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ABSTRACT

The problem of reducing the direct product of two irreducible tensors into irreducible components is dealt with, for the $0_3$ group, through the use of a redundant cartesian representation. Half-integral order tensors are also included. Explicit formulae are stated for the reduction coefficients. A table is given for the simplest cases. Although it sacrifices algebraic elegance, the present treatment is much simpler in some practical applications.

SOMMARIO

Il presente lavoro tratta il problema delle riduzioni del prodotto diretto di due tensori irriducibili nelle sue componenti irriducibili, nel caso del gruppo $0_3$, adoperando una rappresentazione cartesiana ridondante. E' compreso anche il caso dei tensori di ordine semintero. Si danno formule esplicite di coefficienti di riduzione, e una tabella dei casi più semplici. Questa trattazione, pur sacrificando l'eleganza algebrica, riesce di più semplice applicazione pratica in alcuni casi.

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# I. INTRODUCTION

The classical Clebsch-Gordan problem deals with the reduction of the direct product of two irreducible representations of the orthogonal group in three dimensions ($O_3$). An alternative formulation is the following: given two irreducible tensioral sets of degrees $j_1$, $j_2$, to extract from their direct product a third irreducible set of degree j. It is well known that the solution exists if and only if

(1) $$\left| j_1 - j_2 \right| \leq j \leq j_1 + j_2$$

and can be written

(2) $$C_m^{(j)} = \sum_{m_1 m_2} (jm | m_1 m_2)\, a_{m_1}^{(j_1)}\, b_{m_2}^{(j_2)}$$

where the $(jm | m_1 m_2)$ represent the Clebsch-Gordan-Wigner coefficients[1].

The above way of dealing with the Clebsch-Gordan problem is beyond any doubt the most elegant and deep-rooted one, because of its algebraic simplicity and of its group-theoretical meaning. Furthermore, and needless to say, the concept of irreducible tensorial set proves very useful in quantum theory.

There are, however, some cases where irreducible tensorial sets are not the most convenient tool (just as the choice of polar coordinates is not always better than that of cartesian coordinates, even if the former exploit the symmetry properties of a system better that do the latter). But, if one wants to resort to cartesian tensors, the difficulty arises that cartesian tensors are generally reducible.

This paper is devoted to giving an extensive tratment of cartesian irreducible tensors (of integral and half-integral order). The cartesian representation of an irreducible tensor is necessarily redundant, i. e. some identities exist among its components, but definite rules can be given, which describe how to extract all irreducible parts from the product of two irreducible tensors. It will be found that the coefficients involved are much simpler than Clebsch-Gordan-Wigner's, and that the composition rules are rather intuitive.

On account of what we have said above, the following will contain no basically new result. It is mainly intended to give a plain exposition of the Clebsch-Gordan problem in cartesian form and to present some formulae which we expect to be of use in many practical calculations.

Section 2 contains the main definitions and sets the notations. In Section 3 the irreducible tensors are defined and some irreducibility criteria, to be used later, are stated.

Section 4 outlines the structure of the irreducible part of the product of two irreducible tensors, and fixes the procedure to be followed in calculating the coefficients. The final formulae are also given for easy reference.

Sections 5, 6, 7 contain the details of the calculations for the various cases (spinor-spinor, spinor-tensor, tensor-tensor coupling).

In Section 8 we present some examples and discuss some possible extensions.

Finally, in the Appendix, we give some algebraic results about symmetrized products of Pauli matrices which are used in this paper.

## II. CARTESIAN TENSORS OF INTEGRAL AND HALF-INTEGRAL ORDER

We will always be concerned with the group $0_3$, i. e. with the group of rotations in a three-dimensional real space:

(3) $$X_a \rightarrow A_{ab} X_b$$

We need not recall the properties of $A_{ab}$. Only the form for infinitesimal rotations will be given

(4) $$A_{ab} = \delta_{ab} - \varepsilon_{abc} u_c$$

where $u_c$ is the "vector" of the rotation.

A tensor of order m (m integral) is defined in the conventional way: it is a set of $3^m$ numbers[2] $T_{a_1 \dots a_m}$ ($a_i = 1, 2, 3$; $i = 1, \dots m$) transforming according to

(5) - $$T_{a_1 \dots a_m} \rightarrow A_{a_1 b_1} \dots A_{a_m b_m} T_{b_1 \dots b_m}$$

All the usual operations of tensor algebra are understood, as well as the meaning of the Kronecker and Ricci tensors $\delta_{ab}$, $\varepsilon_{abc}$.

A tensor of order m+1/2 (m integral) is defined as a set of $2 . 3^m$ numbers $\chi_{a_1 \dots a_m}$ ($\alpha = 1, 2$; $a_i = 1, 2, 3$; $i = 1, \dots m$) transforming according to

(6) $$\chi_{a_1 \dots a_m} \rightarrow A_{a_1 b_1} \dots A_{a_m b_m} U^{\alpha\beta} \chi_{b_1 \dots b_m}$$

where $U^{\alpha\beta}$ is a representation of order 2 of $0_3$. The explicit form of U is

(7) $$U = e^{-1/2 \sigma_c u_c}$$

$u_c$ being the vector of the rotation and $\sigma_c$ a set of 2 x 2 matrices satisfying

(8) $$\sigma_a \sigma_b = \delta_{ab} + i \varepsilon_{abc} \sigma_c$$

The Pauli matrices are a solution of (8), and any other solution is unitarily equivalent to them.

An important property of U is the following

(9) $$U^{-1} \sigma_a U = A_{ab} \sigma_b$$

(easy to prove for infinitesimal transformations). From Eq. (9) it follows that by applying a $\sigma$ to a tensor of half-integral order one gets another tensor which is one unit higher in order.

In strict analogy with the direct product of two tensors of integral order, the direct product of a tensor of integral order with one of half-integral order can be defined. It is a tensor whose order is the sum of the orders of the factors. The direct product of two tensors of half-integral order, is not a tensor, however, because of its two upper indices. These can be eliminated by performing the following operation

(10) $$T_{a_1 \cdots a_m b_1 \cdots b_n} = B^{\alpha\beta} \chi^{\cdot\alpha}_{a_1 \cdots a_m} \varphi^{\beta}_{b_1 \cdots b_n}$$

and $T_{a_1 \cdots a_m b_1 \cdots b_n}$ is a tensor of order m + n if $B^{\alpha\beta}$ satisfies

(11) $$B^{\alpha\beta} U^{\alpha\rho} U^{\beta\sigma} = B^{\rho\sigma}$$

A necessary and sufficient condition that Eq. (11) hold for every U is

(12) $$B \sigma_a B^{-1} = - \sigma_a^T$$

(in this equation T denotes the transpose). This defines B up to a numerical factor. (If the Pauli representation is used for the $\sigma$'s, B can be taken equal to $\sigma_2$). The tensor which one forms from two tensors of half integral order, according to Eq. (10), will be called their tensor product. It will be often denoted in this paper by the following notatio

(13) $$T_{a_1 \cdots a_m b_1 \cdots b_n} = ( \chi_{a_1 \cdots a_m,} \varphi_{b_1 \cdots b_n})$$

It can be shown that $B^{\alpha\beta}$ is an antisymmetric matrix. It follows that

(14) $$( \chi_{a_1 \cdots a_m} \varphi_{b_1 \cdots b_n}) = - (\varphi_{b_1 \cdots b_n} \chi_{a_1 \cdots a_m})$$

Another useful property of the tensor product is

(15) $$( \sigma_c \chi_{a_1 \cdots a_m} \varphi_{b_1 \cdots b_n}) = - ( \chi_{a_1 \cdots a_m} \sigma_c \varphi_{b_1 \cdots b_n})$$

which is a direct consequence of Eqs. (10), (12). Thus we need not bother about the order of the factors in a tensor product (apart for its sign) and may always understand that a $\sigma$ is applied to its first factor. It should be noted that the l. h. s. 's of Eqs. (14), (15) taken together have just as many independent components as the direct product of $\chi$ and $\varphi$, while being "good" tensors.

## III. IRREDUCIBLE CARTESIAN TENSORS

The order of a tensor can be lowered by contraction with the isotropic tensors $\delta_{ab}$, $\varepsilon_{abc}$; contraction with $\delta_{ab}$ lowers the order by two, contraction with $\varepsilon_{abc}$ by one or by three. For a tensor of half-integral order there is a further possibility: contraction by $\sigma_a^{(3)}$, which lowers the order by one. By irreducible tensor we mean one whose contractions all vanish.

