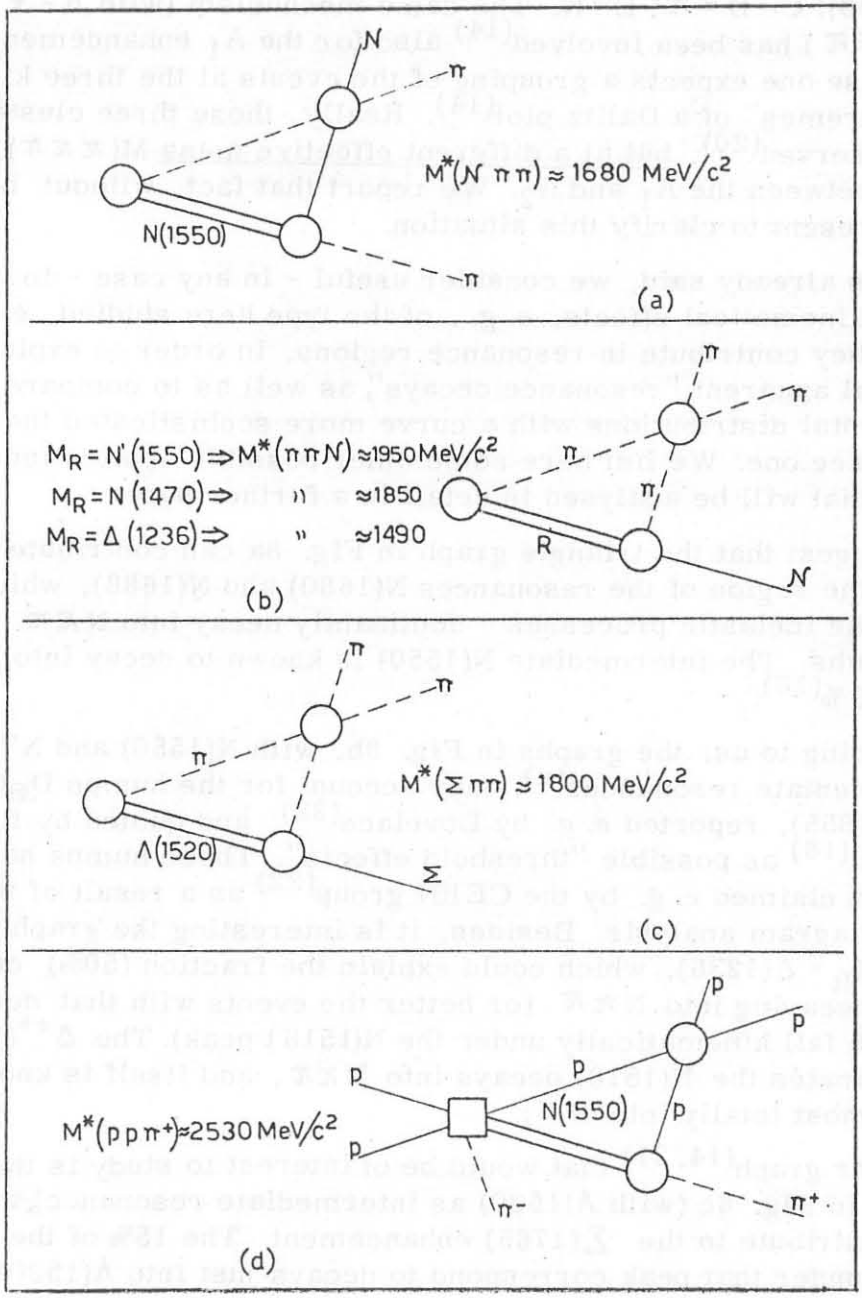

FIG. 8 - Other possible applications (see the text): proposed models for explaining (at least partially) some observed enhancements as possible kinematical effects due to final state interaction. We fit out in these figures only the positions of the peaks predicted by our mechanism according to the approximate formula (5). Graph (a) can contribute in the region of the resonances N(1680) and N(1688); graphs (b) may account for the bumps $D_{35}(1950)$ and $P_{13}(1855)$ quoted in ref. (18, 22), and for a (large) fraction of the N(1518) resonance; graph (c) might give some contribution to the $\Sigma(1765)$ enhancement; the diagram (d) could render account for the "$D^{+++}(2520)$" peak, in the ppπ^+ effective-mass distribution, reported in ref. (23).

preted as probable reflections of the dynamics of pion production(x). A priori, our "kinematical" model could allow to interpret why that peak was observed at some entering momenta and not at higher ones.

