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π^+ - PROTON PHASE SHIFT ANALYSIS UP TO 500 MeV

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1. - INTRODUCTION

After the discovery of the pi-meson in the cosmic radiation, many physicists concentrated their efforts to the study of pion interactions with nuclear matter. From the theoretical point of view, the pion-nucleon scattering phenomena seem to be the most easily understandable and the most useful in order to obtain informations on the effective pion-nucleon potential and on the nuclear forces in general. It is well known that many positive results have been obtained on this field but it cannot be stated that the problem is completely solved whether experimentally or theoretically. In fact, the so called dynamical theories are powerless in the strong interaction problems and the dispersion relations are able to give only qualitative justifications of the experimental results.

On the other hand, experiments provided a large amount of data on the total and differential cross-sections, especially in the low energy region; unfortunately the accuracy of these data was not high enough to give an unambiguous panorama of the phenomena involved. Some phenomenological approaches ([1]) ([2]) to obtain a system of pion-nucleon phase-shifts from a best fit of all the existing experimental data by the aid of theoretical considerations have been performed in the last years but a confirmation, based upon much more precise data, is necessary, according also to the Authors.

The purpose of this work is essentially the search of a possible answer to the following questions:

a) what is the maximum number of pure experimental informations which could be obtained by the analysis of all the available π^+ - proton scattering data;

b) what is the energy range and the maximum tolerable error for new experiments in order to obtain an unambiguous set of phase-shifts with satisfactory precision;

c) what is the better method to perform the phase-shifts analysis. In fact, the calculation of the phase-shifts from an experimental differential cross-section can be performed essentially in two ways, i.e. either by fitting directly the experimental points by a set of phases as parameters ([9]) or (method used by us) by fitting the coefficients A_n in the power series expansion of the angular distribution:

$$(1) \qquad \frac{1}{\lambda\!\!\!^{-2}} \frac{d\sigma}{d\Omega} = \Sigma_{n=0}^{n_{max}} A_n \cdot (\cos\vartheta)^n = \frac{1}{4\pi\lambda\!\!\!^{-2}} \sigma_{tot}^{el} \Sigma_{n=0}^{n_{max}} c_n \cdot (\cos\vartheta)^n .$$

The coefficients, so determined, allow then the evaluation of the phases by solving a suitable system of trascendental equations.

It is important to note that while in the first method it is always possible to find a set of solution, more or less acceptable, depending on the minimum χ^2 achieved, this may be not true in the second method because of the additional conditions imposed by the system of trascendental equations.

In fact, after a suitable choice of the A_n (choice depending on the minimum value of χ^2 achieved in the approximation used in the series expansion), it may happen that, although the A_n so determined have small errors, they are not able to satisfy the system of equations, or that, at some energy, the solutions are too critical with respect to the input data. Besides the method cannot take into account the Coulomb interaction which is dominant at small angles and low energies.

On the other hand there are obvious advantages such the possibility to discuss the solutions and the ambiguities which are always present.

As a first step to answer to the above mentioned questions, we determine the quantity $\frac{1}{4\pi\lambda\!\!\!^{-2}} \sigma_{tot}^{el}$ and the coefficients:

$$c_n = \frac{A_n}{\frac{1}{4\pi\lambda\!\!\!^{-2}} \sigma_{tot}^{el}} \tag{2}$$

as function of energy from a best fit of the experimental values with appropriate trial functions.

Furthermore, the behaviour of a possible phase-shifts set, as function of energy, is searched by using the coefficients A_n obtained from (2).

This procedure is adopted for several reasons; first of all it clearly appears that a direct fit from the phases given by the authors ([3]) is not of great significance because of the large discrepancies existing between them, especially in the case of the small phases. In our opinion these discrepancies, particularly marked near the first resonance, are not due to the experimental errors only. We will return to this point later. Moreover we preferred to do firstly the best fit of the total cross-section and the coefficients c_n defined by (2) instead of the A_n because they are generally determined experimentally by two independent measurements of the scattering process and also because the quantities c_n are slowly varying with energy and can be fitted quite well in all the energy-range 0-700 MeV with a relatively small number of terms in a series power expansion of momentum in the c.m. system.

In chapters 2 and 3 the method used for this fit is outlined with more details. Chapters 4 and 5 concern the detailed explanation of the method and the results on the phase shifts. Chapter 6 is dedicated to a general discussion and to the conclusions which can be drawn from this research. Anyhow we can anticipate that, in our personal feeling, if one really want to improve the knowledge of the behaviour of the phases against energy (a problem which in principle could be criticized if necessary or not) it is absolutely necessary, in several cases, to repeat already performed experiments with a much more high degree of accuracy.

2. - BEST FIT OF THE ELASTIC TOTAL CROSS-SECTION

In this chapter we are dealing with the problem of finding an analytical expression of the total elastic cross-section as function of the energy. Unfortunately theories are of little aid in the search of the from of this function especially for energies greater than 200 MeV. After several tentatives we used a semi-empirical expression determined in such a way to obtain, for the entire range 0-700 MeV, a χ^2 of the same order as the value reached by a fit with the pure Chew-Low formula in the range 0-150 MeV.

Putting:

$$\frac{\sigma^{el}_{tot}}{4\pi\lambda^2} = \mathrm{sen}^2\alpha_3 + \mathrm{sen}^2\alpha_{31} + 2\,\mathrm{sen}^2\alpha_{33} \tag{3}$$

the expression of α_3, α_{31} and $\mathrm{sen}^2\alpha_{33}$ are selected of the following form:

$$\begin{cases} \alpha_3 = a_3 \cdot k + a_3' \cdot k^2 + a_3'' \cdot k^3 + \dots \\ \alpha_{31} = a_{31} \cdot k^3 + a_{31}' \cdot k^4 + a_{31}'' \cdot k^5 + \dots \end{cases} \tag{4}$$

$$\mathrm{sen}^2\alpha_{33} = \frac{h \cdot \frac{k^6}{\omega^2}\left[\frac{1 + h_1 \cdot k^4}{1 + h_2 \cdot k^4}\right]}{\left[1 - \frac{\omega}{\omega_0} - h_3 \frac{a^2}{[1 - \frac{h^4}{k^4}]^2 + h_5}\right]^2 + h \cdot \frac{k^6}{\omega^2}\left[\frac{1 + h_1 . k^4}{1 + h_2 \cdot k^4}\right]} \tag{5}$$

with k = momentum in the c.m. system in units $c = m_\pi = 1$.

$$\omega = \sqrt{M^2 + k^2} + \sqrt{1 + k^2} - M \text{ and } M = \text{proton mass.}$$

The quantities a_3, a_3', ... a_{31}, ... h, h_1, h_2, h_3, h_4, h_5 and ω_0 are the parameters to be determined by the fit with the experimental data [4] [5] [6]. The best fit was split into two parts; first of all the parameters a_3, h and ω_0 are determined by fitting the experimental data between $E = \omega - 1 = 0$ and a suitable energy E_{max} by the semplified formula:

$$\frac{\sigma^{el}_{tot}}{4\pi\lambda^2} = \mathrm{sen}^2\,(a_3 \cdot k) + \mathrm{sen}^2\,(a_{31} \cdot k^3) + \frac{2h \cdot \frac{k^6}{\omega^2}}{\left(1 - \frac{\omega}{\omega_0}\right)^2 + h\,\frac{k^6}{\omega^2}} \tag{6}$$

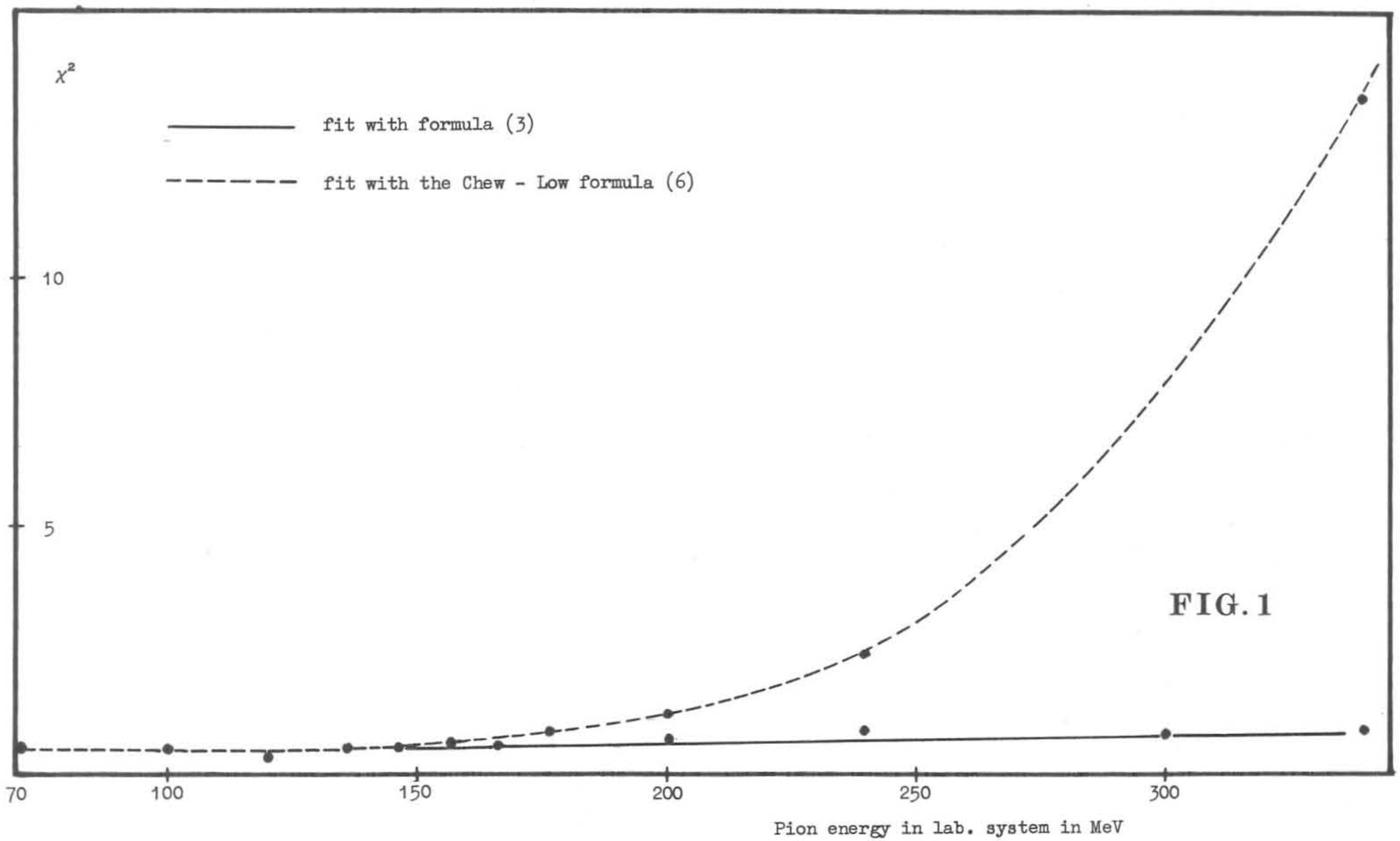
with $a_{31} = -0.04$.