Quite obviously, any tensor of order less than or equal to 1 is irreducible. For tensors of higher order, irreducibility is not a trivial property and we can state the following criteria:

IC1. Irreducibility criterion for tensors of integral order

IC1. 1: The tensor must be symmetrical in all indices
IC1. 2: Its trace must vanish.

IC1. 1 is necessary and sufficient in order that the contraction with $\varepsilon_{abc}$ vanish.

IC1. 2 (stated for any one of the traces, which are all equal on account of IC1. 1) ensures that the contraction with $\delta_{ab}$ is zero.

IC2. Irreducibility criterion for tensors of half integral order

IC2. 1: The tensor must be symmetrical in all lower indices
IC2. 2: The contraction with $\sigma_a$ must vanish.

IC2. 1 has the same meaning as IC1. 1. IC2. 2 is obvious and entails also the vanishing of the contraction with $\delta_{ab}$, on account of IC2. 1 and Eq. (8).

## IV. REDUCING THE PRODUCT OF TWO IRREDUCIBLE TENSORS

Our problem can be solved in two steps:

1) writing down all tensors of a given order, which are bilinear in the given tensors;
2) finding a linear combination of them, which is irreducible.

It results as a consequence of the procedure we are going to outline, that the irreducible combination, when it exists, is unique up to a factor.

On account of the different form of representation which we choose for integral and half-integral order tensors, the treatment will not be the same for both; so we shall distinguish three cases:

A: both tensors of half-integral order;
B: one tensor of half-integral and one of integral order:
C: both tensors of integral order.

Let us start with case A.

A: <u>both tensors of half integral order.</u> - Let m + 1/2 and n + 1/2 (m n) be the orders of the two tensors. Our first problem is how to build up, through linear operations on their direct product, a tensor of order p. We have already noted at the end of Sec. 2 that ( $\chi$ , $\varphi$ ) and ( $\sigma_c \chi$ , $\varphi$ ) have just as many independent components as the direct product of $\chi$ and $\varphi$ , while being already tensors of orders m + n and m + n + 1 respectively. In the following we shall use for ( $\sigma_c \chi$ , $\varphi$ ) the notation $\sigma_c$( $\chi$ , $\varphi$ ) which is handier, though quite equivalent to the former. It will be always understood that any number of $\sigma$'s outside the parenthesis should be applied to the first factor inside, before effecting the tensor product. Thanks to Eq. (15), no need will ever arise of applying $\sigma$'s to the second factor in a tensor product.

In order to get a tensor of order p from ( $\chi$ , $\varphi$ ) and $\sigma_c$ ( $\chi$, $\varphi$ ) we may only multiply one of these tensors by some $\delta$'s and/or $\varepsilon$'s, perhaps contracting some indices. But it is not difficult to show from Eq. (8) that both $\delta_{ab}$ and $\varepsilon_{abc}$ can be expressed as a sum of products of $\sigma$'s. Then all independent tensors we can obtain will be of the form $\sigma \dots \sigma$( $\chi$ , $\varphi$ ) (indices not being shown). Some of the indices will be free, others will be dummy; we are now going to discuss this point more thoroughly.

First of all, no contraction is allowed between indices of $\chi$ (one would get zero), and the same holds for $\varphi$ . Second, it is useless to contract between two $\sigma$'s. If the contracted $\sigma$'s are consecutive, the result is an expression with two $\sigma$'s less. If they are not consecutive, we can move one of them near the other, and add the resulting anticommutators: these are again expressions with two $\sigma$'s less. Third, no contraction is needed between one $\sigma$ and $\chi$ (or $\varphi$ ) for much the same reasons: if the contracted $\sigma$ is the rightmost one, the result is zero, since $\chi$ is irreducible; if this is not the case, we have only to add some anticommutators, which give terms with two $\sigma$'s less. So the only contractions allowed are those between $\chi$ and $\varphi$ . Hereafter, for cases B and C as well, we will denote the number of contracted pairs by $\lambda$ . If s is the number of $\sigma$'s, we must obviously have

(16) $$p = m + n + s - 2\lambda$$

For given p, we have many possibilities for s and $\lambda$ . Since $\lambda$ must always be less than or equal to n, it will be convenient to introduce a new variable t, defined by

(17) $$t = n - \lambda$$

In terms of t, s is given by

(18) $$s = k - 2t$$

where

(19) $$k = p - m + n$$

is the greatest value allowed for s. It is obvious that s has always the parity of k; we will refer later to this result. As a consequence of Eq. (19), we see that p must not be less than m-n, otherwise no tensor of order p can be built.

Both $\lambda$ and s must be greater than or equal to zero; therefore the upper bound of t is

(20) $$r = \min\left( [k/2],\ n\right)$$

(the square bracket meaning "the largest integer not greater than"). We shall take t in the following as the independent variable, ranging from zero to r; $\lambda$ and s can be expressed through t by Eqs. (17) and (18).

Thus there are r + 1 different ways of forming a tensor of order p. Since only symmetric tensors are needed, the general form thereof is

(21) $$V^{(p,t)}_{a_1 \dots a_p} = \sum \Delta^{(s)}_{b_1 \dots b_s} \left( \chi_{b_{s+1} \dots b_{s+m-\lambda}\, c_1 \dots c_\lambda}\ \varphi_{b_{s+m-\lambda+1} \dots b_p\, c_1 \dots c_\lambda} \right)$$

where $b_1, \dots b_s$, $b_{s+1} \dots b_{p-t}$, $b_{p-t+1} \dots \dots b_p$ is a ternary combination of $a_1, \dots a_p$, and the sum is over all such combinations. The number of terms in the sum is $(p!)/(s!(m-\lambda)!(n-\lambda)!)$. The symbol $\Delta^{(s)}$ denotes the symmetrized product of s $\sigma$'s; its properties are given in the Appendix.

The most general symmetric tensor of order p is a linear combination of (21)

(22) $$W^{(p)}_{a_1 \dots a_p} = \sum_{t=0}^{r} \xi^{(p)}_t\, V^{(p,t)}_{a_1 \dots a_p}$$

We must now choose the $\xi$'s in order that $W^{(p)}_{a_1 \dots a_p}$ be irreducible, i.e. - - on account of ICI.2 - that its trace vanish. The details of the calculations are given in Sec. 5; the result is that if $m - n \leq p \leq m + n + 1$ there is one irreducible tensor, given by Eq. (22) with the following values of the $\xi$'s

(23) $$\xi^{(p)}_t = (-1/2)^t \begin{cases} \dfrac{k!!\,(2p - k + 2t - 1)!!}{(k - 2t)!!\,(2p - k - 1)!!} & \text{even } k \\[2ex] \dfrac{(k - 1)!!\,(2p - k + 2t)!!}{(k - 2t - 1)!!\,(2p - k)!!} & \text{odd } k \end{cases}$$

B: <u>one tensor of half-integral and one of integral order.</u> In this case, through an analysis quite similar to the one we made in A, we can readily show that two different kinds of tensors are obtained, according to whether no $\sigma$ is contracted with the tensor of integral order, or one $\sigma$ is contracted. These two kinds of tensors will be denoted by subscripts $\alpha$ and $\beta$ respectively. If m, n+1/2 are the orders of the original tensors, equations analogous to Eq. (16) may be written

$$(24)\qquad \begin{cases} p = m + n + s - 2\lambda & (\alpha) \\ p = m + n + s - 2\lambda - 2 & (\beta) \end{cases}$$

the order of the product tensor being p + 1/2. Eqs. (24) can be summarized into

$$(25)\qquad p = m + n + s - 2\lambda - 2$$

where $\mu = 0$ for ($\alpha$), $\mu = 1$ for ($\beta$).
If we define

$$(26)\qquad \begin{cases} f = \max(m - \mu, n) \\ g = \min(m - \mu, n) \end{cases}$$

$$(27)\qquad e = f - g = |m - n - \mu|$$

$$(28)\qquad t = g - \lambda$$

$$(29)\qquad k = p - e$$

we have

$$(30)\qquad s = k - 2t + \mu$$

and t is allowed to assume the values 0, 1,....., r with

$$(31)\qquad r = \min([k/2], g).$$

In the following we shall denote by $k_\alpha$, $r_\alpha$, $k_\beta$, $r_\beta$ the values of k, r for $\mu = 0, 1$ respectively.