Many other kinematical effects may be supposed to influence the effective-mass-distributions: for examplification, diagrams of the type of Fig. 5, with $M_R = \Lambda(1405)$; $m_1 = m_2 = \pi$; $m_3 = \Sigma$, or with $M_R = \Lambda(1520)$; $m_1 = \Sigma$; $m_2 = m_3 = \pi$, might bring contributions in the region of $\Sigma(1660)$ resonance.

At this point, we want clarify that the (purely kinematical) conditions for actual "anomalous singularity" intervention impose really that the triangle degenerates into a "stright line". The Landau (logarithmic) singularity then arises when that degenerate triangle graph describes a three-step reaction involving particles on the mass-shell. First (see Fig. 5), in the c. m. s. particles M_1 and M_2 are (of course) produced on the same line, and then particle M_1 breaks up into paticles M_3 and μ, where M_3 moves backwards relative to the flight direction of μ, it reaches particle M_2 and rescatters or reacts(o).

In order that this mechanism be physically possible, the life-time τ of resonance M_1 should be sufficiently short ($\lesssim 10^{-18}$ s), and obviously the c. m. s. velocity of the primary-emitted particle M_2 must be smaller than the subsequently-emitted M_3. As one can expect, resonance M_1 lifetime has been shown to be obtainable from m_1, m_2 angular correlation and effective mass(24). For instance the lifetimes of some ^{13}N levels have been "measured"(25) by means of reaction $^{12}C + d \rightarrow {}^{12}C + n + p$.

Let us notice that a priori the intermediate resonances are allowed to decay also weakly, provided that the aformentioned condition ($\tau \lesssim 10^{-18}$ s) is approximately satisfied. But we mention that Valuev(16, 26) considered the possibility of determining even the Σ^o-lifetime, through the reaction $K^- + N_Z \rightarrow \Sigma^o + N_{Z-1} \rightarrow$ $\rightarrow \Lambda + \gamma + N_{Z-1}$ (with pair production in the final-nucleus field). Moreover, if we would "forget" the previous physical meaning or limitations of the triangle interactions, we might arrive to consider other weak graphs corresponding to "mass-formula" type(+) relations

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(x) - Analogous kinematical enhancements might be predicted for the $NN\pi$ system also (e. g.) at about 2480 MeV/c^2, 2630 MeV/c^2 and so on.

(o) - For instance, the contribution of a mechanism of this type to reaction (2) may be desumed also from the decay angular distribution for the $d\pi^+$ system, plotted by Buchner et al. (5).

(+) - Of course, one may write a series of "mass-formula" also in correspondence to above considered graphs; e. g.: $2(K^{*2}+K^2)-\pi^2 = E^2$; $2(K^{*2}+\pi^2)-K^2 = K_{3/2}(1175)^2$, $2(\Delta^2+\pi^2)-N^2 = N'(1470)^2$; $2\varrho^2+\pi^2 = A_1^2$. For instance, the last one is similar to the mass-formulas reported by S. Weinberg, Phys. Rev. Letters 18, 507 (1967) and by C. Lovelace, Phys. Letters 28B, 264 (1968).

like the following: $2(\Sigma^2+\pi^2)-N^2 = \Lambda(1405)^2$; $2(\Xi^{(-)2}+\pi^{(\pm)2})-\Lambda^2 = \Xi(1530)^2$, verified within about 1%.