The last term in (6), which is obtained from (5) if $h_1 = h_2 = h_3 = 0$, is simply the Chew-Low formula with $h = \left(\frac{4}{3} f^2\right)^2$.

The fit is performed with increasing values of E_{max}; we verified that up to E_{max}= 150 MeV the expression (6) represents a very good approximation for the total elastic cross-section as it can be seen from Fig. 1 where the χ^2-values as a function of E_{max} are reported (dashed line).

The weighted mean value of the quantities a_3, ω_0, h and f^2 are:

$$\begin{aligned} a_3 &= -0.089 \pm 0.004 \\ \omega_0 &= 2.19 \pm 0.02 \\ h &= 0.015 \pm 0.001 \end{aligned} \qquad f^2 = 0.092 \pm 0.004 \tag{7}$$



χ^2
fit with formula (3)
fit with the Chew - Low formula (6)
10
5
70
100
150
200
250
300
Pion energy in lab. system in MeV

FIG. 1

The value of a_3 agrees with the Hamilton and Woolcock (7) value, while f^2 is certainly larger than the value determined by them.

The a_{31} parameter is kept fixed at the value - 0.04 as suggested by many authors(7) (8); this was done because of the scarce experimental points at very low energies are not able to determine it better. In Fig. 2 the function (6), calculated with the parameters (7), is drawn (dashed line).

The remaining parameters of (4) are obtained by the fit of the experimental data in the range 0-700 MeV keeping fixed the values (7).

The final result is:

$$(8) \qquad \left\{ \begin{array}{lll} h_1 = (0.97 \pm 1.20)\cdot 10^{-4} & \qquad & a_3' = (2.48 \pm 2.38)\cdot 10^{-2} \\ h_2 = (2.79 \pm 2.00)\cdot 10^{-2} & & a_3'' = -(5.86 \pm 4.30)\cdot 10^{-2} \\ h_3 = (1.40 \pm 0.71)\cdot 10^{-2} & & a_3''' = (1.51 \pm 1.14)\cdot 10^{-2} \\ h_4 = 32.5 \pm 3.1 & & a_{31}' = (0.86 \pm 1.27)\cdot 10^{-2} \\ h_5 = 6.95 \pm 3.52 & & a_{31}'' = (1.36 \pm 3.40)\cdot 10^{-3} \; . \end{array} \right.$$

In Fig. 2 the graphical results and the corridor of errors are shown (solid line). The χ^2-value is pratically constant over the entire range of energy and is of the order of 0.8 (Fig. 1 solid line).

Although the errors of the parameters (8) are rather large, the calculated values of the total elastic cross-section have errors of the order of some percent only; this fact is due to the strong correlation existing between the (8) which however does not have a direct physical meaning.

We point out that α_3, α_{31} and α_{33} in (3), (4) and (5), determined by our fit, can be considered as approximate values of the phases only in the energy region where d-wave effects are not relevant. Nevertheless we used the formula (3) (combined with (4) and (5)) for the entire energy-range because our purpose was to find a trial function only, regardless of any physical meaning of the quantities α_3, α_{31} and α_{33} involved.

3. - BEST FIT OF THE COEFFICIENTS OF THE ELASTIC DIFFERENTIAL CROSS-SECTION

The coefficients c_n in the expression (2) are calculated assuming a series expansion in power of k as trial function from which a best fit of the experimental values (3) is performed. The criterion used is to stop the expansion at the power where the first minimum of χ^2 is achieved.

Five coefficients only are evaluated i.e. c_0, c_1, c_2, c_3 and c_4, the last two being available from the experiments above 300 MeV although with rather large errors. This does not mean that d-waves are effective only above this energy but the experimental points at lower energies does not have the required precision and sensitivity to allow small effects of $l \geq 2$ states to be measurable.

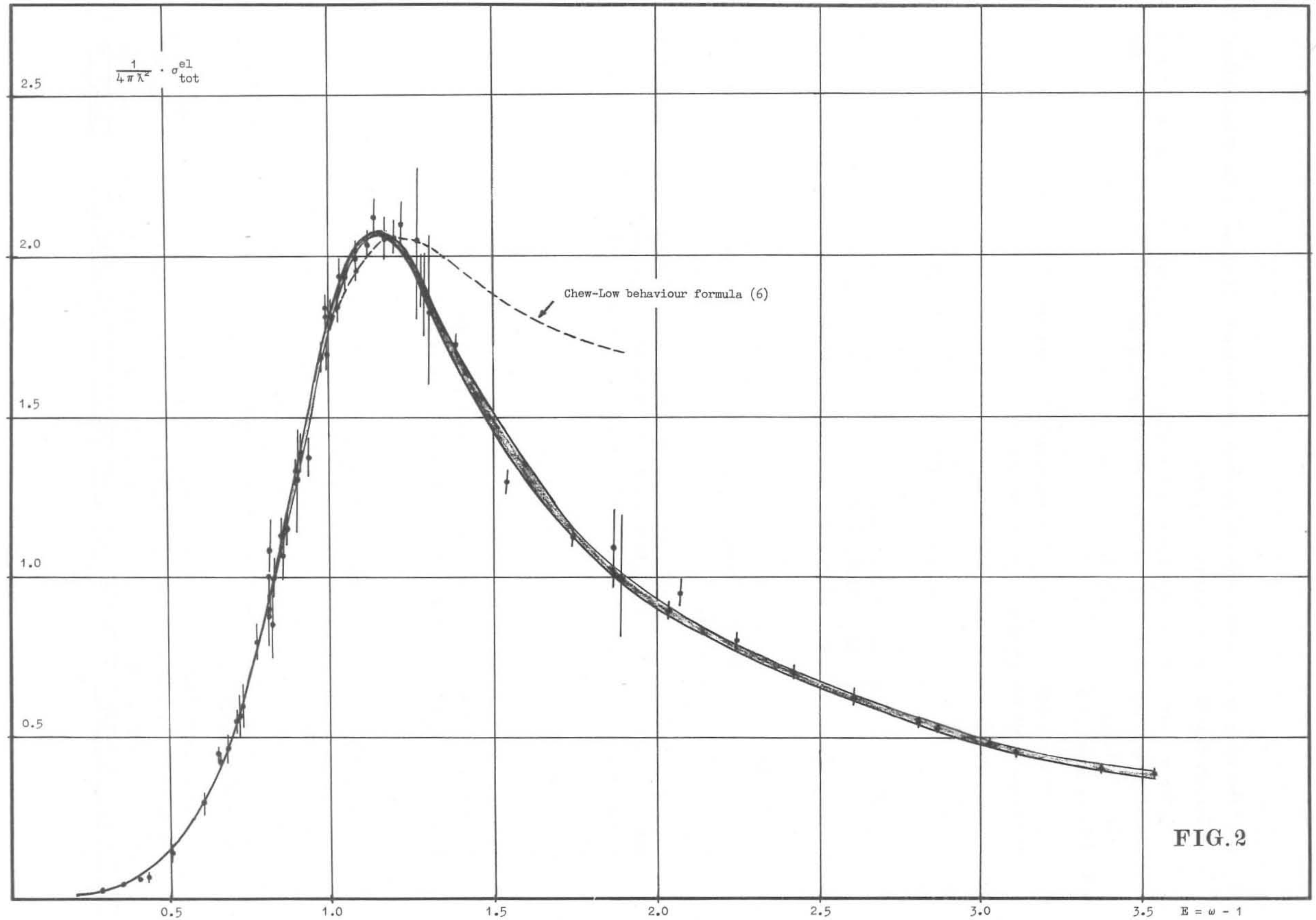

$\frac{1}{4\pi\lambda^2} \cdot \sigma^{el}_{tot}$
2.5
2.0
1.5
1.0
0.5
Chew-Low behaviour formula (6)
0.5
1.0
1.5
2.0
2.5
3.0
3.5
$E = \omega - 1$

FIG. 2

An important remark must be done at this point. It is clear that the coefficients of the terms in the polinomial (1), corresponding to higher powers (in our case A_3 and A_4), are more important near 0° and 180° in the differential cross-section. Due to the practical impossibility to measure directly the cross-section at these angles and due to the Coulomb effects in the forward direction, the determination of the A_i (or c_i) can be made cutting-off the points below some minimum angle which obviously decreases with the increasing energy. For this reason the large errors found for A_3 and A_4 (or c_3 and c_4) reflect the indetermination of the forward differential cross-section which can be only evaluated by extrapolation of the data at higher angles.