It is not difficult to show that no tensor of order p + 1/2 can be built unless p satisfies at least one of the conditions

$$p \geq |m - n|$$

$$p \geq |m - n - 1|$$

or, that is the same thing, unless

$$p + 1/2 \geq |m - n - 1/2|$$

which is what we expected.

The symmetric $\alpha$ - type tensor is

$$V^{(p,t)}_{\alpha a_1 \dots a_p} = \sum_P \Delta^{(s)}_{b \dots b_s} T_{b_{s+1} \dots b_{s+m-\lambda}\, c_1 \dots c_\lambda}\,, \tag{32}$$

$$\chi_{b_{s+m-\lambda+1} \dots b_p\, c_1 \dots c_\lambda}$$

the $\beta$ - type is

$$V^{(p,t)}_{\beta a_1 \dots a_p} = \sum_P \Delta^{(s-1)}_{b_1 \dots b_{s-1}} \sigma_d T_{b_s \dots b_{s+m-\lambda-2}\, c_1 \dots c_\lambda d} \tag{33}$$

$$\chi_{b_{s+m-\lambda-1} \dots b_p\, c_1 \dots c_\lambda}$$

The general symmetric tensor of order p + 1/2 is

$$W^{(p)}_{a_1 \dots a_p} = \sum_{t=0}^{\alpha} \xi^{(p)}_{\alpha t} V^{(p,t)}_{\alpha a_1 \dots a_p} + \sum_{t=0}^{\beta} \xi^{(p)}_{\beta t} V^{(p,t)}_{\beta a_1 \dots a_p} \tag{34}$$

The details of the ensuing calculations are given in Sec. 6; the result is that if $| m - n - 1/2 | \leq p + 1/2 \leq m + n + 1/2$ there is one irreducible tensor, given by Eq. (34) with the following values of the $\xi$'s

$$m > n, \quad h = p + n - m$$

$$\xi^{(p)}_{\alpha t} = (-1/2)^t \begin{cases} \dfrac{h!!(2p - h + 2t + 1)!!}{(2p - h + 1)!!(h - 2t)!!} & \text{(even } h) \\[2ex] \dfrac{(h-1)!!(2p - h + 2t)!!}{(h - 2t - 1)!!(2p - h)!!} & \text{(odd } h) \end{cases} \qquad \xi^{(p)}_{\beta t} = (-1/2)^t \begin{cases} -\dfrac{h!!(2p - h + 2t - 1)!!}{(2p - h + 1)!!(h - 2t)!!} \\[2ex] \dfrac{(h-1)!!(2p - h + 2t)!!}{(h - 2t + 1)!!(2p - h)!!} \end{cases} \tag{35}$$

$$m \leq n, \quad h' = p - n + m$$

$$\xi^{(p)}_{\alpha t} = (-1/2)^t \begin{cases} \dfrac{h'!!(2p - h' + 2t + 1)!!}{(2p - h' + 1)!!(h' - 2t)!!} & \text{(even } h') \\[2ex] \dfrac{(h'-1)!!(2p - h' + 2t)!!}{(2p - h')!!(h' - 2t - 1)!!} & \text{(odd } h') \end{cases} \qquad \xi^{(p)}_{\beta t} = (-1/2)^{t+1} \begin{cases} -\dfrac{h'!!(2p - h' + 2t + 1)!!}{(2p - h' + 1)!!(h' 2t - 2)!!} \\[2ex] \dfrac{(h' - 1)!!(2p - h' + 2t + 2)!!}{(2p - h')!!(h' - 2t - 1)!!} \end{cases}$$

C: <u>both tensors of integral order.</u> This case deserves special discussion due to the fact that no $\sigma$-matrices are involved in constructing the composed tensor. Let us examine the operations by which a tensor can be constructed from two tensors T and U (of order m, n ($m \leq n$), respectively):

1) Contracting an index of T with an index of U.

2) Saturating and index of T and an index of U with a Ricci tensor $\varepsilon_{ijk}$.
3) Multiplying by any number of Kronecker $\delta$'s.

No Ricci tensor with more than one free index can occur because of the requirement of symmetry of the tensor. Moreover, only one Ricci tensor is needed at most, for a product of an even number of Ricci tensors can be reduced to a linear combination of products of Kronecker $\delta$'s. Therefore the possible forms for the composed tensor W, of order p, are

e-type: $W_e \propto$ (any number of $\delta$'s)$_{a_1 \cdots a_s}$ $T_{b_1 \cdots b_i\, d_1 \cdots d_e}$

$$, U_{c_j \cdots c_1\, d_1 \cdots d_e}$$

o-type: $W_o \propto$ (any number of $\delta$'s)$_{a_1 \cdots a_s e}$ $\varepsilon_{efg}$.

$$\cdot T_{b_1 \cdots b_i\, d_1 \cdots d_k f}\, U_{c_1 \cdots c_j d_1 \cdots d_k g}.$$

It is immediately seen that in the o-type, no substantial limitation is imposed by having saturated all the indices of the Ricci tensor.

It is shown, in the Appendix, that a $\Delta^{(s)}$-function with even s behaves, apart from multiplication by the spinor identity, as a symmetrized product of Kronecker $\delta$'s. Therefore we shall use an even-order $\Delta$ in the place of the product of $\delta$'s, the index a of the $\Delta^{(a)}$ being chosen so that s always indicates the number of free indices of the $\Delta$. Therefore the relation

$$p = m + n + s - 2\lambda - 2\mu \tag{36}$$

holds, where $\lambda$ has the usual significance and $\mu = 0, 1$ for e- and o- types, respectively.

It clearly follows that in the e-type, s equals a and is even, therefore p-m-n is even; in the o-type, s=a-1 and is odd, therefore p-m-n is odd. This is equivalent to saying that if for the composed tensor p-m-n is even, only e-type terms will occur in W, while, if p-m-n is odd, only o-type terms will occur there.

In analogy with the preceding cases, one can define

$$t = g - \lambda = m - \mu - \lambda \tag{37}$$

$$k = p + m - n - \mu \tag{38}$$

and therefore

$$s = k - 2t + \mu \tag{39}$$

and

$$0 \leq t \leq \min\left([k/2],\ m - \mu\right) \tag{40}$$

which (taking account of the condition $m \leq n$) are exactly the same formulae of case B.

The general symmetric tensor of order p such that p-m-n is even is

(41)
$$W^{(p)}_{a_1 \dots a_p} = \sum_{t=0}^{r} \xi^{(p)}_t \sum_P \Delta^{(s)}_{b_1 \dots b_s} T_{b_{s+1} \dots b_{s+m-\lambda} c_1 \dots c_\lambda} \cdot U_{b_{s+m-\lambda+1} \dots b_p c_1 \dots c_\lambda}$$

where $\sum_P$ means sum over the combinations of $a_1, \dots\dots, a_p$ according to the partition $(s, m-\lambda, n-\lambda)$.

The coefficients $\xi$ are given by

(42)
$$\xi^{(p)}_t = (-1/2)^t \frac{k!!\,(2p - k + 2t - 1)!!}{(k - 2t)!!\,(2p - k - 1)!!}$$

The general symmetric tensor of order p such that p-m-n is odd is given by

(43)
$$W^{(p)}_{a_1 \dots a_p} = \sum_{t=0}^{r} \xi^{(p)}_t \sum_P \Delta^{(s+1)}_{b_1 \dots b_s e} \varepsilon_{efg} \cdot T_{b_{s+1} \dots b_{s+m-\lambda-1} c_1 \dots c_\lambda f} U_{b_{s+m-\lambda} \dots b_p c_1 \dots c_\lambda g}.$$

where $\sum_P$ means sum over the combinations of indices $a_1, \dots\dots, a_p$ according to the partition $(s, m-\lambda-1, n-\lambda-1)$. The coefficients $\xi$ are given by

(44)
$$\xi^{(p)}_t = (-1/2)^t \frac{(k+2)!!\,(2p - k + 2t - 1)!!}{(k - 2t + 2)!!\,(2p - k - 1)!!}$$

## 5. EVALUATION OF THE $\xi$-COEFFICIENTS FOR CASE A

The general form of the tensor of order p which can be obtained from two tensors $\chi$ and $\varphi$ of order $m + 1/2$ and $n + 1/2$ respectively, is given by Eq. (22), where

(45)
$$V^{(p,t)}_{a_1 \dots a_p} = \sum_P \Delta^{(s)}_{b_1 \dots b_s} (\chi_{b_{s+1} \dots b_{s+m-\lambda} c_1 \dots c_\lambda}, \varphi_{b_{s+m-\lambda+1} \dots b_p c_1 \dots c_\lambda})$$

The sum in Eq. (45) is made over all the combinations of indices $a_1, \dots, a_p$

according to the partition $(s, m-\lambda, n-\lambda)$. It contains therefore $p!/ /(s!(m-\lambda)!(n-\lambda)!)$ terms.

