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Abstract.

It is proved that the analytic continuation of Wightman function, the vacuum expectation value of product of field operators, due to local commutativity is single-valued in the union of the extended tubes which correspond to the Wightman functions obtained by permuting the order of the field operators in the product, and that the extended tubes, the union of them and the intersection of any two are simply-connected.

1. Introduction.

In the systematic analysis of the frame of quantum field theory, the investigation of the analytic property of the vacuum expectation value, called Wightman function (denoted W-function hereafter), of product of field operators turned out to be important: Wightman has shown[(1)] that a set of analytic functions with certain properties, such as suitable invariance properties and boundedness, is equivalent to quantum field theory with certain axioms (see below), identifying the boundary values of these analytic functions with the W-functions of the theory.

We take the following axioms[(2)] as the basis of the theory:

(I) Invariance under the proper[(3)] inhomogeneous Lorentz group;

(II) Spectral condition, i. e., existence of the Hilbert space spanned by the physical state vectors, non-negativity of the energy spectrum of these states and the existence of the vacuum as the lowest energy state;

(III) Existence of field operators as temperate distribution operators;

(IV) Local commutativity, i. e., field operators commute or anticommute for space-like separation.

From axioms (I) - (III) it follows that the W-function

(1) $$W_{\nu}^{N}(\xi_1, \ldots\ldots, \xi_N) = \langle 0 | \psi_{\nu_0}^{(0)}(x_0)\, \psi_{\nu_1}^{(1)}(x_1) \ldots \psi_{\nu_N}^{(N)}(x_N) | 0 \rangle ,$$

where $\xi_j = x_j - x_{j-1}$ $(j = 1, 2, \ldots, N)$, $\psi_{\nu_j}^{(j)}(x_j)$ are field operators, ν_j are the spin indices and ν stands for the set $\nu_0, \nu_1, \ldots\ldots, \nu_N$, is a temperate distribution which is the boundary value of a function analytic in the forward tube $\mathcal{T}_N = \{\{\zeta_j\};\ \mathrm{Im}\,\zeta_j \in V_+\}$, where V_+ is the forward cone. The Bargmann-Hall-Wightman Theorem(4) enables us to enlarge the analyticity domain of the W-function: $W^N(\zeta_1, \ldots\ldots, \zeta_N)$ is a single-valued analytic function in the extended tube

$$\mathcal{T}'_N = \{\{\zeta_j\}\ ;\ \zeta_j = L_+(C)\,\zeta'_j\,,\ \{\zeta_j\} \in \mathcal{T}_N\} ,$$

where $L_+(C)$ is the totality of the proper homogeneous complex Lorentz transformations (with determinant +1).

Local commutativity (axiom IV) then relates the W-functions which correspond to various permutations of the field operators in the product, and gives us analytic continuation in the union $\bigcup_{g \in S_{N+1}} P(g)\,\mathcal{T}'_N$ of the extended tubes $P(g)\,\mathcal{T}'_N$. Here we adopt the following notation:

$g = \begin{pmatrix} 0, 1, \ldots\ldots, N \\ i_0, i_1, \ldots\ldots, i_N \end{pmatrix}\downarrow$ is an element of the symmetric group of degree N+1, S_{N+1} ; the set $\{\tilde{\xi}_j\} = P(g)\{\xi_j\} = \{P(g)\,\xi_j\}$ is the set of the transformed variables of $\{\xi_j\}$ induced by the permutation g operating on the suffix of $(x_0, x_1, \ldots\ldots, x_N)$, i. e.,

$$\tilde{\xi}_j = P(g)\,\xi_j = x_{i_j} - x_{i_{j-1}} = \sum_{k=1}^{N} p_{jk}(g)\,\xi_k =$$

(2) $$= \begin{cases} \xi_{i_j} + \xi_{i_j-1} + \ldots\ldots + \xi_{i_{j-1}+1}\,, & i_j > i_{j-1}\,, \\ -(\xi_{i_{j-1}} + \xi_{i_{j-1}-1} + \ldots\ldots + \xi_{i_j+1})\,, & i_j < i_{j-1}\,. \end{cases}$$