In some papers ([5]) ([6]) the coefficients of the differential cross-section are calculated putting the values at zero degree equal to the dispersion relations one; it is obvious that, in these cases, the c_n cannot be considered of a pure experimental origin, the effects being stronger on c_3 and c_4. It is our personal feeling that these forward values are lower than one could predict from the experiments especially from 300 to 400 MeV. This fact could explain the disagreement between c_3 and c_4 value of Foote et al. ([9]) with the Ogden ([6]) value at 310 MeV.

The c_n given, by fitting the available experimental data, are then:

$$
(9) \quad \begin{cases}
c_0 = 1 + (0.148 \pm 0.029)\cdot k - (0.952 \pm 0.100)\cdot k^2 + (0.708 \pm 0.100)\cdot k^3 - \\
\qquad - (0.218 \pm 0.040)\cdot k^4 + (0.025 \pm 0.005)\cdot k^5 \qquad \text{for } 0.3 < k < 3.4 \\
c_1 = - (5.76 \pm 0.32)\cdot k + (8.65 \pm 0.90)\cdot k^2 - (5.04 \pm 0.84)\cdot k^3 + \\
\qquad + (1.43 \pm 0.32)\cdot k^4 - (0.16 \pm 0.04)\cdot k^5 \qquad \text{for } 0.5 < k < 3.4 \\
c_2 = - (0.53 \pm 0.10)\cdot k \pm (3.19 \pm 0.35)\cdot k^2 - (2.48 \pm 0.37)\cdot k^3 + \\
\qquad + (0.78 \pm 0.15)\cdot k^4 - (0.088 \pm 0.020)\cdot k^5 \qquad \text{for } 0.3 < k < 3.4 \\
c_3 = - (1.12 \pm 1.06) + (0.94 \pm 0.82)\cdot k - (0.18 \pm 0.15)\cdot k^2 \\
\qquad \text{for } 2.0 < k < 3.4 \\
c_4 = 5 \cdot (1 - c_0 - \frac{c_2}{3}) \qquad \text{for } 2.0 < k < 3.4
\end{cases}
$$

For $k < 0.25$ (~6 MeV) there are no experiments; also in the case of c_1 there are still uncertainties for $k < 0.5$ (~20 MeV). These uncertainties clearly indicate the great difficulty in evaluating the coefficients at very low energies because the Coulomb effects are present also at larger angles. The curves corresponding to (9) are drawn in Fig. 3, 4, 5 and 6 with the corridor of errors and the experimental points (3).

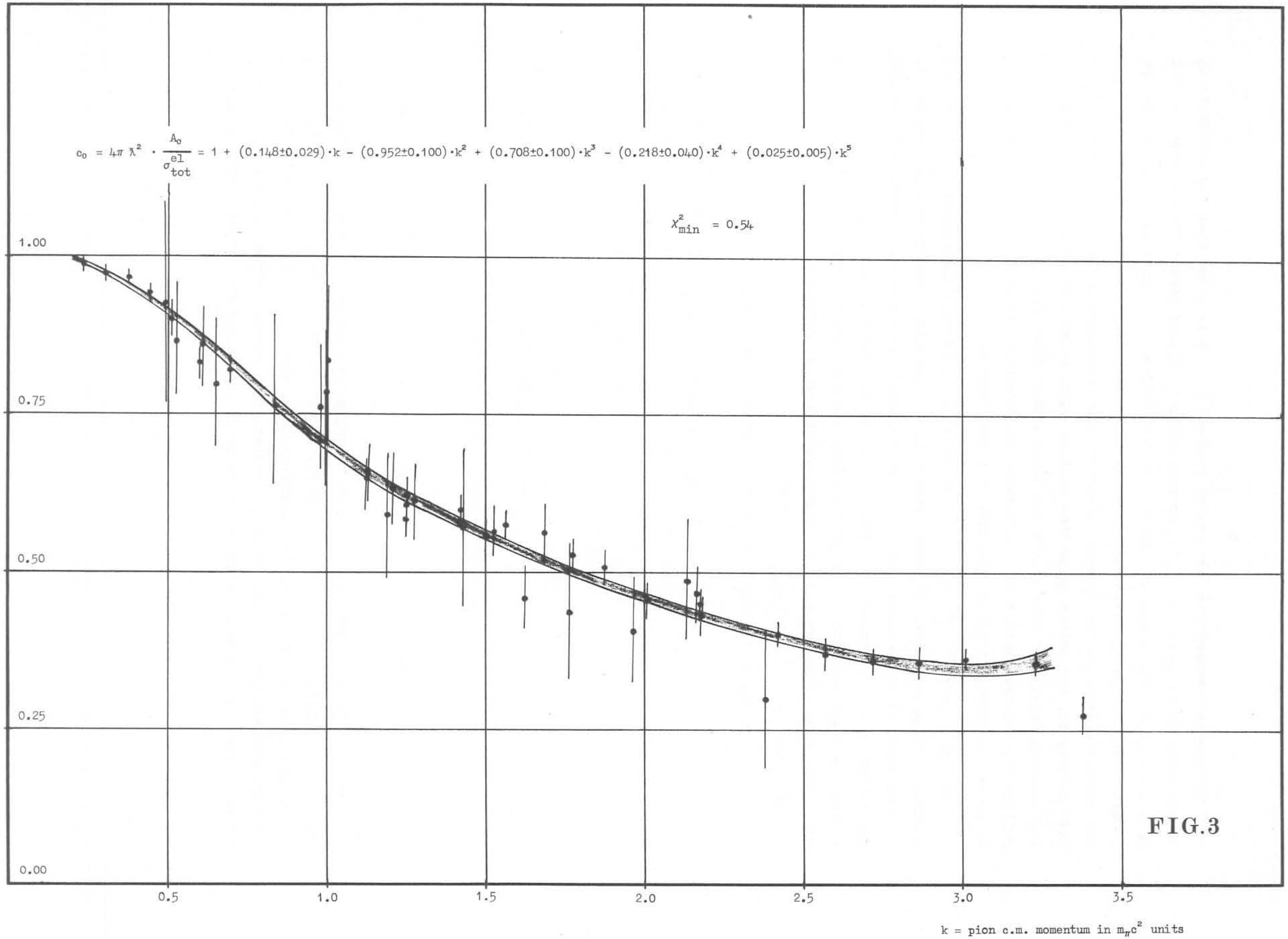
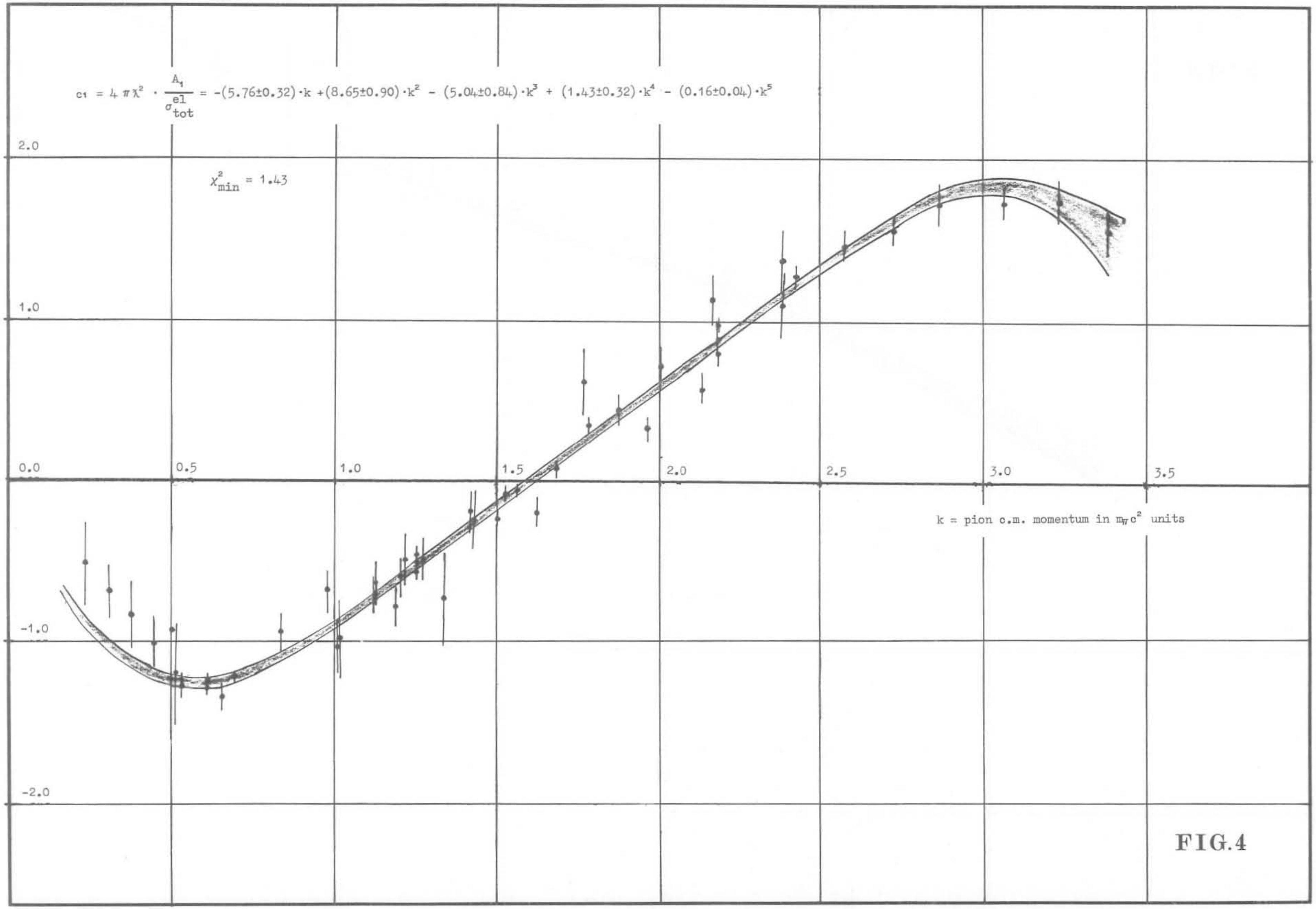