The problem is to find the coefficients $\xi_t^{(p)}$ of the irreducible tensor of order p. The tensor given in Eq. (22) with the $V^{(p,t)}$ given by Eq. (45) is symmetric. After the general method described in Sec. 4, it remains only to use the condition that the contraction of $W^{(p)}$ is zero. This contraction gives the equation

$$\begin{aligned} \overline{W}^{(p)}_{a_3\ldots a_p} &= \delta_{a_1 a_2} W^{(p)}_{a_1 a_2\ldots a_p} = \sum_t \xi_t^{(p)} \overline{V}^{(p,t)}_{a_3\ldots a_p} = \\ &= \sum_t \xi_t^{(p)} V^{(p,t)}_{a_1 a_2 \ldots a_p} \delta_{a_1 a_2} = 0 \end{aligned} \tag{46}$$

Let us call first group the group of indices $b_1, \ldots\ldots, b_s$, second group $b_{s+1}, \ldots\ldots b_{p-t}$, third group the indices $b_{p-t+1}, \ldots\ldots, b_p$. The sum (46) splits into six parts according to the position of the indices $a_1$, $a_2$ in the three groups. If $a_1$, $a_2$ are both in the first group, that is if they are indices of a $\Delta^{(s)}$, use of Eq. (A6) of the appendix gives the contribution

$$\begin{Bmatrix} s(s+1) \\ (s-1)(s+2) \end{Bmatrix} V^{(p-2,t)}_{a_3\ldots a_p} \tag{47}$$

the upper line referring to the case of even s (that is of even k), the lower to the case of odd s. The preceding notation will be used wherever such a distinction is to be done.

If $a_1$ is in the first group and $a_2$ is in the second group, or $a_2$ in the first and $a_1$ in the second, one gets

$$2 \begin{Bmatrix} s \\ s-1 \end{Bmatrix} (m-\lambda) V^{(p-2,t)}_{a_3\ldots a_p} \tag{48}$$

the factor 2 arising because the terms with $a_1$ in the first group and those with $a_2$ in the first group give the same result. The first coefficient in Eq. (48) is the one found in Eq. (A7) of the Appendix. The factor $(m-\lambda)$ is accounted for by comparing the number of terms of $\overline{V}$ and V. In the case of $\overline{V}$ the number of terms (compare Eq. (45)) is given by

$$\frac{(p-2)!}{(s-1)!(m-\lambda-1)!(n-\lambda)!}.$$

In the case of V the number of terms is

$$\frac{(p-2)!}{(s-2)!(m-\lambda)!(n-\lambda)!}.$$

We might expect therefore a factor $(m - \lambda)/(s - 1)$, but the $(s - 1)$ is furnished by the sum on i in Eq. (A7) of the Appendix.

If $a_1$ is in the first and $a_2$ is in the third group or viceversa, the result follows from (A8) by only exchanging m with n,

(49) $$2 \begin{Bmatrix} s \\ s-1 \end{Bmatrix} (n - \lambda) V^{(p-2,t)}_{a_3 \dots a_p}$$

The cases with $a_1$, $a_2$ both in the second or in the third group do not contribute on account of the irreducibility of $\chi$ and $\varphi$. If however $a_1$ is in the second group and $a_2$ is in the third or viceversa, the result is

(50) $$2 \quad V^{(p-2,\ t-1)}_{a_3 \dots a_p}$$

the counting of the number of terms in Eq. (50) giving no factor but the factor 2 which is accounted for by the interchange of $a_1$ with $a_2$.

The contraction of the tensor $W^{(p)}_{a_1 \dots a_p}$ is therefore obtained by summing the contributions (47) to (50). (It is worth while to point out that (47), (48) and (49) give zero if $s \leq 2$. Moreover, (48) gives no contribution if $t = 0$, for there are no indices to contract in the second group; a similar thing happens for (49) and (50) when $s = p-m+n$).

The condition that the contraction of W be zero is therefore

(51) $$\sum_t \xi^{(p)}_t \left[ \begin{Bmatrix} s(2p - s+1) \\ (s-1)(2p-s+2) \end{Bmatrix} V^{(p-2,t)}_{a_3 \dots a_p} + 2\, V^{(p-2,t-1)}_{a_3 \dots a_p} \right] = 0$$

where the sum goes over all possible values of t. Let us substitute for s its expression k-2t. We find

(52) $$\sum_t \xi^{(p)}_t \begin{Bmatrix} (k-2t)(2p-k+2t+1) \\ (k-2t-1)(2p-k+2t+2) \end{Bmatrix} V^{(p-2,t)}_{a_s \dots a_p} + 2 \sum_t \xi^{(p)}_t V^{(p-2,t-1)}_{a_3 \dots a_p} = 0$$

We must distinguish two cases:

1) $p \leq m+n+1$. In this case $r = [k/2]$. The first sum goes therefore from 0 to $[k/2]-1$ on account of the fact that p has decreased by 2. The second sum goes from 1 to $[k/2]$ on account of the fact that t-1 must be greater than or equal to zero. The Eq. (52) reduces to

(53) $$\sum_{t=0}^{[k/2]-1} \xi^{(p)}_t \begin{Bmatrix} (k-2t)(2p-k+2t+1) \\ (k-2t-1)(2p-k+2t+2) \end{Bmatrix} V^{(p-2,t)}_{a_3 \dots a_p} + 2 \sum_{t=1}^{[k/2]} \xi^{(p)}_t V^{(p-2,t-1)}_{a_3 \dots a_p} = 0$$

Shifting the origin of the index t in the first sum we obtain

$$\text{(54)} \qquad \sum_{t=1}^{[k/2]} \left[ \xi_{t-1}^{(p)} \left\{ \begin{matrix} (k-2t+2)(2p-k+2t-1) \\ (k-2t+1)(2p-k+2t) \end{matrix} \right\} +2\, \xi_t^{(p)} \right] V_{a_3 \ldots a_p}^{(p-2,t-1)} = 0$$

The preceding condition is equivalent to a system of equations in the $\xi$ 's, on account of the fact that the

$$V_{a_3 \ldots a_p}^{(p-2,t-1)}$$

are linearly independent. The system is composed of $[k/2]-1$ equations in the $[k/2]$ variables $\xi_t^{(p)}$. We can solve it by setting arbitrarily $\xi_0^{(p)} = 1$, and obtain

$$\text{(55)} \qquad \xi_t^{(p)} = (-1/2)^t \left\{ \begin{matrix} \dfrac{k!!\,(2p-k+2t-1)!!}{(k-2t)!!\,(2p-k-1)!!} \\[2ex] \dfrac{(k-1)!!\,(2p-k+2t)!!}{(k-2t-1)!!\,(2p-k)!!} \end{matrix} \right\}$$

where, as we already said, the upper line refers to even k, the lower to odd k.

2) $p \geq m+n+1$. In this case it is not difficult to show, from the definitions of r and k, that $r=n < [k/2]$. The first sum in Eq. (52) goes therefore from zero to n. In the second sum the term with t=0 drops off on account of the fact that t-1 must be greater than zero. Therefore Eq. (52) takes the form

$$\text{(56)} \qquad \sum_{t=0}^{n} \xi_t^{(p)} \left\{ \begin{matrix} (k-2t)(2p-k+2t+1) \\ (k-2t-1)(2p-k+2t+2) \end{matrix} \right\} V_{a_3 \ldots a_p}^{(p-2,t)} + 2 \sum_{t=1}^{n} \xi_t^{(p)} V_{a_3 \ldots a_p}^{(p-2,t-1)} = 0$$

By the procedure of case (1), Eq. (56) can be reduced to

$$\text{(57)} \qquad \sum_{t=1}^{n+1} \left[ \xi_{t-1}^{(p)} \left\{ \begin{matrix} (k-2t+2)(2p-k+2t-1) \\ (k-2t+1)(2p-k+2t) \end{matrix} \right\} +2\,(1-\delta_{t,n+1})\, \xi_t^{(p)} \right] \cdot V_{a_3 \ldots a_p}^{(p-2,t-1)} = 0$$

This is equivalent to a system of n+1 equations in the n+1 variables $\xi_o^{(p)} \ldots \ldots \xi_n^{(p)}$. The solution is therefore determined unambiguously. From the equation for t=n+1 we obtain $\xi_n^{(p)} = 0$; for t=n, we get

$$\left\{\begin{matrix}(k-2n+2)(2p-k+2n-1)\\(k-2n+1)(2p-k+2n)\end{matrix}\right\} \xi^{(p)}_{n-1} + 2\,\xi^{(p)}_{n} = 0$$

and therefore $\xi^{(p)}_{n-1} = 0$, and so on. The solution of the system is therefore the trivial one. This shows that no irreducible tensor can be constructed in this case, and this completes the proof.