Similarly we write $\{\tilde{\zeta}_j\} = P(g)\{\zeta_j\} = \{P(g)\zeta_j\}$; the permuted forward tube $P(g)\mathcal{T}_N$ and the permuted extended tube $P(g)\mathcal{T}'_N$ are defined as follows, writing the variables $\{\zeta_j\}$ explicitly,

(3a) $$P(g)\mathcal{T}_N(\{\zeta_j\}) = \mathcal{T}_N(P(g)\{\zeta_j\}) = \{\{\zeta_j\};\ \mathrm{Im}(P(g)\zeta_j) \in V_+\}$$

and

$$P(g)\mathcal{T}'_N(\{\zeta_j\}) = [P(g)\mathcal{T}_N]'(\{\zeta_j\}) = \mathcal{T}'_N(P(g)\{\zeta_j\}) =$$

(3b) $$= \{P(g)\{\zeta_j\};\ P(g)\zeta_j = L_+(C)P(g)\zeta'_j,\ \mathrm{Im}(P(g)\zeta'_j) \in V_+$$

$$= \{\{\zeta_j\};\quad \zeta_j = L_+(C)\zeta'_j,\ \{\zeta'_j\} \in P(g)\mathcal{T}_N\},$$

where $L_+(C)P(g)\zeta_j = P(g)L_+(C)\zeta_j$ (if we write $\zeta_j = \zeta_j^\mu$, μ denoting the component of the 4-vector ($\mu = 0, 1, 2, 3$), $P(g)$ operates only on j and $L_+(C)$ operates only on μ).

The aim of this article is to prove that the analytic continuation of the W-function due to local commutativity, mentioned above, is single-valued in $\bigcup_{g \in S \subseteq S_{N+1}} P(g)\mathcal{T}'_N$ (Sect. 2), where S is an arbitrary subset of S_{N+1}, and that the domain $\bigcup_{g \in S \subseteq S_{N+1}} P(g)\mathcal{T}'_N$ is simply-connected[5] (Sect. 3). It is also proved that the intersection $P(g_1)\mathcal{T}'_N \cap P(g_2)\mathcal{T}'_N$ is simply-connected (Sect. 4).

2. Single-valuedness of the analytic continuation of the W-function.

The set J_N of the real points $\{\xi_j\}$, called the Jost points, of the extended tube $\mathcal{T}'_N$ is characterized by the following condition[9,10]

(4) $$\Big(\sum_{j=1}^{N} \lambda_j\, \xi_j\Big)^2 > 0$$

for

(5) $$\sum_{j=1}^{N} \lambda_j = 1\ , \qquad \lambda_j \geq 0\ .$$

The set $P(g)J_N$ of the Jost points of the permuted extended tube $P(g)\mathcal{T}'_N$

is given by

$$P(g)J_N(\{\xi_j\}) = J_N(P(g)\{\xi_j\}) =$$

(6)

$$= \left\{\{\xi_j\};\ (\sum_{j=1}^{N} \lambda_j P(g)\ \xi_j)^2 > 0;\ \sum_{j=1}^{N} \lambda_j = 1,\ \ \lambda_j \geq 0\right\}.$$

L e m m a 1.

$$\bigcap_{\alpha=1,2,3} P(g_\alpha)J_N \text{ is non-empty for } \forall g_\alpha \in S_{N+1},\ \ (\alpha = 1, 2, 3).$$

P r o o f.