$c_0 = 4\pi \lambda^2 \cdot \frac{A_0}{\sigma^{el}_{tot}} = 1 + (0.148\pm0.029)\cdot k - (0.952\pm0.100)\cdot k^2 + (0.708\pm0.100)\cdot k^3 - (0.218\pm0.040)\cdot k^4 + (0.025\pm0.005)\cdot k^5$
$\chi^2_{min} = 0.54$
1.00
0.75
0.50
0.25
0.00
0.5
1.0
1.5
2.0
2.5
3.0
3.5
k = pion c.m. momentum in $m_\pi c^2$ units

FIG.3



$c_1 = 4\pi\lambda^2 \cdot \frac{A_1}{\sigma^{el}_{tot}} = -(5.76\pm0.32)\cdot k + (8.65\pm0.90)\cdot k^2 - (5.04\pm0.84)\cdot k^3 + (1.43\pm0.32)\cdot k^4 - (0.16\pm0.04)\cdot k^5$
$\chi^2_{min} = 1.43$
2.0
1.0
0.0
-1.0
-2.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
k = pion c.m. momentum in $m_\pi c^2$ units

FIG.4

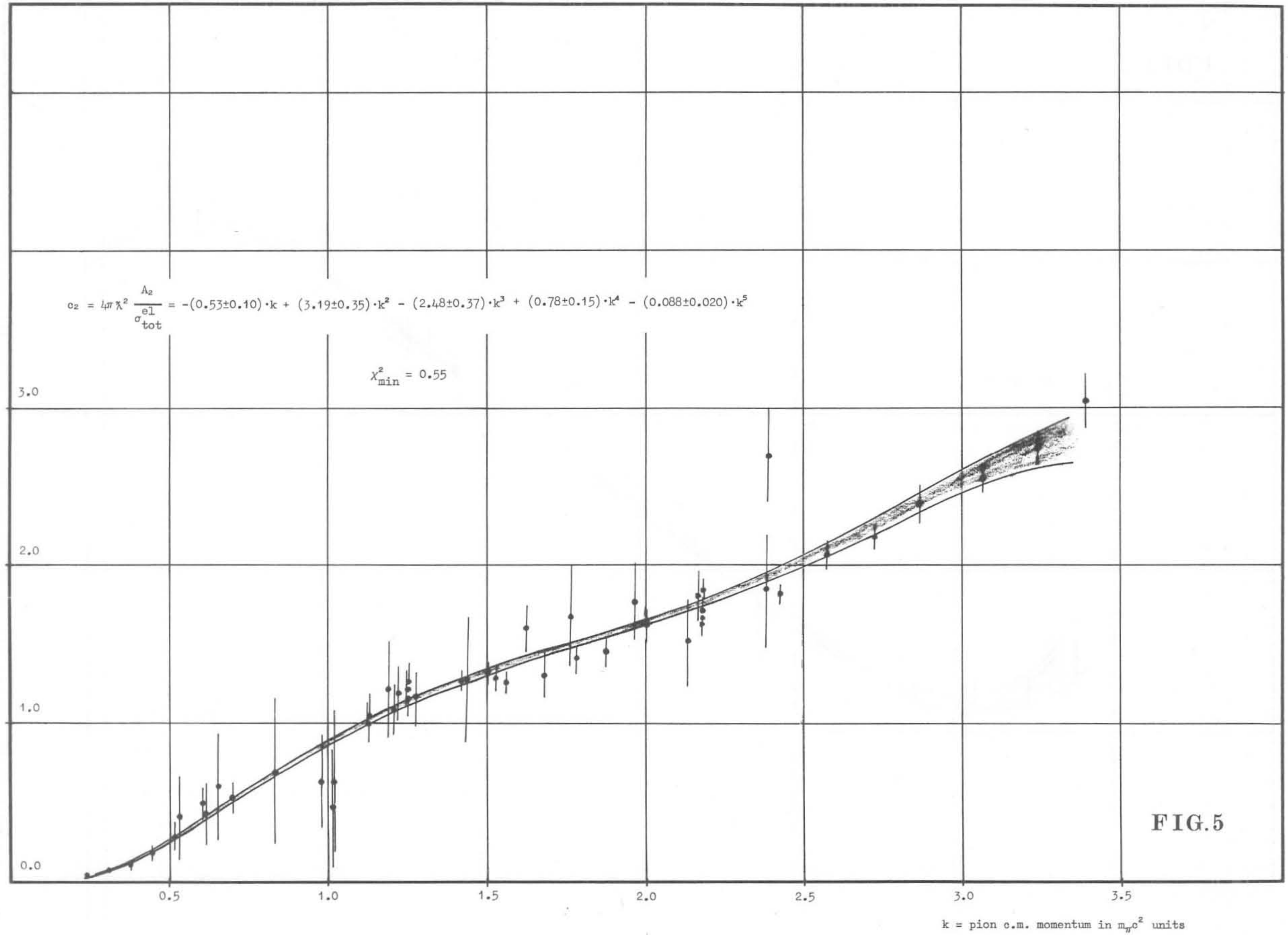
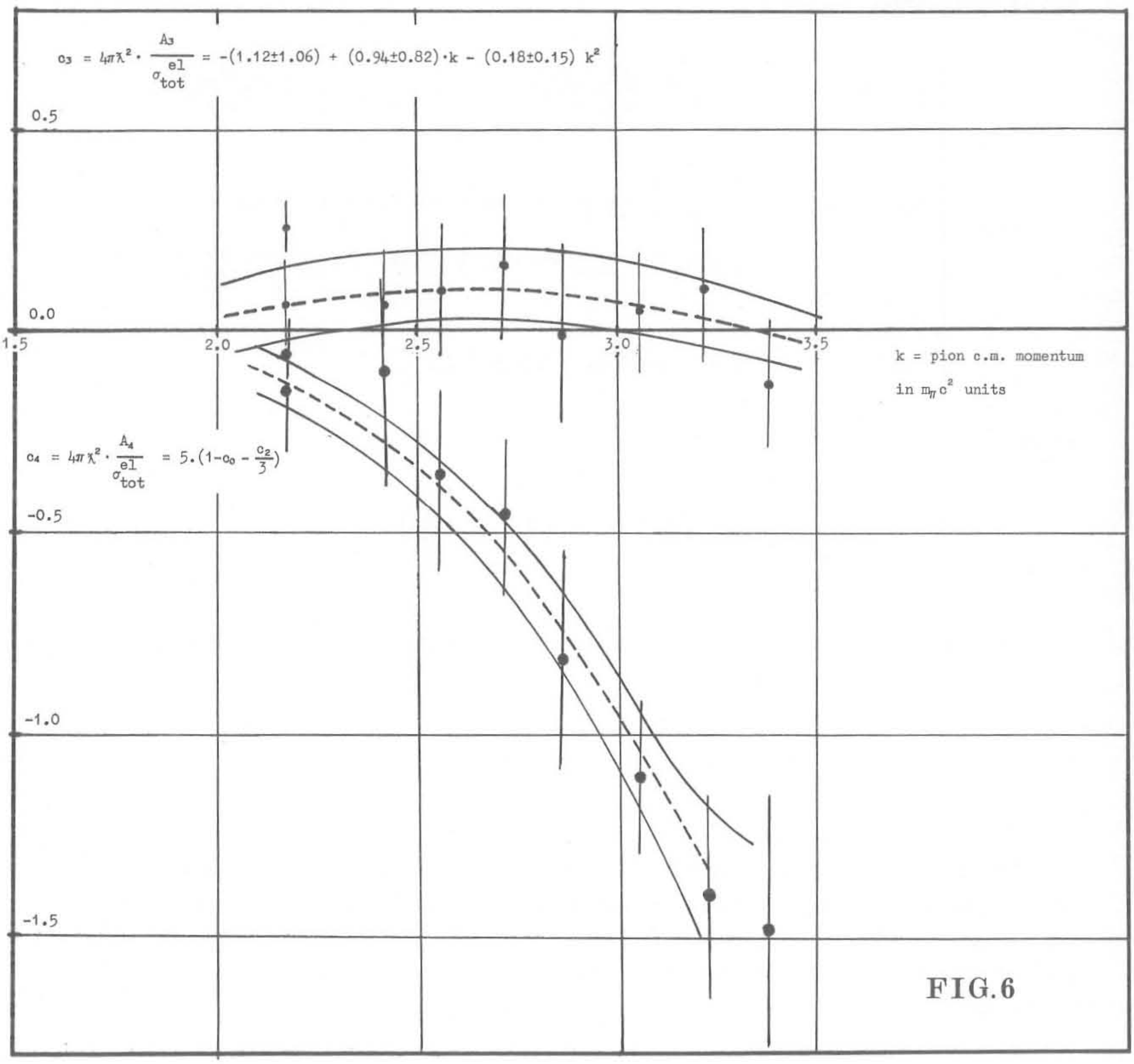

$c_2 = 4\pi \bar{\lambda}^2 \frac{A_2}{\sigma^{el}_{tot}} = -(0.53\pm0.10)\cdot k + (3.19\pm0.35)\cdot k^2 - (2.48\pm0.37)\cdot k^3 + (0.78\pm0.15)\cdot k^4 - (0.088\pm0.020)\cdot k^5$
$\chi^2_{min} = 0.55$
3.0
2.0
1.0
0.0
0.5
1.0
1.5
2.0
2.5
3.0
3.5
k = pion c.m. momentum in $m_\pi c^2$ units