## 6. EVALUATION OF THE $\xi$-COEFFICIENTS FOR THE CASE B

The general form of the tensor of order p+1/2 built from a tensor T of order m and a tensor $\chi$ of order n+1/2 is given by

$$W^{(p)}_{a_1\dots a_p} = \sum_{t=0}^{r_\alpha} \xi^{(p)}_{\alpha t} V^{(pt)}_{\alpha a_1\dots a_p} + \sum_{t=0}^{r_\beta} \xi^{(p)}_{\beta t} V^{(pt)}_{\beta a_1\dots a_p} \tag{58}$$

where the first sum refers to $\alpha$-type tensors, which are of the form

$$V^{(pt)}_{\alpha a_1\dots a_p} = \sum_P \Delta^{(s)}_{b_1\dots b_s} T_{b_{s+1}\dots b_{s+m-\lambda} c_1\dots c_\lambda} \cdot \chi_{b_{s+m-\lambda+1}\dots b_p c_1\dots c_\lambda} \tag{59}$$

this sum extending over all the combinations of the indices $a_1,\dots, a_p$ according to the partition $(s, m-\lambda, n-\lambda)$; this sum contains

$$\frac{p!}{s!(m-\lambda)!(n-\lambda)!}$$

terms. The second sum in Eq. (58) refers to $\beta$-type tensors which are of the form

$$V^{(pt)}_{\beta a_1\dots a_p} = \sum_P \Delta^{(s-1)}_{b_1\dots b_{s-1}} \sigma_d T_{b_s\dots b_{s+m-\lambda-2} c_1\dots c_\lambda d} \cdot \chi_{b_{s+m-\lambda-1}\dots b_p c_1\dots c_\lambda} \tag{60}$$

the sum extending over the

$$\frac{p!}{(s-1)!(m-\lambda-1)!(n-\lambda)!}$$

combinations of the indices $a_1,\dots, a_p$ according to the partition $(s-1, m-\lambda-1, n-\lambda)$.

The condition of irreducibility of the tensor, because of the symmetry of its components, amounts to setting equal to zero a contraction of $W^{(p)}$

with $\sigma_a$, that is

$$\overline{W}^{(p)}_{a_2\ldots a_p} = \sigma_{a_1} W^{(p)}_{a_1\ldots a_p} =$$

(61) $$= \sum_t \xi^{(p)}_{\alpha t} \overline{V}^{(pt)}_{\alpha a_2\ldots a_p} + \sum_t \xi^{(p)}_{\beta t} \overline{V}^{(pt)}_{\beta a_2\ldots a_p} =$$

$$= \sum_t \xi^{(p)}_{\alpha t} \sigma_{a_1} V^{(pt)}_{\alpha a_1 a_2\ldots a_p} + \sum_t \xi^{(p)}_{\beta t} \sigma_{a_1} V^{(pt)}_{\beta a_1 a_2\ldots a_p} = 0$$

In analogy with Sec. 5, let us call first group the indices of the factor $\Delta^{(s)}$, 2nd group the free indices of T, 3rd group the free indices of $\chi$. Each sum in Eq. (61) splits therefore into three parts, with $a_1$ in the 1st, 2nd and 3rd group. Let us see them separately:

a) $\alpha$ - type term-$a_1$ in the 1st group.

By direct application of the Eq. (A5) of the Appendix we get

(62) $$\left\{ \begin{matrix} s \\ s+2 \end{matrix} \right\} V^{(p-1,\ t)}_{\alpha a_2\ldots a_p}$$

b) $\alpha$ - type term-$a_1$ in the 2nd group.

Use of Eq. (A8) of the Appendix gives

(63) $$\left\{ \begin{matrix} V^{(p-1,\,t-\varepsilon)}_{\beta a_2\ldots a_p} \\ \\ 2\,(m-\lambda)\, V^{(p-1,\,t)}_{\alpha a_2\ldots a_p} - V^{(p-1,\,t-\varepsilon)}_{\beta a_2\ldots a_p} \end{matrix} \right\}$$

where $\varepsilon = 0,\ 1$ according that $m > n$, $m \leq n$ respectively.

c) $\alpha$ - type term-$a_1$ in the 3rd group.

By the same procedure as (63) we have

(64) $$\left\{ \begin{matrix} 0 \\ \\ 2\,(n-\lambda)\, V^{(p-1,\,t)}_{\alpha a_2\ldots a_p} \end{matrix} \right\}$$

d) $\beta$ - type term-$a_1$ in the 1st group.

$$\left\{ \begin{matrix} s+1 \\ s-1 \end{matrix} \right\} V^{(p-1,\,t)}_{\beta a_2\ldots a_p}$$

e) $\beta$ - type term-$a_1$ in the 2nd group.

(66)
$$\left\{ \begin{array}{c} 2\,(m - \lambda - 1)\, V^{(p-1,t)}_{\beta a_2 \ldots a_p} \\ 0 \end{array} \right\}$$

f) $\beta$ - type term-$a_1$ in the 3rd group.

(67)
$$\left\{ \begin{array}{l} 2(n-\lambda)\, V^{(p-1,t)}_{\beta a_2 \ldots a_p} - 2\, V^{(p-1,t-1+\varepsilon)}_{\alpha a_2 \ldots a_p} \\ 2\, V^{(p-1,\, t-1+\varepsilon)}_{\beta a_2 \ldots a_p} \end{array} \right\}$$

In expressions (62) to (67) the upper line refers to even ($k+\mu$), the lower line to odd ($k+\mu$). (Note that s has the same parity as $k+\mu$).

By summing expressions (62) to (64) we obtain, for the $\alpha$ - part of $\overline{W}$ (the variable s has been substituted with t, as defined by Eq. (30):

(68)
$$\overline{V}^{(pt)}_{\alpha a_2 \ldots a_p} = \left\{ \begin{array}{c} k_\alpha - 2t \\ 2p - k_\alpha + 2t + 2 \end{array} \right\} V^{(p-1,t)}_{\alpha a_2 \ldots a_p} + \\ + \left\{ \begin{array}{c} 1 \\ -1 \end{array} \right\} V^{(p-1,t-\varepsilon)}_{\beta a_2 \ldots a_p}$$

and summing from (65) to (67) we obtain for the $\beta$-part of $\overline{W}$

(69)
$$\overline{V}^{(pt)}_{\beta a_2 \ldots a_p} = \left\{ \begin{array}{c} -2 \\ +2 \end{array} \right\} V^{(p-1,\, t-1+\varepsilon)}_{\alpha a_2 \ldots a_p} + \\ + \left\{ \begin{array}{c} 2p - k_\beta + 2t + 2 \\ k_\beta - 2t \end{array} \right\} V^{(p-1,t)}_{\beta a_2 \ldots a_p}$$

Substitution of expressions (68) and (69) into Eq. (61) gives the equation (tensor indices in the V's are omitted)

(70)
$$\sum_{t=0}^{r_\alpha} \xi^{(p)}_{\alpha t} \left[ \left\{ \begin{array}{c} k_\alpha - 2t \\ 2p - k_\alpha + 2t + 2 \end{array} \right\} V^{(p-1,t)}_{\alpha} + \left\{ \begin{array}{c} 1 \\ -1 \end{array} \right\} V^{(p-1,t-\varepsilon)}_{\beta} \right. + \\ + \sum_{t=0}^{r_\beta} \xi^{(p)}_{\beta t} \left[ \left\{ \begin{array}{c} -2 \\ 2 \end{array} \right\} V^{(p-1,t-1+\varepsilon)}_{\alpha} + \left\{ \begin{array}{c} 2p - k_\beta + 2t + 2 \\ k_\beta - 2t \end{array} \right\} V^{(p-1,t)}_{\beta} \right. = 0$$