Take a set $Q(g_1, g_2, g_3)$ of points $\{\xi_j\}$, such that

$$Q(g_1, g_2, g_3) = \Big\{\{\xi_j\};\ \ \xi_j^{\circ} = 0,\ P(g_\alpha)\,\xi_j^{\alpha} > 0 \ \text{ for } \forall j$$

(7)

$$\text{or } < 0 \ \text{ for } \forall j,\ \ \ (\alpha = 1, 2, 3)\Big\},$$

which is non-empty because $P(g)\{\xi_j\}$ affords a representation[11] of S_{N+1}, and thus the eq. (2) is solvable in terms of $\{\tilde{\xi}_j\}$ (or see eq. (2') after Lemma 3). Since

(8) $$(\sum_{j=1}^{N} \lambda\ P(g_\alpha)\ \xi_j)^2 = \sum_{\beta=1}^{3}(\sum_{j=1}^{N} \lambda_j P(g_\alpha)\ \xi_j^{\beta})^2 > 0$$

for a point $\{\xi_j\} \in Q(g_1, g_2, g_3)$, for $\alpha = 1, 2, 3$, and for $\{\lambda_j\}$ satisfying (5), we get

(9) $$Q(g_1, g_2, g_3) \subset \bigcap_{\alpha=1,2,3} P(g_\alpha)J_N\ .$$ (q. e. d.)

Now, $P(g)J_N \subset P(g)\,\mathcal{J}'_N$, and so we have the following Theorem:

T h e o r e m 1.

The intersection of any three of the permuted extended tubes $P(g)\,\mathcal{J}'_N$ is non-empty.

L e m m a 2.

Any arbitrary point belonging to $P(g_1)\mathcal{J}'_N \cap P(g_2)\mathcal{J}'_N$ is connected by a path inside $P(g_1)\mathcal{J}'_N \cap P(g_2)\mathcal{J}'_N$ to a point belonging to

$P(g_1)\mathcal{T}'_N \cap P(g_2)\mathcal{T}_N$ and to a point belonging to $P(g_1)\mathcal{T}_N \cap P(g_2)\mathcal{T}'_N$.

Proof.

Take a point $\{\zeta'_j\} \in P(g_1)\mathcal{T}'_N \cap P(g_2)\mathcal{T}'_N$. By the definition of the extended tube, we can find complex Lorentz transformations Λ_1, $\Lambda_2 \in L_+(C)$ such that

$$(10) \qquad \{\Lambda_\alpha \zeta'_j\} = \{(\zeta_j)_\alpha\} \in P(g_\alpha)\mathcal{T}_N, \qquad (\alpha = 1, 2).$$

Since $L_+(C)$ is a connected set, we can find continuous paths $\Lambda_\alpha(t)$ such that

$$(11) \qquad \begin{aligned} &\Lambda_\alpha(t) \in L_+(C), && 0 \le t \le 1, \\ &\Lambda_\alpha(0) = 1, \quad \Lambda_\alpha(1) = \Lambda_\alpha, && (\alpha = 1, 2). \end{aligned}$$

From the invariance of $P(g)\mathcal{T}'_N$ under the operation of $L_+(C)$, and from eqs. (10) and (11) it follows that the continuous curves

$$(12) \qquad \begin{aligned} &\{\zeta_j(t)_\alpha\} = \{\Lambda_\alpha(t)\zeta'_j\} \subset P(g_1)\mathcal{T}'_N \cap P(g_2)\mathcal{T}'_N \\ &0 \le t \le 1, \qquad \alpha = 1, 2, \end{aligned}$$

give the required paths to connect the points of the Lemma. (q. e. d.)

According to this Lemma, the question of the connectedness of $P(g_1)\mathcal{T}'_N \cap P(g_2)\mathcal{T}'_N$ was reduced to that of $P(g_1)\mathcal{T}'_N \cap P(g_2)\mathcal{T}_N$. Clearly it is sufficient to discuss the case with $g_1 = 1$, and $g_2 = g$ an arbitrary permutation of S_{N+1}, since

$$(13) \qquad \begin{aligned} &P(g_1)\mathcal{T}'_N(\{\zeta_j\}) \cap P(g_2)\mathcal{T}_N(\{\zeta_j\}) = \\ &= \mathcal{T}'_N(P(g_1)\{\zeta_j\}) \cap P(g_2 g_1^{-1})\mathcal{T}_N(P(g_1)\{\zeta_j\}), \end{aligned}$$

and we can take $P(g_1)\{\zeta_j\}$ as the set of new variables. Here we have used the definition (3) of the permuted forward and extended tubes, and the relation

$$(14) \qquad P(g_2 g_1)\mathcal{T}_N = P(g_2)P(g_1)\mathcal{T}_N$$

or

$$(14') \qquad P(g_2 g_1)\mathcal{T}'_N = P(g_2)P(g_1)\mathcal{T}'_N$$

which follows from the fact that $P(g)\{\zeta_j\}$ affords a representation[11] of S_{N+1}.