FIG.5



$c_3 = 4\pi ƛ^2 \cdot \frac{A_3}{\sigma^{el}_{tot}} = -(1.12\pm1.06) + (0.94\pm0.82)\cdot k - (0.18\pm0.15)\ k^2$
$c_4 = 4\pi ƛ^2 \cdot \frac{A_4}{\sigma^{el}_{tot}} = 5.(1-c_0 - \frac{c_2}{3})$
0.5
0.0
-0.5
-1.0
-1.5
1.5
2.0
2.5
3.0
3.5
k = pion c.m. momentum
in $m_\pi c^2$ units

FIG.6

4\. - s and p PHASE-SHIFTS CALCULATION

In the energy range 0-300 MeV we calculate the s and p phases only because there are no experimental evidence of the existence of significant c_3 and c_4 values. In this case one can write down the cosines of the phases as simple algebric functions of A_0, A_1 and A_2 as follows:

$$(10)\quad \begin{cases} (\cos 2\alpha_3)_{\pm} = (u-2)\dfrac{u^2+v^2-4(\omega+1)}{2[(u-2)^2+v^2]} \pm |v|\cdot\left|\sqrt{\dfrac{1}{(u-2)^2+v^2}-\left[\dfrac{u^2+v^2-4(\omega+1)}{2[(u-2)^2+v^2]}\right]^2}\right| \\[2ex] \cos 2\alpha_{33} = \dfrac{1}{4(\cos 2\alpha_3-\omega)-5}\cdot\Big\{(\cos 2\alpha_3-u)(2-\cos 2\alpha_3+\omega) + \\ \qquad + \sqrt{[(\cos 2\alpha_3-\omega)^2][(\cos 2\alpha_3-u)^2+4(\cos 2\alpha_3+\omega)-5]}\Big\} \\[2ex] \cos 2\alpha_{31} = 2u - \cos 2\alpha_3 - 2\cos 2\alpha_{33} \end{cases}$$

where:

$$(11)\quad \begin{cases} u = 2\left(2-\dfrac{1}{4\pi\lambda^2}\,\sigma^{el}_{tot}\right) = 2\left(2-A_0-\dfrac{A_2}{3}\right) \\[2ex] v = \pm 2\sqrt{\dfrac{1}{\lambda^2}\left[\dfrac{d\sigma}{d\Omega}\right]_{\vartheta=0} - \left(\dfrac{1}{4\pi\lambda^2}\,\sigma^{el}_{tot}\right)^2} = \pm 2\sqrt{A_0+A_1+A_2-\left(A_0+\dfrac{A_2}{3}\right)^2} \\[2ex] \omega = 2\left[1-\dfrac{1}{\lambda^2}\left(\dfrac{d\sigma}{d\Omega}\right)_{\vartheta=90^\circ}\right] = 2(1-A_0) \end{cases}$$

In the above equations only Fermi solutions are considered for the p-phases; this means that the plus sign is taken for the square root in the $\cos(2\alpha_{33})$ expression. The α_3 has an ambiguity which was already pointed out by Pisent [10] and comes from the sign of the double square root in the first equation (11). The sign can be selected by the following simple criterion: in the energy range from 45 to 150 MeV, the solution with the minus sign gives an α_3 which is negative, very high and too rapidly varying with energy (at about 42 MeV $\alpha_3^+ = -4°, 6$ and $\alpha_3^- = -7°, 3$; at 128 MeV $a_3^+ = -13°, 8$ and $\alpha_3^- = -138°$); we discarded this solution.

Beyond 150 MeV, the double square root goes to zero two times because of the quantity v and of the second square root. These zeroes are present in the zone between 170 and 205 MeV i.e. near the α_{33}-resonance. Continuity arguments select the plus sign between the zeroes and the minus sign beyond the second zero. We point out that, in the region between the two zeroes the two α_3 solutions does not differ very much in cosine (but not in angle) and can be confused. Besides, near the zeroes, the double square root is rapidly varying with energy and, accordingly, the solutions are critical and very sensitive to the A_i values. Therefore we conclude that much more precise data are required in this range of energy in order to get acceptable solutions. Probably, at least in the range 150-300 MeV, one should consider also d-wave effects which, because of the too rough experimental data, has been neglected. This hypothesis is suggested a posteriori from the spd-analysis at higher energies (see par. 5).

The following very simple argument gives an idea of how critical can be the solutions in the case of the non resonant phases with respect to the experimental data. Let us consider the energy where α_{33} is resonant ($2 \cdot \sin^2\alpha_{33} = 2$) and suppose, for semplicity, that the errors of α_3 and α_{31} arise from the error in the total cross-section only. In this very optimistic case we have:

$$\frac{\Delta(\mathrm{sen}\ \alpha_3)}{\mathrm{sen}\ \alpha_3} = \frac{\Delta\ \sigma^{el}_{tot}}{\sigma^{el}_{tot}} \ \frac{4-u+2\,A_0}{2\,A-u-2} = \frac{\Delta\ \sigma^{el}_{tot}}{\sigma^{el}_{tot}} \ \frac{4-u+2\,A_0}{2\ \mathrm{sen}^2\alpha_3} \simeq 20 \cdot \frac{\Delta\ \sigma^{el}_{tot}}{\sigma^{el}_{tot}}$$

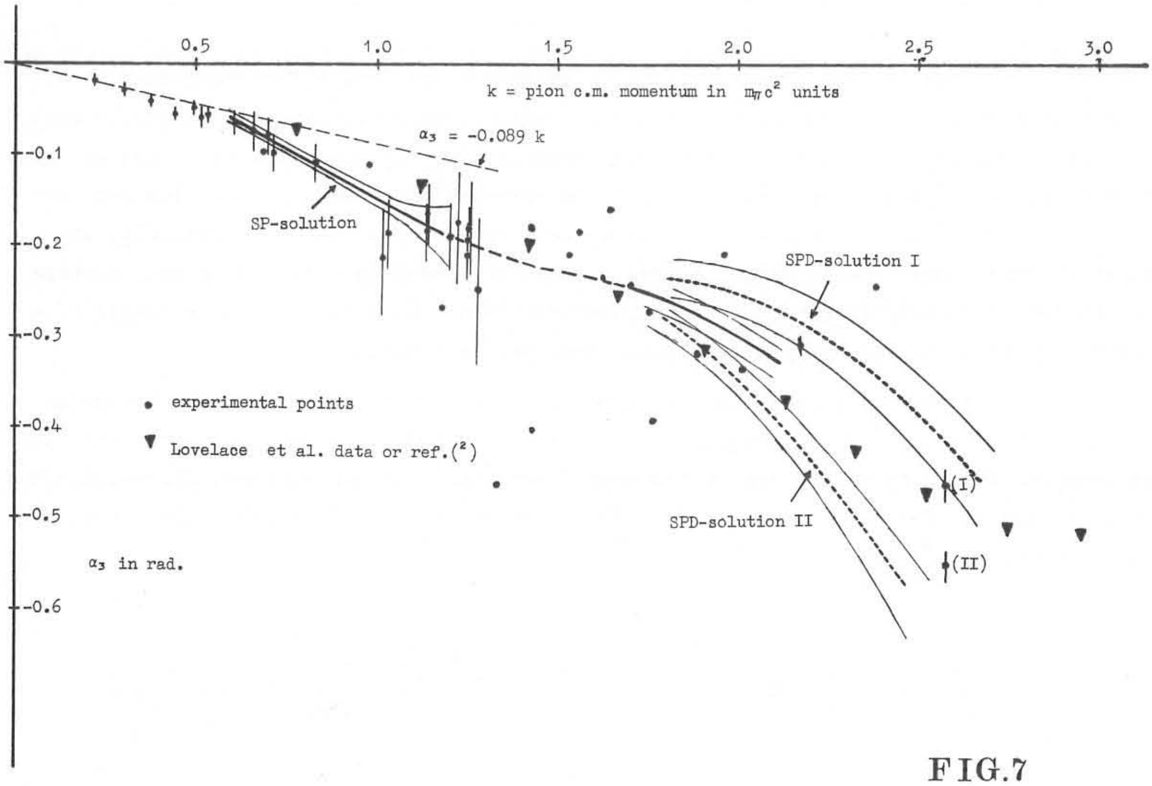
$$\frac{\Delta(\mathrm{sen}\ \alpha_{31})}{\mathrm{sen}\ \alpha_{31}} = \frac{\Delta\ \sigma^{el}_{tot}}{\sigma^{el}_{tot}} \ \frac{4-u-2\,A_0}{2-u-2\,A_0} = \frac{\Delta\ \sigma^{el}_{tot}}{\sigma^{el}_{tot}} \ \frac{4-u-2\,A_0}{2 \cdot \mathrm{sen}^2\alpha_{31}} \gg \frac{\Delta\ \sigma^{el}_{tot}}{\sigma^{el}_{tot}} .$$

It is clear that, in order to reduce the relative errors of α_3 and α_{31} to a few per cent it would be necessary to know the total cross-section with a relative error (statistical and systematic) of the order of 0.1% which is at least by one order of magnitude lower than the present accuracy of the total cross-section experimental data.

In conclusion, the large errors of the experimental cross-sections (both total and differential), the sensitivity of the phases to the A_i values, the α_3-ambiguity and, probably, the importance of the $l = 2$ state explain the dispersion of the data on α_3 and α_{31} (see fig. 7 and 8) near the resonance region and do not allow an unambiguous determination of these phases in the 150-220 MeV energy range.