After rearrangement of the four terms in Eq. (70) and a change of indices in the 2nd and the 3rd sums, we obtain

(71)
$$\left[ \sum_{t=0}^{r_\alpha} \xi^{(p)}_{\alpha t} \left\{ \begin{array}{c} k_\alpha - 2t \\ 2p - k_\alpha + 2t + 2 \end{array} \right\} + \sum_{t=0}^{r_\beta - 1 + \varepsilon} \xi^{(p)}_{\beta,\, t+1-\varepsilon} \left\{ \begin{array}{c} -2 \\ 2 \end{array} \right\} \right] V^{(p-1,t)}_{\alpha} +$$

$$+\left[\sum_{t=0}^{r_\alpha-\varepsilon} \xi^{(p)}_{\alpha,t+\varepsilon}\begin{Bmatrix}1\\-1\end{Bmatrix} + \sum_{t=0}^{r_\beta} \xi^{(p)}_{\beta t}\begin{Bmatrix}2p-k_\beta+2t+2\\k_\beta-2t\end{Bmatrix}\right] V^{(p-1,t)}_\beta = 0$$

Eq. (71) is equivalent to a set of linear equations in the variables $\xi^{(p)}_{\alpha t}$, $\xi^{(p)}_{\beta t}$. We shall obtain the upper bound of p, that is the largest order of the tensor that can be constructed from T and $\chi$, from the solvability condition of such a system.

It is immediately seen that the number of unknowns is $N_1 = r_\alpha + r_\beta + 2$, where $r_\alpha$, $r_\beta$ are defined by Eq. (31). The number $N_2$ of independent equations which can be furnished by (71) is given by

(72) $$N_2 = \min([(k_\alpha - 1)/2], m, n) + \min([(k_\beta - 1)/2], m-1, n) + 2$$

By the comparison of $N_2$ with $N_1$, as given by

$$N_1 = \min([k_\alpha/2], m, n) + \min([k_\beta/2], m-1, n) + 2$$

it is possible to see that $N_2$ can be only less than or equal to $N_1$ but not less than $N_1 - 1$. If $N_1 = N_2$ there is only the trivial solution. If $N_2 \leq N_1$, i. e. $N_2 = N_1 - 1$, a system of $\xi^{(p)}_{\alpha t}$, $\xi^{(p)}_{\beta t}$ can be constructed which is unique apart for an arbitrary multiplicative factor.

We shall now discuss what $N_2 = N_1$ implies for the value of p. Let us define (see Eqs. (28), (29)).

$$A = \min\left(\left[\frac{p-|m-n|}{2}\right], m, n\right) \qquad C = \min\left(\left[\frac{p-|m-n|-1}{2}\right], m, n\right)$$

$$B = \min\left(\left[\frac{p-|m-n-1|}{2}\right], m-1, n\right) \quad D = \min\left(\left[\frac{p-|m-n-1|-1}{2}\right], m-1, n\right)$$

It is immediately seen that $A \geq C$, $B \geq D$. The condition $N_2 = N_1$ reads, with this notation, as

$$A + B = C + D$$

and this relation can be satisfied only if $A = C$, $B = D$. These conditions can be satisfied only if the following pairs of relations hold

1) $\left[\frac{p-|m-n|}{2}\right] > \min(m,n)$ and $p - |m-n-1|$ is odd

2) $\left[\frac{p-|m-n-1|}{2}\right] > \min(m-1,n)$ and $p - |m-n|$ is odd

3) $\left[\frac{p-|m-n|}{2}\right] > \min(m,n)$ and $\left[\frac{p-|m-n-1|}{2}\right] > \min(m-1,n)$

Relations 1) and 2) give the strongest conditions. These are

1) $p \geq m+n+2$ 2) $p \geq m+n+1$

This shows that only if $p \leq m + n$ the system derived from Eq. (71) has non-trivial solutions and therefore a tensor W can be constructed.

The simplest way to find a solution to that system is to split the cases $m > n$, $m \leq n$: in such a way the index $\varepsilon$ can be given a fixed value and the system simply reduces to a system of linear equations, with all indices completely defined.

Case $m > n$:

In this case $k_\alpha = p - |m - n| = p - m + n = h$; $k_\beta = p - |m - n - 1| =$
$= p - m + n + 1 = h + 1$

and the system is

$$(73) \quad \begin{cases} \xi^{(p)}_{\alpha t} \begin{Bmatrix} h - 2t \\ 2p - h + 2t + 2 \end{Bmatrix} + \xi^{(p)}_{\beta t+1} \begin{Bmatrix} -2 \\ 2 \end{Bmatrix} = 0 \\ \xi^{(p)}_{\alpha t} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} + \xi^{(p)}_{\beta t} \begin{Bmatrix} 2p - h + 2t + 1 \\ h - 2t + 1 \end{Bmatrix} = 0 \end{cases}$$

The solution is immediately found (with the condition $\xi^{p}_{\alpha_0} = 1$) and is

$$(74) \qquad \xi^{(p)}_{\alpha t} = (-1/2)^t \begin{Bmatrix} \dfrac{h!!(2p - h + 2t + 1)!!}{(2p - h + 1)!!(h - 2t)!!} \\[2ex] \dfrac{(h - 1)!!(2p - h + 2t)!!}{(h - 2t - 1)!!(2p - h)!!} \end{Bmatrix}$$

$$(75) \qquad \xi^{(p)}_{\beta t} = (-1/2)^t \begin{Bmatrix} -\dfrac{h!!(2p - h + 2t - 1)!!}{(2p - h + 1)!!(h - 2t)!!} \\[2ex] \dfrac{(h - 1)!!(2p - h + 2t)!!}{(h - 2t + 1)!!(2p - h)!!} \end{Bmatrix}$$

It is worth while to remember that the first line of Eqs. (74) and (75) refers to even values, the second line to odd values of $[p - (m - n)]$.

Case $m \leq n$:

In this case $k_\alpha = p - |m-n| = p-n+m = h'$; $k_\beta = p - |m-n-1| =$
$= p-n-1+m = h' - 1$. Eq. (71) gives the following system

$$(76) \quad \begin{cases} \xi^{(p)}_{\alpha t} \begin{Bmatrix} h' - 2t \\ 2p - h' + 2t + 2 \end{Bmatrix} + \xi^{(p)}_{\beta t} \begin{Bmatrix} -2 \\ 2 \end{Bmatrix} = 0 \\ \xi^{(p)}_{\alpha t+1} \begin{Bmatrix} 1 \\ -1 \end{Bmatrix} + \xi^{(p)}_{\beta t} \begin{Bmatrix} 2p - h' + 2t + 3 \\ h' - 2t - 1 \end{Bmatrix} = 0 \end{cases}$$

The solution of system (76) is given, with the condition $\xi^{(p)}_{\alpha 0} = 1$, by

(77)
$$\xi^{(p)}_{\alpha t} = (-1/2)^t \begin{cases} \dfrac{h'!!(2p - h' + 2t + 1)!!}{(2p - h' + 1)!!(h' - 2t)!!} \\[2ex] \dfrac{(h' - 1)!!(2p - h' + 2t)!!}{(2p - h')!!(h' - 2t - 1)!!} \end{cases}$$

and

(78)
$$\xi^{(p)}_{\beta t} = (-1/2)^{t+1} \begin{cases} -\dfrac{h'!!(2p - h' + 2t + 1)!!}{(2p - h' + 1)!!(h' - 2t - 2)!!} \\[2ex] \dfrac{(h' - 1)!!(2p - h' + 2t + 2)!!}{(2p - h')!!(h' - 2t - 1)!!} \end{cases}$$

the first line referring to even values, the second to odd values of p-n+m.

## 7. EVALUATION OF THE $\xi$ - COEFFICIENTS FOR CASE C.

As explained in Sec. 4, the case C can be splitted into two parts, according to the parity of p-m-n which is the difference between the order of the composed tensor and those of the component tensors.