L e m m a 3. (Jost[9])

$$\mathcal{T}'_N = P(g_I)\mathcal{T}'_N, \text{ where } g_I = g_I^{-1} = \binom{0, 1, \ldots,}{N, N-1, \ldots, 0}\downarrow.$$

P r o o f.

g_I induces the transformation $\{\zeta_j\} \longrightarrow P(g_I)\{\zeta_j\} = \{-\zeta_j\}$.

Take $\Lambda(-1) = \begin{pmatrix} -1 & & & \\ 0 & -1 & & \\ & -1 & 0 & \\ & & & -1 \end{pmatrix} \in L_+(C)$.

Using $L_+(C)\Lambda(-1) = L_+(C)$, and eq. (3), we have

$$P(g_I)\mathcal{T}'_N = \left\{ \{\zeta_j\};\ \zeta_j = L_+(C)\zeta'_j,\ \mathrm{Im}(-\zeta'_j) \in V_+ \right\}$$

$$= \left\{ \{\zeta_j\};\ \zeta_j = L_+(C)(\Lambda(-1)\zeta'_j),\ \mathrm{Im}(\Lambda(-1)\zeta'_j) \in V_+ \right\}$$

$$= \mathcal{T}'_N . \qquad \text{(q. e. d.)}$$

According to this Lemma we need to discuss the connectedness of $\mathcal{T}'_N \cap P(g)\mathcal{T}_N$ only for the cases $g \neq 1, g_I$.

A point $\{\zeta_j\} \in \mathcal{T}'_N \cap P(g)\mathcal{T}_N$ has the following properties:

a) $\exists\, \Lambda \in L_+(C)$ such that $\mathrm{Im}(\Lambda\zeta_j) \in V_+$,

b) $\mathrm{Im}\, \tilde{\zeta}_j \in V_+$ where $\tilde{\zeta}_j = P(g)\zeta_j = \sum_{k=1}^{N} p_{jk}(g)\zeta_k$

(see eq. (2)).

Using $g = \binom{0, 1, \ldots\ldots, N}{i_0, i_1, \ldots\ldots, i_N}\downarrow = \binom{l_0, l_1, \ldots\ldots\ldots, l_N}{0, 1, \ldots\ldots\ldots, N}\downarrow$, the inverse of eq. (2) (with $\zeta_j \longrightarrow \zeta_j$) is given by

$$\zeta_j = P(g^{-1})\tilde{\zeta}_j = \sum_{k=1}^{N} p_{jk}(g^{-1})\tilde{\zeta}_k =$$

$$(2') \qquad = \begin{cases} \tilde{\zeta}_{l_j} + \tilde{\zeta}_{l_j-1} + \ldots\ldots\ldots + \tilde{\zeta}_{l_{j-1}+1}, & l_j > l_{j-1}, \\ -\left(\tilde{\zeta}_{l_{j-1}} + \tilde{\zeta}_{l_{j-1}-1} + \ldots\ldots + \tilde{\zeta}_{l_j+1}\right), & l_j < l_{j-1} . \end{cases}$$

According to the property b) and the relation (2'), we can classify j, the suffix of the component 4-vector ζ_j of the point $\{\zeta_j\} \in P(g)\mathcal{T}_N$ into two classes $Z_{\pm}(g)$ as follows:

(15) $$j \in Z_+(g) \quad \text{or} \quad Z_-(g) \quad \text{if} \quad p_{jk}(g^{-1}) \geq 0 \quad \text{or} \quad \leq 0 \ , \qquad (k = 1, 2, \ldots, N) \ .$$

Thus a necessary condition of the property b) is that

(16) $$\operatorname{Im} \zeta_j \in V_{\pm} \qquad \text{for} \qquad j \in Z_{\pm}(g) \ ,$$

according to eq. (2') and the fact that the $V_{\pm}$ are convex sets. Incidentally both sets $Z_{\pm}(g)$ are non-empty unless $g = 1$, g_I.