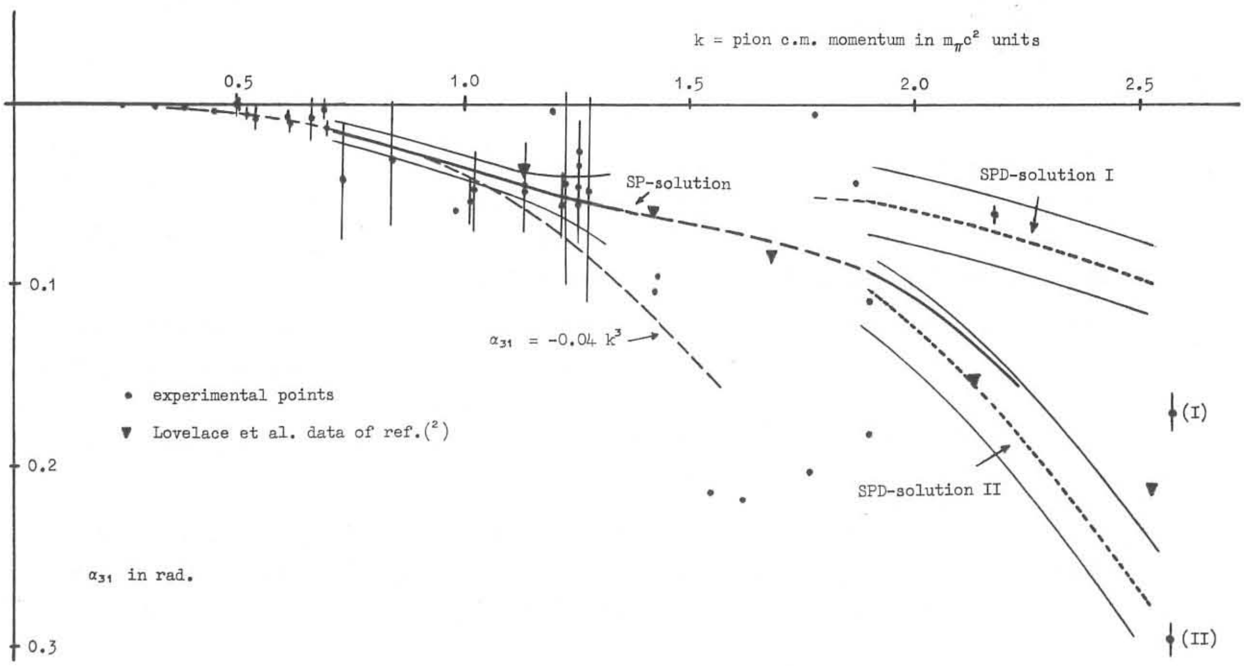
The behaviour of the phase-shifts with the calculated errors as a function of the center of mass momentum in $m_\pi c^2$ units has been reported in Figs. 7, 8 and 9 for the energy ranges 40-140 MeV and 220-300 MeV (solid line).

The intermediate values (dashed line) has been extrapolated from the calculated ones. They give a total cross-section which is in good agreement with the previously fitted values.



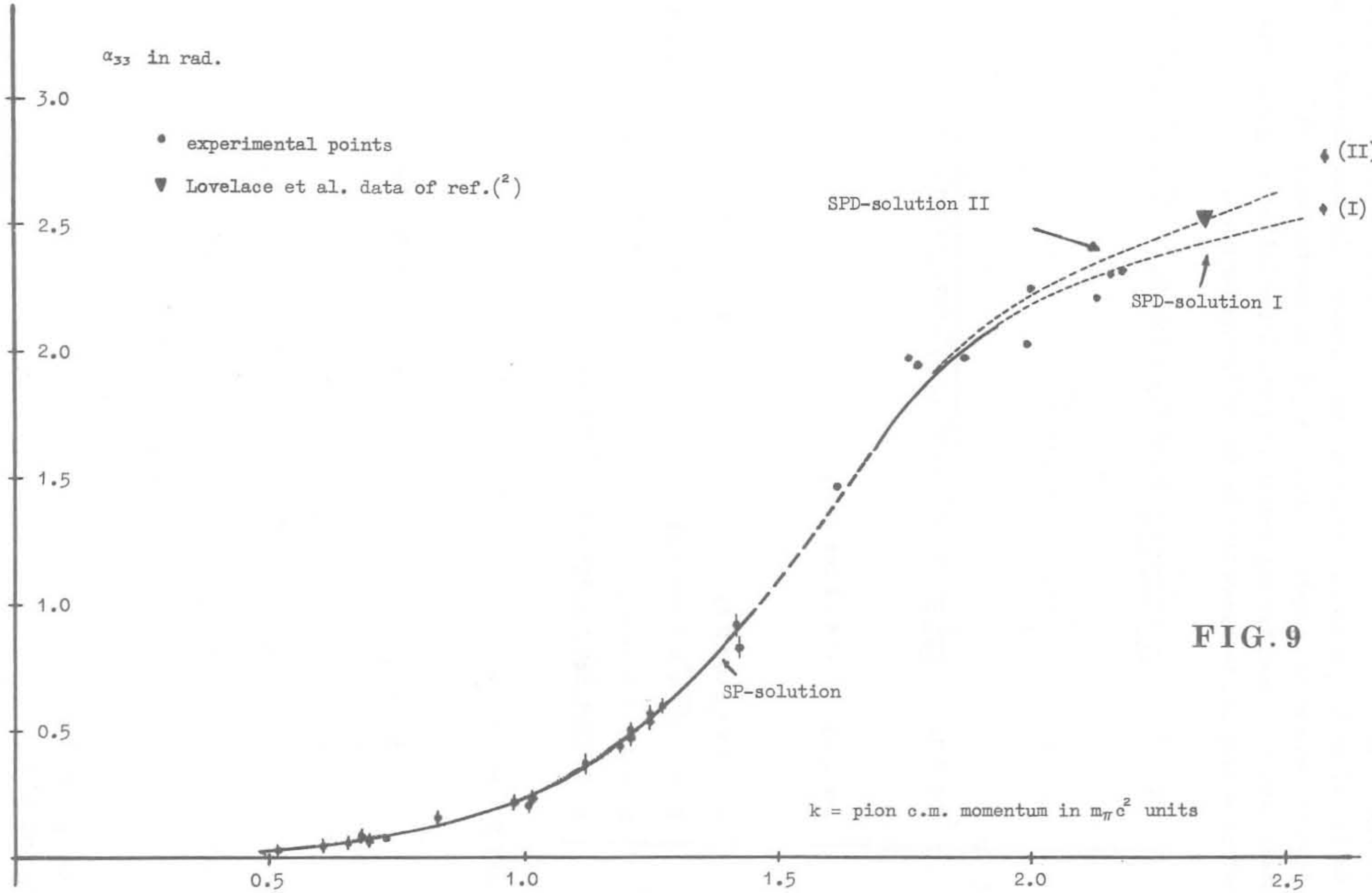
0.5
1.0
1.5
2.0
2.5
3.0
k = pion c.m. momentum in $m_\pi c^2$ units
α_3 = -0.089 k
-0.1
-0.2
-0.3
-0.4
-0.5
-0.6
SP-solution
SPD-solution I
SPD-solution II
experimental points
Lovelace et al. data or ref.(2)
α_3 in rad.
(I)
(II)

FIG.7



k = pion c.m. momentum in $m_\pi c^2$ units
0.5
1.0
1.5
2.0
2.5
0.1
0.2
0.3
SP-solution
SPD-solution I
SPD-solution II
α_{31} = -0.04 k^3
experimental points
Lovelace et al. data of ref.(2)
α_{31} in rad.
(I)
(II)

FIG.8



α_{33} in rad.
3.0
2.5
2.0
1.5
1.0
0.5
experimental points
Lovelace et al. data of ref.(2)
SPD-solution II
SPD-solution I
SP-solution
(II)
(I)
k = pion c.m. momentum in $m_\pi c^2$ units
0.5
1.0
1.5
2.0
2.5

FIG. 9

5. - SPD-SOLUTIONS

For energy larger than 300 MeV the experimental knowledge of the coefficients c_3 and c_4 allow us to calculate the contributions of the d-phases. In this case it is impossible to write down the phases as algebric functions of the coefficients only. For this reason computers have been used for the search of possible solutions by parametric methods. Our parametric method can be summarized in the following formulas:

$$(12) \quad \begin{cases} \cos 2\beta_5 = \dfrac{1}{45(13-12x)}\left\{(3-2x)\,[45(3-2x)-8A_4] \pm 4\sqrt{1-x^2}\,\sqrt{45(45+12A_4)-(45x+4A_4)^2}\right\} \\[2ex] \cos 2\alpha_3 = \dfrac{1}{C}\left[A\cdot B - (v-f)\sqrt{2C-B^2}\right] \\[2ex] \cos 2\alpha_{33} = \dfrac{1}{4N-5}\left\{M(2-N) + \sqrt{M^2\cdot(2-N)^2-(4N-5)(1-M^2-N^2)}\right\} \\[2ex] \cos 2\alpha_{31} = -(M+2\cos 2\alpha_{33}) \end{cases}$$

where:

$$(13) \quad \begin{cases} z = 2x+3\cos 2\beta_5 & A = u+3-2z \\ f = \pm 2\sqrt{1-x^2}+3\,\mathrm{sen}\,2\beta_5 & B = (u-z)^2+(v-f)^2-4(\omega+1) \\ M = \cos 2\alpha_3 - u + z & C = 2\,[A^2+(v-2f)^2] \\ N = \dfrac{z\cdot\cos 2\alpha_3 + f\cdot\mathrm{sen}\,2\alpha_3 - 3\cos 2\alpha_3}{2} - \omega & \end{cases}$$

where β_3 and β_5 are the d-phases and the generalized (11) are now of the form:

$$(14) \quad \begin{cases} u = 9 - 2\dfrac{1}{4\pi\lambda^2}\,\sigma^{el}_{tot} = 9 - 2\left(A_0 - \dfrac{A_2}{3} - \dfrac{A_4}{5}\right) \\[2ex] v = -2\sqrt{A_0+A_1+A_2+A_3+A_4-\left(A_0+\dfrac{A_2}{3}+\dfrac{A_4}{5}\right)^2} \\[2ex] \omega = 2\left[1-\dfrac{1}{\lambda^2}\left(\dfrac{d\sigma}{d\Omega}\right)_{\vartheta=90^\circ}\right]+\dfrac{2}{15}A_4 = 2\left(1-A_0+\dfrac{A_4}{15}\right) \end{cases} .$$