1) p-m-n even. The form of the general tensor is a sum of e-type terms and reads

(79)
$$W^{(p)}_{a_1 \dots a_p} = \sum_t \xi^{(p)}_t V^{(pt)}_{a_1 \dots a_p}$$

where

(80)
$$V^{(pt)}_{a_1 \dots a_p} = \sum_P \Delta^{(s)}_{b_1 \dots b_s} T_{b_{s+1} \dots b_{s+m-\lambda} c_1 \dots c_\lambda} \cdot U_{b_{s+m-\lambda-1} \dots b_p c_1 \dots c_\lambda} :$$

The condition of irreducibility of tensor W is the vanishing of its trace. We can divide the indices of the V's into three groups, according to the partition $(s, m-\lambda, n-\lambda)$ and call these groups 1st, 2nd and 3rd group.

a) both indices are in the first group. The contribution of this part is

$$s\,(s + 1)\, V^{(p-2,\, t)}$$

b) one index in the first and one in the second group:

$$2\, s\, (m - \lambda)\, V^{(p-2,\, t)}$$

c) one index in the first and one in the third group:

$$2\, s\, (n - \lambda)\, V^{(p - 2,\, t)}$$

d) one index in the second and one in the third group:

$$2\, V^{(p - 2,\, t - 1)}$$

Summing the contributions from a) to d) and expressing s through the variable t one gets

$$\sum_t \xi_t^{(p)} \left[ (k - 2t)(2p - k + 2t + 1)\, V^{(p-2,\, t)} + + 2\, V^{(p - 2,\, t - 1)} \right] = 0 \tag{81}$$

which is equivalent to a set of min $(m, [(k-2)/2])$ equations in the min$(m, [k/2])$ unknowns $\xi_t^{(p)}$. The system has nontrivial solution only if

$$[k/2] < m$$

The preceding inequality gives

$$2\, [k/2] \leq 2m, \quad \text{that is} \quad k \leq 2m$$

and therefore $p \leq m+n$. In this case we have the system

$$(k - 2t)(2p - k + 2t+1)\, \xi_t^{(p)} + 2\, \xi_{t-1}^{(p)} = 0 \tag{82}$$

With the condition $\xi_o^{(p)} = 1$, the solution of system (82) is

$$\xi_t^{(p)} = (-1/2)^t \quad \frac{k!!(2p - k + 2t - 1)!!}{(k - 2t)!!(2p - k - 1)!!} \tag{83}$$

It is worth while to remember that, in this case, $k = p+m-n$.

2) p-m-n odd. The general tensor W is a sum of o-type terms. It can be written

$$W^{(p)}_{a_1 \ldots a_p} = \sum_t \xi_t^{(p)}\, V^{(pt)}_{a_1 \ldots a_p} \tag{84}$$

where

$$V^{(p,t)}_{a_1 \ldots a_p} = \sum_P \Delta^{(s+1)}_{b_1 \ldots b_s e}\, \varepsilon_{efg} \cdot \cdot T_{b_{s+1} \ldots b_{s+m-\lambda-1} c_1 \ldots c_\lambda f}\, U_{b_{s+m-\lambda} \ldots b_p c_1 \ldots c_\lambda g} \tag{85}$$

By a procedure quite similar to the one of case (1), a system of linear equations for the unknowns $\xi_t^{(p)}$ can be writt en which has nontrivial solution only if $p \leq m+n$. The system is the following

$$(k - 2t + 2)(2p - k + 2t + 1)\, \xi_t^{(p)} + 2\, \xi_{t+1}^{(p)} = 0 \tag{86}$$

With the condition $\xi_0^{(p)} = 1$, its solution is

$$\xi_t^{(p)} = (-1/2)^t \quad \frac{(k+2)!!(2p-k+2t-1)!!}{(k-2t+2)!!(2p-k-1)!!} \tag{87}$$

(Note that in this case $k = p+m-n-1$).

## 8. CONCLUSION

This paper solves one side of the complete Clebsch-Gordan problem, that is the building up of irreducible tensors from the product of two irreducible tensors of any order.

Inspection of Section 4 is sufficient to make clear that it is possible to give dummy rules for the explicit construction of the irreducible tensors. These rules are quite straightforward and will be given in a abridged version of this paper which will be submitted to Il Nuovo Cimento. We hope that these rules will be of practical use in any case where a physical problem will be of easier solution in terms of a cartesian point of view. In Table I we give therefore, as an example and for easy reference, a list of the first few cases of composed irreducible tensors.

The solution of the complete Clebsch-Gordan problem requires another step, that is the construction of the series equivalent to the inversion of Eq. (2). This problem is easily solved for the irreducible tensorial sets thanks to the orthogonality of Clebsch-Gordan-Wigner coefficients.

In our scheme, this is not the case. We must find anew how the direct product of two tensors of any order can be written in terms of its irreducible components. The main difficulty lies in the following: the cartesian representation for an irreducible tensor is redundant, since it involves more than $2n+1$ components for a tensor of order n. So the set of cartesian irreducible tensors is not a true basis in the vector space of all tensors. The way out of this difficulty should be in a clever use of the isotropic tensors, $\delta_{ab}$ and $\varepsilon_{abc}$, and of the isotropic "vector" $\sigma_a$.

Another related part not settled so far is the normalization of our tensors. The difficulty is the same as before: since we do not have a natural orthonormal basis, a separate definition of the metrics must be given, consistently with the usual one. Research on both problems is under way.

The techniques developed in this paper could possibly be extended to the similar problem for the Lorentz Group. In that case the usual group theoretical approach is not so straightforward[4]; we believe that the cartesian method could better exhibit its usefulness in that connection.

Thanks are due to Miss. M. Batini for checking all the formulae and working out the examples of Table I. The authors are also indebted to L. Lovitch for helpful comments.

FOOTNOTES

(1) - Notations are as in Fano-Racah's book: Irreducible Tensorial Sets. (Academic Press Publ. New York, 1959).
(2) - We mean complex numbers: hereafter this will always be understood.
(3) - On account of Eq. (9), $\sigma_a$ is a kind of isotropic vector matrix; it is the only one, to within a factor.
(4) - See, for instance, A. J. Mcfarlane, Revs. Modern Phys. 34, 41 (1962).

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APPENDIX

We want to collect here some of the properties of symmetrized products of $\sigma$-matrices which have been used in this paper. We define the $\Delta$ symbol by

(A1) $$\Delta^{(s)}_{a_1 \dots a_s} = \sum_P \sigma_{b_1} \dots \sigma_{b_s}$$

where the sum goes over the s! permutations of the indices $a_1, \dots\dots a_s$.

It is immediately evident that

(A2) $$\Delta^{(s)}_{a_1 \dots a_s} = \sum_{i=1}^{s} \sigma_{a_i} \Delta^{(s-1)}_{a_1 \dots (a_i) \dots a_s}$$

Let us now show the result of the contraction of a $\sigma$ with a $\Delta$. We want to calculate $\sigma_a \Delta^{(s)}_{aa_2 \dots a_s}$. The result must be proportional to the $\Delta^{(s-1)}$ and the therefore we set

(A3) $$\sigma_a \Delta^{(s)}_{aa_1 \dots a_{s-1}} = c_s \Delta^{(s-1)}_{a_1 \dots a_s -1}$$

We use now Eq. (A2) separating the contribution of $\sigma_a$. We have

$$\sigma_a \Big(\sum_{i=1}^{s-1} \sigma_{a_i} \Delta^{(s-1)}_{a \dots (a_i) \dots a_{s-1}} + \sigma_a \Delta^{(s-1)}_{a_1 \dots a_{s-1}}\Big) =$$

$$= 2(s-1) \Delta^{(s-1)}_{a_1 \dots a_{s-1}} - \sum_{i=1}^{s} \sigma_{a_i} \sigma_a \Delta^{(s-1)}_{aa_1 \dots (a_i) \dots a_{s-1}} + 3 \Delta^{(s-1)}_{a_1 \dots a_{s-1}} =$$

$$= (2s+1) \Delta^{(s-1)}_{a_1 \dots a_{s-1}} - c_{s-1} \sum_{i=1}^{s-1} \sigma_{a_i} \Delta^{(s-2)}_{a_1 \dots (a_i) \dots a_{s-1}} =$$

$$= (2s+1-c_{s-1}) \Delta^{(s-1)}_{a_1 \dots a_{s-1}}$$

We obtain therefore

(A4) $$c_s = 2s + 1 - c_{s-1}$$

The preceding equation can be iterated to yield

$$c_s = c_{s-2} + 2$$

which has the solution

$$c_s = s + x$$

where x must be fixed by comparison with some known case. It is easily shown that $c_1 = 3$, $c_2 = 2$ and therefore

$$c_s = \begin{cases} s & \text{even } s \\ s+2 & \text{odd } s \end{cases}$$

Eq. (A3) then has the form

$$\text{(A5)} \qquad \sigma_a \Delta^{(s)}_{aa_1\ldots a_{s-1}} = \begin{Bmatrix} s \\ s+2 \end{Bmatrix} \Delta^{(s-1)}_{a_1\ldots a_{s-1}}$$