A complex Lorentz transformation $\Lambda \in L_+(C)$ can be expressed in the normal form

(17) $$\Lambda = L_1 M L_2$$

where L_1, $L_2 \in L_+^{\uparrow}$ ($L_+^{\uparrow}$ being the proper homogeneous real Lorentz group) and $M \in L_+(C)$ has one of two possible forms[4]

(18a) $$M_1(\varphi, \chi) = \begin{vmatrix} \cos\varphi & i\sin\varphi & 0 & 0 \\ i\sin\varphi & \cos\varphi & 0 & 0 \\ 0 & 0 & \cosh\chi & i\sinh\chi \\ 0 & 0 & -i\sinh\chi & \cosh\chi \end{vmatrix}, \quad \varphi, \chi \ \text{real}$$

or

(18b) $$M_2^{\pm}(\tau) = \pm \begin{vmatrix} 1 & 0 & \tau & i\tau \\ 0 & 1 & \tau & i\tau \\ \tau & -\tau & 1 & 0 \\ i\tau & -i\tau & 0 & 1 \end{vmatrix}, \quad \tau \ \text{real} \ .$$

Since $L_+^{\uparrow}$ is conncted and leaves $P(g)\mathcal{T}_N$ and $P(g)\mathcal{T}'_N$ invariant, we can ignore L_1 and L_2. (There exists a continuous curve which connects $\{\zeta_j\}$ and $\{L_2 \zeta_j\}$ inside $\mathcal{T}'_N \cap P(g)\mathcal{T}_N$. We write $\{L_2 \zeta_j\}$ as $\{\zeta_j\}$ for simplicity. As for L_1, if $\{\Lambda \zeta_j\} \in \mathcal{T}_N$, then $\{L_1^{-1} \Lambda \zeta_j\} \in \mathcal{T}_N$.

Lemma 4.

The second normal form $M_2^{\pm}(\tau)$ cannot transform any 4--vector ζ_j with $\mathrm{Im}\,\zeta_j \in V_{\mp}$ into $\mathcal{T}_1$, i.e., into a 4-vector ζ_j' with $\mathrm{Im}\,\zeta_j' \in V_+$.

Proof.

Take the case of $M_2^+(\tau)$ and $\mathrm{Im}\,\zeta_j = \eta_j \in V_-$. We can readily get[4]

$$\zeta_j(\tau) = M_2^+(\tau)\zeta_j = \xi_j(\tau) + i\,\eta_j(\tau)\,, \tag{19}$$

$$\eta_j^0(\tau) = \eta_j^0 + \tau(\eta_j^2 + \xi_j^3)\,, \tag{20a}$$

$$\begin{aligned} -(\eta_j(\tau))^2 = (\eta_j^0(\tau))^2 - (\underset{\sim}{\eta}_j(\tau))^2 &= -(\eta_j)^2 + 2\tau\Big[\xi_j^3(\eta_j^0 - \eta_j^1) - \\ &- \eta_j^3(\xi_j^0 - \xi_j^1)\Big] - \tau^2\Big[(\eta_j^0 - \eta_j^1)^2 + (\xi_j^0 - \xi_j^1)^2\Big]. \end{aligned} \tag{20b}$$