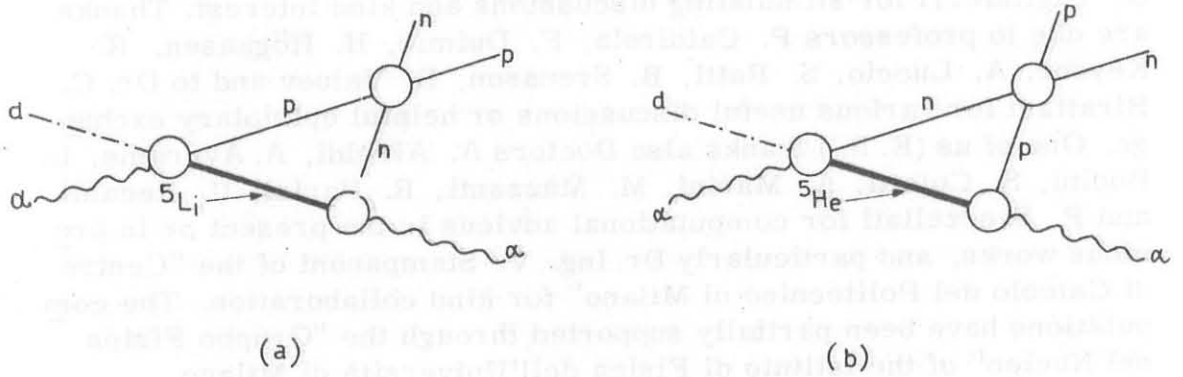
At last, we want to point out further applications in the very field of nuclear physics. In nuclear physics the triangle singularities seem to be more frequent and more active[2]: our preliminary evaluations stress that "triangle" peaks can usually appear in the lab. kinetic - energy range between few MeV and a few tenths MeV. It is already well known[27] that – for example in the projectile-induced break-up of a target nucleus X into two constituents: A+X → A+B+C (where B, C are the nucleus fragments) – final state interactions between pairs of particles can strongly modify the energy spectra of detected particles[2, 10]. Almost all existing measurements appear to show that this mechanism is the predominant one for bombarding energies around 10 MeV and for target nucleus mass $\mu_X > 2$ a. m. u. (In the case of X being a deuteron both processes seem to be present)[28]. We would like to suggest here, as an example, the diagrams represented in Fig. 9 and relative to reaction α(d, pn)α. One might predict some possible peaks in the total cross-section as function of the lab. kinetic energy. For instance, if we assumed ^{5}Li and ^{5}He in their ground levels, we should have peakings at about 5.5 MeV and 4 MeV, in the cases of Fig. 9a and 9b respectively. In Fig. 9c we show some points (connected with a dashed line only for indicative purpose(x)) of the theoretical cross-section for case (a). The phase-space behaviour is shown. It is worthwhile to add that, in the non-relativistic limit for the case of three final particles like in Fig. 5, the total cross-section peaking is expected to appear according to formula (5) at the lab. incoming kinetic energy:

$$T \approx \left[(\mu_1+\mu_2-Q+\Delta)^2+\frac{m_2}{m_1}\Delta(2m_3+\Delta)\right]^{1/2} - \tag{14}$$

$$-(\mu_1+\mu_2) \approx \Delta - Q, \tag{14'}$$

where Q is the reaction Q-value and where Δ is the kinetic energy at which the resonant state R, between m_1 and m_3, is observed (in its partial c. m. s.). In nuclear physics one can use the approximate very simple formula (14'), which differs from (14) within 1% in the usual situations. It may be noticed that, when the intermediate resonance is a target-nucleus excited state, the eventual enhancement according to (14') should appear at energies just enough for the target excitation.

(x) - More details will be given in further papers, when supports will be available to perform other computer elaborations.