By the symmetry of the $\Delta$'s, the anticommutation properties of the $\sigma$-matrices and the use of Eq. (A5) one gets also

$$\text{(A6)} \qquad \Delta^{(s)}_{aaa_1\ldots a_{s-2}} = \begin{Bmatrix} s(s+1) \\ (s-1)(s+2) \end{Bmatrix} \Delta^{(s-2)}_{a_1\ldots a_{s-2}}$$

By a procedure similar to the preceding one it is not difficult to arrive to

$$\text{(A7)} \qquad \Delta^{(s)}_{ab_1\ldots b_{s-1}} \chi_{ac_1\ldots c_\lambda} = \begin{Bmatrix} s \\ s-1 \end{Bmatrix} \sum_{i=1}^{s-1} \Delta^{(s-2)}_{b_1\ldots(b_i)\ldots b_{s-1}} \chi_{b_i c_1\ldots c_\lambda}$$

Two other properties of the $\Delta$-symbols are of interest in this paper. The first is the commutation property of $\Delta^{(s)}$ with $\sigma$. The result is

$$\text{(A8)} \qquad \sigma_a \Delta^{(s)}_{b_1\ldots b_s} = (-)^s \Delta^{(s)}_{b_1\ldots b_s} \sigma_a - \left((-)^s - 1\right) \sum_{i=1}^{s} \delta_{ab_i} \Delta^{(s-1)}_{b_1\ldots(b_i)\ldots b_s}$$

saying that for even s the two factors commute. If s is odd, the anticommutator reduces to a function of the $\Delta^{(s-1)}$.

The second property is a reduction property of the $\Delta$'s. By using the Eq. (A2) twice, one easily obtains

$$\text{(A9)} \qquad \Delta^{(s)}_{a_1 \ldots a_s} = 2 \sum_{i<j}^{1,s} \delta_{a_i a_j} \Delta^{(s-2)}_{a_1 \ldots (a_i) \ldots (a_j) \ldots a_s}$$

Eq. (A9) shows that for even s, a $\Delta$ can be reduced to the direct product of a symmetrized function of the Kronecker $\delta$'s and the spinor identity; this property has been used in Sec. 7.

TABLE I

(All tensors are defined up to an arbitrary multiplicative factor and are not normalized. The factor has been disposed of in order to make all coefficients integral numbers (a direct application of formulae in Sect. 5 would give some fractional coefficients). The expression Symm [....] means the complete symmetrization on the free indices of the expression in brackets. Partial symmetries must be taken into account.
Ex. : Symm $[\sigma_a \delta_{bc}] = \sigma_a \delta_{bc} + \sigma_b \delta_{ca} + \sigma_c \delta_{ab})$.

A: spinor-spinor coupling

| m | n | p | Irreducible compound tensor |
|---|---|---|---|
| 0 | 0 | 0 | $(\chi, \varphi)$ |
| 0 | 0 | 1 | $\sigma_a(\chi, \varphi)$ |
| 1 | 0 | 1 | $(\chi_a, \varphi)$ |
| 1 | 0 | 2 | $\sigma_a(\chi_b, \varphi) + \sigma_b(\chi_a, \varphi)$ |
| 1 | 1 | 0 | $(\chi_a, \varphi_a)$ |
| 1 | 1 | 1 | $\sigma_a(\chi_b, \varphi_b)$ |
| 1 | 1 | 2 | $2\,\delta_{ab}(\chi_c, \varphi_c) - 3\,\mathrm{Symm}\,[(\chi_a, \varphi_b)]$ |
| 1 | 1 | 3 | $2\,\mathrm{Symm}\,[\sigma_a \delta_{cb}]\,(\chi_d, \varphi_d) - 5\,\mathrm{Symm}\,[\sigma_a(\chi_b, \varphi_c)]$ |

B: tensor-spinor coupling

| m | n | p | Irreducible compound tensor |
|---|---|---|---|
| 1 | 0 | 0 | $\sigma_a T_a \chi$ |
| 1 | 0 | 1 | $T_a \chi$ |
| 1 | 1 | 0 | $T_a \chi_a$ |

Table I - (continued ; case B).

| m | n | p | Irreducible compound tensor |
|---|---|---|---|
| 1 | 1 | 1 | $2\,\sigma_a T_b \chi_b - 3\,\sigma_b T_b \chi_a$ |
| 1 | 1 | 2 | $2\,\delta_{ab} T_c \chi_c - 5\,\mathrm{Symm}\,[T_a \chi_b] + \mathrm{Symm}\,[\sigma_a \sigma_c T_c \chi_b]$ |
| 2 | 1 | 0 | $\sigma_a T_{ab} \chi_b$ |
| 2 | 1 | 1 | $3\,T_{ab} \chi_b - \sigma_a \sigma_b T_{bc} \chi_c$ |
| 2 | 1 | 2 | $2\,\mathrm{Symm}\,[\sigma_a T_{bc} \chi_c] + 2\,\delta_{ab} \sigma_c T_{cd} \chi_d -$ $- 5\,\mathrm{Symm}\,[\sigma_c T_{ac} \chi_b]$ |
| 2 | 1 | 3 | $10\,\mathrm{Symm}\,[\delta_{ab} T_{cd} \chi_d] - 35\,\mathrm{Symm}\,[T_{ab} \chi_c] -$ $- 2\,\mathrm{Symm}\,[\delta_{ab} \sigma_c]\, \sigma_d T_{de} \chi_e + 5\,\mathrm{Symm}\,[\sigma_a \sigma_d T_{bd} \sigma_c]$ |
| 1 | 2 | 1 | $T_b \chi_{ab}$ |
| 1 | 2 | 2 | $2\,\mathrm{Symm}\,[\sigma_a T_c \chi_{bc}] - 5\,\sigma_c T_c \chi_{ab}$ |
| 1 | 2 | 3 | $\mathrm{Symm}\,[\sigma_a \sigma_d T_d \chi_{bc}] + 2\,\mathrm{Symm}\,[\delta_{ab} T_d \chi_{cd}] -$ $- 7\,\mathrm{Symm}\,[T_a \chi_{bc}]$ |

C: <u>tensor-tensor coupling</u>

| m | n | p | Irreducible compound tensor |
|---|---|---|---|
| 1 | 1 | 0 | $T_a U_a$ |
| 1 | 1 | 1 | $\varepsilon_{abc} T_b U_c$ |
| 1 | 1 | 2 | $2\,\delta_{ab} T_c U_c - 3\,\mathrm{Symm}\,[T_a U_b]$ |
| 2 | 1 | 1 | $T_{ab} U_b$ |
| 2 | 1 | 2 | $\mathrm{Symm}\,[\varepsilon_{acd} T_{bc} U_d]$ |
| 2 | 1 | 3 | $2\,\mathrm{Symm}\,[\delta_{ab} T_{cd} U_d] - 5\,\mathrm{Symm}\,[T_{ab} U_c]$ |
| 2 | 2 | 0 | $T_{ab} U_{ab}$ |

Table I - (continued : case C).

| m | n | p | Irreducible compund tensor |
|---|---|---|---|
| 2 | 2 | 1 | $\varepsilon_{abc} T_{bd} U_{cd}$ |
| 2 | 2 | 2 | $2\,\delta_{ab} T_{cd} U_{cd} - 3\,\mathrm{Symm}\left[T_{ac} U_{bc}\right]$ |
| 2 | 2 | 3 | $2\,\mathrm{Symm}\left[\delta_{ab}\,\varepsilon_{cde} T_{df} U_{ef}\right] - 5\,\mathrm{Symm}\left[\varepsilon_{ade} T_{bd} U_{ce}\right]$ |
| 2 | 2 | 4 | $4\,\mathrm{Symm}\left[\delta_{ab}\,\delta_{cd}\right] T_{ef} U_{ef} - 10\,\mathrm{Symm}\left[\delta_{ab} T_{ce} U_{de}\right] +$ $+ 35\,\mathrm{Symm}\left[T_{ab} U_{cd}\right]$ |