All the phase-shifts in (12) are so expressed as functions of the experimental quantities A_i and of the parameter:

$$x = \cos 2\beta_3 .$$

This parameter has to be determined in such a way that the condition

(15) $$15 - 5\,(\cos 2\,\alpha_{31} + 2\cos 2\,\alpha_{33}) - 3\,z\;3\,(z\cos 2\,\alpha_{33} + f\,\mathrm{sen}\,2\,\alpha_{33}) + \\ + 5\,[\cos 2\,\beta_5\,(\cos 2\,\alpha_{31} - \cos 2\,\alpha_{33}) + \mathrm{sen}\,2\,\beta_5\,(\mathrm{sen}\,2\,\alpha_{31} - \mathrm{sen}\,2\,\alpha_{33})] = \\ = \frac{4}{3} A_3$$

is satisfied.

Owing to the presence of the second root in the first equation of (12) the minimum value of the parameter x (which is positive as far as β_3 is smaller than 90°) is

$$(\cos 2\,\beta_3)_{\mathrm{min}} = \sqrt{1 + \frac{4}{15} A_4} - \frac{4}{45} A_4 \;,$$

which means that, in the spd-approximation, the phase β_3 cannot be (in absolute value) less than a quantity fixed by the experimental value of A_4.

The parametic method, above outlined, can be only applied as far as the anelastic scattering cross-section is vanishing. From the experimental data on the total cross-sections one can see that this can be considered true only up to 350 MeV (k = 2.3). However, if an anelastic parameter only contributes to the scattering processes, it can be determined by a independent fit of the anelastic cross-section and these value allows again the determination of the phases by formulas which are sligtly different from the (12) ones.

The procedure was followed in the present paper for the determination of spd sets of phase-shifts which appear in Figs. 7, 8, 9 and 10 (dotted lines) from k = 1.9 (about 250 MeV) up to k = 2.6 (about 420 MeV). From k = 1.9 up to k = 2.3 anelastic contributions have been neglected; from k = 2.4 up to k = 2.6 the phases are calculated in the hypothesis that the elastic parameter of the α_3 only contributes to the scattering process. For k greather than 2.6 no solutions has been found in this hypothesis. This fact is probably due whether to the increasing importance of other anelastic parameters (which can be calculated by the aid of polarization experiments) or to the effect of states with l larger than two.

As one can see from the figures, we found two solutions; the first one labelled by I, has positive β_3 and negative β_5 as pointed out by other authors ([9]). The second one (labelled II) the d-phase have apposite sign with respect to the solution I; further more, the α_3 and α_{31} are more rapidly increasing in the solution II than in the I one. In Fig. 11 the spd-ambiguity is illustrated grafically by the Clementel-Villi method, in the case of 310 MeV.

This ambiguity (arising from the spd-analysis) has the property that, in the limit of vanishing d-waves, corresponds to the Fermi-type sp-solution.

Our sets of phases I and II then are equivalent to the so called Fermi I and Fermi II solutions of Foote et al. ([9]).

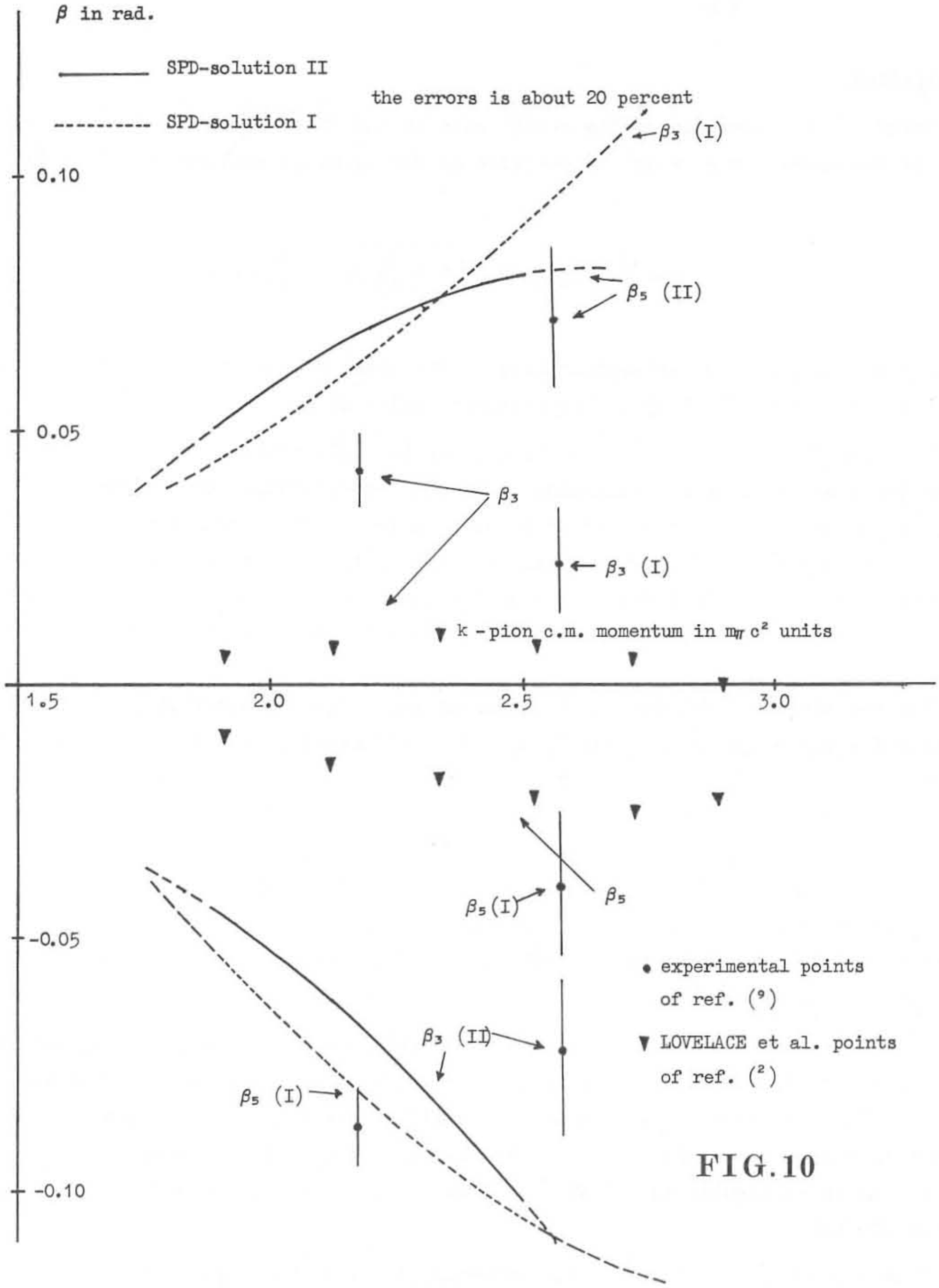
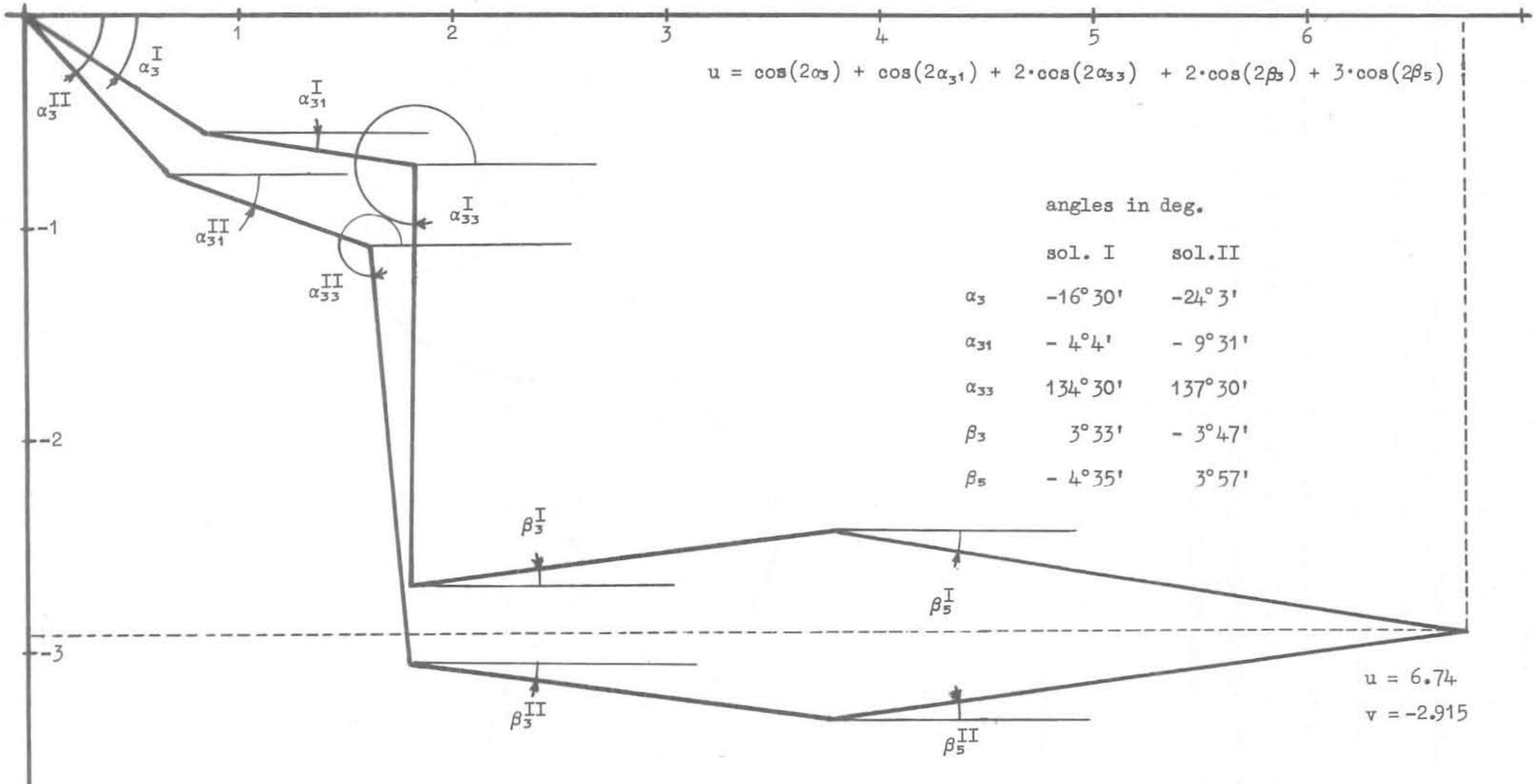