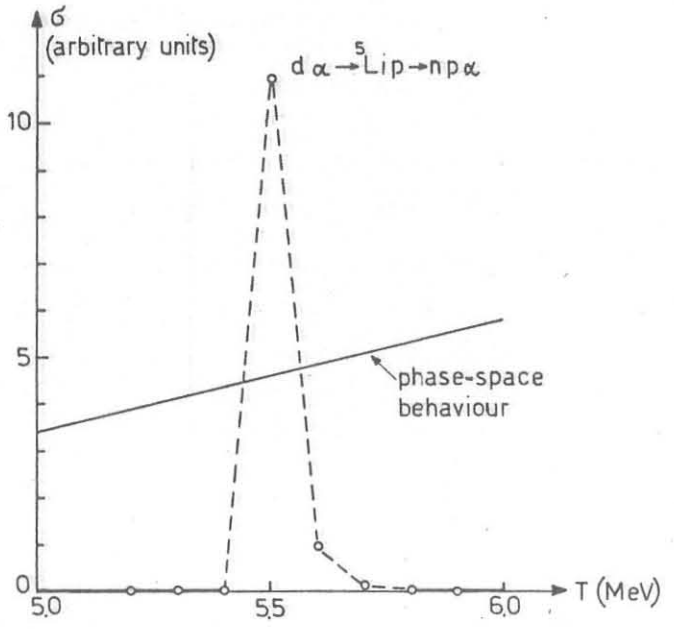


FIG. 9 - A possible application in nuclear physics. Fig. (a) and (b) refer to different intermediate states. Considering their ground levels, we should have peakings in the total cross-section e. g. at about 5. 5 and 4 MeV, respectively, of lab. initial kinetic energy. Fig. 9(c) shows some points of the theoretical cross-section for diagram (a) and the Lorentz-invariant "phase-space" behaviour. The dashed line is drawn only for indicative purpose.

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APPENDIX A

Our case considers two initial and three final (relativistic) particles. In our model (see Fig. 5) we can choose the variables:

$$s \equiv (p_1 + p_2 + p_3)^2$$

$$E_1 \equiv p_{10}$$

$$\cos\theta \equiv \hat{\vec{p}}_1 \cdot \hat{\vec{p}}_3$$

In our particular case we need three invariants (instead of five); we used:

$$s = s$$

$$z = m_2^2 - m_1^2 + 2E_1(\sqrt{s} - E_3) + 2\cos\theta \sqrt{(E_1^2 - m_1^2)(E_3^2 - \mu^2)}$$

$$v \equiv (p_2 + p_3)^2 = s + m_1^2 - 2E_1\sqrt{s}\ ,$$

where obviously $E_3 = E_3(s, E_1, \cos\theta)$.

For every fixed value of s the remaining variables E_1 and $\cos\theta$ may vary in the connected domain[17]:

$$(\cos\theta < 0 \cap m_1 \leq E_1 \leq b(s, \cos\theta)) \cup (\cos\theta \geq 0 \cap m_1 \leq E_1 \leq a(s)).$$

The conservation laws allow E_3 to assume in the whole (previous) definition region the value $E_3^{(-)}$ (see eq. (13) of the text). Besides, E_3 may assume a second value, $E_3^{(+)}$, in the restricted region[17]:

$$(\cos\theta \leq 0 \cap a(s) \leq E_1 \leq b(s, \cos\theta))\ .$$

APPENDIX B.

Following Valuev[16], the explicit expression of $\mathcal{M}$ (see formula (9) of the text) may be written, for $\sqrt{s} > M_1 + M_2$:

$$\text{(B1)}\qquad \frac{2i}{\pi^2}\mathcal{M}(s,z) = A_0 + \frac{i\pi P_1}{2\sqrt{\lambda}}\left\{\frac{\ln A_1}{\sqrt{P_1^2 - i\gamma_1 z}} + \frac{\ln A_3}{\sqrt{P_3^2 - i\gamma_1 s}}\right\},$$

where ($z_1 \equiv z$; $z_2 \equiv \mu^2$; $z_3 \equiv s$):

$$A_j = \frac{z_j \Phi + i \gamma_1 z_j}{\left(\sqrt{\lambda R_j} + \sqrt{P_j - i \gamma_1 z_j}\right)^2} \quad , \qquad (j = 1 \text{ and } 3)$$

$$A_o = \frac{1}{4} \sum_{i=1}^{3} \left\{ \int_0^{R_i/z_i} \frac{dt}{\Phi - \lambda t} \frac{P_i}{(R_i - z_i t)} \right. \qquad x$$

$$x \ \ln \left(\frac{\zeta_i + t + (R_i - z_i t)^{1/2}}{\zeta_i + t - (R_i - z_i t)^{1/2}} \right) \qquad +$$

$$\left. + \int_{R_i/z_i}^{\infty} \frac{dt}{\Phi - \lambda t} \frac{2P_i}{(z_i t - R_i)^{1/2}} \ \text{arctg} \frac{(z_i t - R_i)^{1/2}}{\zeta_i + t} \right\}$$

$$\gamma_1 \equiv P_1 M_R \Gamma$$

$$4\lambda \equiv z^2 + \mu^2 + s^2 - 2\left[(\mu^2 + s)z + \mu^2 s\right]$$

$$R_j \equiv \zeta_j^2 - M_i^2 M_k^2 \qquad \text{(j, i, k) cyclic permutation of (1, 2, 3)}$$

$$P_j \equiv M_i^2 M_k^2 - \zeta_j^2 + \zeta_j^2 (\zeta_j + \zeta_k) - M_i^2 \zeta_i - M_k \zeta_k$$

(B2) $$\zeta_j \equiv \frac{1}{2}(M_k + M_i - z_j) = M_k M_i y_j$$

$$\Phi \equiv \frac{\lambda R_2 - P_2^2}{\mu^2} \ .$$

The quantities A_i, (i = 0, 1 and 3), are complex; the determination such that $-\pi < \text{Im ln } A < +\pi$ is to be taken for the values of the natural logaritms appearing in (B1). Formula (B1) has been derived under the conditions $y_2 > 1$ and $y_1, \ y_3 < -1$, in order to be able to represent[3, 16] the "singular" amplitude corresponding to our diagram (see Fig. 5): that is to say in the region $\sqrt{s} > M_R + M_2$ (see sect. III).

APPENDIX C.

In this appendix we want forward the energy distribution of particle $m_3 \equiv \mu$ at a given total energy.

To fix the ideas and for semplicity, let us consider the particular reaction:

$$p + p \rightarrow p + n + \pi^+,$$

(in which $m_1 = m_2 = \mu_1 = \mu_2 = m$), under the usual condition $\sqrt{s} > M_\Delta + m$.

In our model (see Fig. 5), the pion energy distribution will be expressed as follows ($\mathcal{E}$ = c. m. pion total energy) at fixed $\sqrt{s}$:

$$\frac{d\sigma}{d\mathcal{E}} = \frac{(2\pi)^4 (g_1 g_2 g_3)^2 \varepsilon}{|s^2 - 4sm^2|^{1/2}} \left\{ \theta(\mathcal{E}_o - \mathcal{E}) \left[\Sigma_1^- + \Sigma_1^+ \right] + \right.$$

(C1)

$$\left. + \theta(\mathcal{E} - \mathcal{E}_o) \left[\Sigma_2^- + \Sigma_2^+ \right] \right\}$$

where :

$$\varepsilon \equiv \sqrt{\mathcal{E}^2 - \mu^2}$$

$$\Sigma_j^{\mp} = \Sigma_j^{\mp}(\mathcal{E}) \equiv (\pm 1)^j \int_0^{d_j^{\pm}(\mathcal{E})} d\cos\theta \left[E_1^{(\mp)2} - m^2 \right]^{1/2} \times$$

$$\times \frac{|\mathcal{M}^{\mp}|^2}{\sqrt{s} - E_1^{(\mp)} - \mathcal{E}}$$

$$d_1^{\pm}(\mathcal{E}) = \pm 1; \qquad d_2^{\pm}(\mathcal{E}) = d(\mathcal{E})$$

$$d(\mathcal{E}) \equiv \left[\frac{4m^2\alpha^2 - \beta^2}{4m^2\varepsilon^2} \right]^{1/2}$$

$$\mathcal{M}^{\mp} \equiv \mathcal{M}(s,z) \Big|_{E_2 = \sqrt{s} - E_1^{(\mp)} - \mathcal{E};\ \vec{p}_2 = -\vec{p}_1 - \vec{p}_3} =$$

$$= \mathcal{M}(s, \mathcal{E}, E_1^{(\mp)}, \cos\theta)$$

$$E_1^{(\bar{+})} = E_1^{(\mp)}(\mathcal{E}, \cos\theta) \equiv \frac{\alpha\beta \pm 2m\varepsilon^2\sqrt{\cos^2\theta(\cos^2\theta - d)}}{2\left[\varepsilon^2\cos^2\theta - \alpha^2\right]}$$

$$\mathcal{E}_o \equiv \frac{s - 2m\sqrt{s} + \mu^2}{2(\sqrt{s} - m)}$$

$$\alpha \equiv \mathcal{E} - \sqrt{s}; \quad \beta \equiv s - 2\mathcal{E}\sqrt{s} + \mu^2.$$

The quantities $\mathcal{M}$ and z are defined as in Appendixes A and B. The kinematical limits of $\mathcal{E}$ are:

$$\mu \leq \mathcal{E} \leq \frac{(s - 4m^2 + \mu^2)}{2\sqrt{s}} .$$

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