β in rad.
SPD-solution II
SPD-solution I
the errors is about 20 percent
β3 (I)
0.10
β5 (II)
0.05
β3
β3 (I)
k - pion c.m. momentum in mπ c² units
1.5
2.0
2.5
3.0
β5 (I)
β5
-0.05
• experimental points of ref. (9)
β3 (II)
▼ LOVELACE et al. points of ref. (2)
β5 (I)
-0.10

FIG. 10



angles in deg.

	sol. I	sol. II
α_3	-16° 30'	-24° 3'
α_{31}	- 4° 4'	- 9° 31'
α_{33}	134° 30'	137° 30'
β_3	3° 33'	- 3° 47'
β_5	- 4° 35'	3° 57'

$v = \sin(2\alpha_3) + \sin(2\alpha_{31}) + 2\cdot\sin(2\alpha_{33}) + 2\cdot\sin(2\beta_3) + 3\cdot\sin(2\beta_5)$

graphical representation of the SPD-ambiguity at 310 MeV.

FIG.11

The choice between the two set (I and II) can be made by polarization experiments. The data of Foote [9] at 310 MeV, which we reported in Fig. 12 with the polarization curves, seem to be more in favour of the solution I.

This conclusion cannot be regarded as definitive because small f-wave effects (combined with the large errors in our results), could effect the situation strongly as was shown by the spd-f-analysis of Foote et al. [9].

We conclude this chapter by a last remark; our analysis is limited to the d-waves only because we believe that the present experimental data on differential cross-sections, although quite good in the energy-range considered by us, does not still have the required degree of precision in order to allow an unambiguous f-wave analysis.

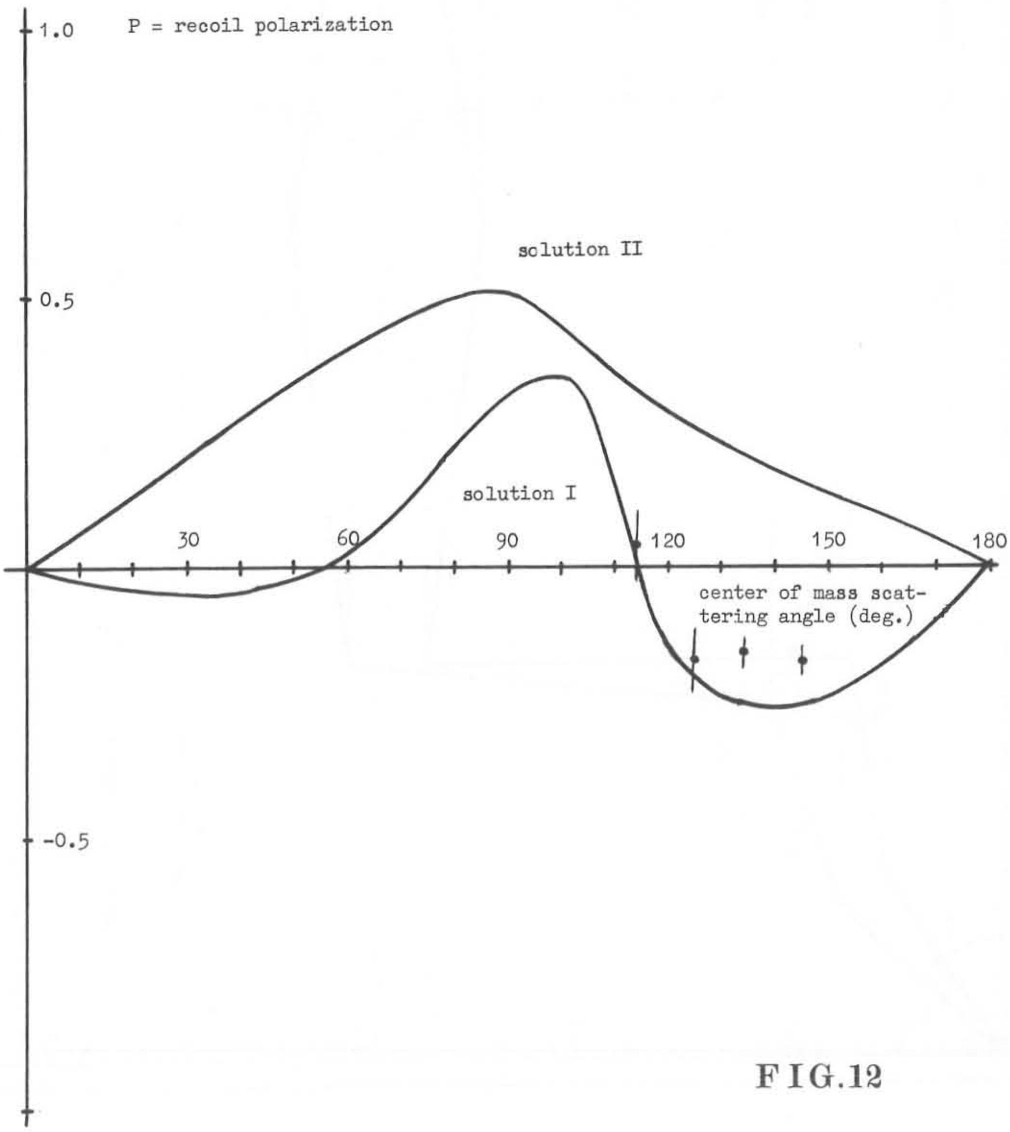


FIG.12

6. - CONCLUSION

The conclusions that one can draw from the present analysis are essentially the following:

a) for the energies smaller than 45 MeV no definite experimental data are, up to now, available in order to obtain the correct values of the scattering lenghts. In fact the values $a_3 = -0.089$ and $a_{31} = -0.04$ cannot be considered as definitive; they have been determined by extrapolation of data at energies too large while, in our opinion(supported by the actual experimental situation), the linear behaviour [12] of α_3 and the k^3-behaviour of α_{31} are probably not valid below 40 MeV;

b) in the range of energies between 40 and 150 MeV a large amount of data is available; however they have been obtained several years ago when the experimental techniques were not so refined as they are today; the repetition of the experiments in this range of energies would be very useful specially because this is the zone where the phase-shifts analysis is more simple;

c) in the range 150-300 MeV the difficulties arise essentially from the presence of the resonance which makes very hard the evaluation of the small phases; in this range the relative errors should be very small and the differential cross-section should be determined in such a way to allow the spd-analysis to be significant; in fact in our opinion, the d-wave are not vanishing in this range or, at least, not smaller than the α_{31} phase for instance. Polarization experiments would be useful for the choice of the correct spd-solutions;

d) at 310 MeV there are two independent accurate measurements [6] [9]. The experimental data were treated by the authors in two different ways: in the analysis reported in ref. [9] the Coulomb interaction was taken directy into account the resulting phases giving $\frac{1}{\lambda^2}\left(\frac{d\sigma}{d\Omega}\right)_{\vartheta=0^\circ} = 3.74$, $A = 0.305$ and $A_4 = -0.078$ while, in the Ogden analysis, the fit, obtained by adding to the experimental points the dispersion relation value at zero degree (equal to 3.34), gives $\frac{1}{\lambda^2}\left(\frac{d\sigma}{d\Omega}\right)_{\vartheta=0^\circ} = 3.59$, $A_3 = 0.07$ and $A_4 = -0.170$.

Obviously our analysis, obtained by A_i-values which derives from the fit of chapter 3 gives results very close to the Ogden's ones. In fact his experiment covers all the energies from 310 to 650 MeV and in the fit of the A_i the value of Foote does not effect practically the above analysis. Neverthless we give more confidence to the first type of analysis (pure experimental data with Coulomb correction) firstly because it represents the correct result of the experiment, secondly because the values of the coefficients A_i are strongly influenced by the minimum cut-off angle of the differential cross-section.

e) Finally for energies greather than about 350 MeV, the presence of anelastic scattering and, probably, of states with l greather than two, makes the analysis very difficult and requires the necessity of very good data also on the polarization $P(\vartheta)$ at several angles in order to obtain the coefficients B_i of the series rexpansion of $P(\vartheta)$. These data are available at some energies [13] and tentative analysis have been done by the authors. We will come back to this argument in a following paper.

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