Since $\eta_j^0(0) = \eta_j^0 < 0$, the condition $\eta_j^0(\tau) > 0$ gives the range $\tau > \tau_0 > 0$ or $\tau < \tau_0 < 0$ according to whether $\tau_0 > 0$ or $\tau_0 < 0$, where $\eta_j^0(\tau_0) = 0$. On the other hand $-(\eta_j(\tau))^2 \le 0$ for $\tau > \tau_0 > 0$ or $\tau < \tau_0 < 0$ respectively, since $-(\eta_j(\tau))^2$ is at most a quadratic function of τ in which the coefficient of τ^2 is ≤ 0, and $-(\eta_j(0))^2 > 0$ and $-(\eta_j(\tau_0))^2 \le 0$. Thus

$$\eta_j(\tau) \notin V_+\,, \quad \text{for } \forall\tau \text{ real and } \forall\,\eta_j \in V_-\,.$$

The case of $M_2^-(\tau)$ and $\eta_j \in V_+$ can be proved quite similarly.

Corollary.

The second normal form $M_2^{\pm}(\tau)$ cannot transform a point $\{\zeta_j\} \in P(g)\mathcal{T}_N$ into $\mathcal{T}_N$ for $g \ne 1, g_I$.

Proof.

For $g \ne 1, g_I$ the classes $Z_{\pm}(g)$ are non-empty. According to Lemma 4 and the definition (16), ζ_j where $j \in Z_{\pm}(g)$ cannot be transformed into $\mathcal{T}_1$ by $M_2^{\pm}(\tau)$. This establishes the statement (q. e. d.).

Thus a point $\{\zeta_j\} \in \mathcal{F}'_N \cap P(g)\mathcal{T}_N$, $(g \neq 1, g_I)$, must be transformed into $\mathcal{F}_N$ by the first normal form[12] $M_1(\varphi, \chi)$.

Lemma 5.

Assume that $\{M_1(\varphi, \chi)\,\zeta_j\} \in \mathcal{T}_N$ for a point $\{\zeta_j\} \in \mathcal{F}'_N \cap P(g)\mathcal{T}_N$ and for a particular (φ, χ). Define

$$\zeta_j(\rho) = \begin{vmatrix} \rho\,\xi_j^0 \\ \xi_j^1 \\ \rho\,\xi_j^2 \\ \rho\,\xi_j^3 \end{vmatrix} + i \begin{vmatrix} \eta_j^0 \\ \rho\,\eta_j^1 \\ \rho\,\eta_j^2 \\ \rho\,\eta_j^3 \end{vmatrix} \tag{21}$$

Then

$$\{\zeta_j(\rho)\} \in \mathcal{F}'_N \cap P(g)\mathcal{T}_N \quad \text{and}$$
$$\{M_1(\varphi, \chi)\,\zeta_j(\rho)\} \in \mathcal{F}_N \quad \text{for } -1 \leq \rho \leq 1 \;. \tag{22}$$

Proof.

Since

$$\widetilde{\zeta_j(\rho)} = P(g)\,\zeta_j(\rho) = \tilde{\zeta}_j(\rho) = \tilde{\xi}_j(\rho) + i\,\tilde{\eta}_j(\rho) \;, \tag{23}$$

and since

$$\tilde{\eta}_j \in V_+ \implies \tilde{\eta}_j(\rho) \in V_+ \qquad \text{for } -1 \leq \rho \leq 1 \;,$$

we get

$$\{\zeta_j\} \in P(g)\mathcal{F}_N \implies \{\zeta_j(\rho)\} \in P(g)\mathcal{T}_N \qquad \text{for } -1 \leq \rho \leq 1 \,. \tag{24}$$

The following formulae and definitions are self- explanatory:

$$\eta_j(\varphi,\chi) = \operatorname{Im}\zeta_j(\varphi,\chi) = \operatorname{Im}(M_1(\varphi,\chi)\,\zeta_j) = \begin{vmatrix} \eta_j^0\cos\varphi & +\,\xi_j^1\sin\varphi \\ \eta_j^1\cos\varphi & +\,\xi_j^0\sin\varphi \\ \eta_j^2\cosh\chi & +\,\xi_j^3\sinh\chi \\ \eta_j^3\cosh\chi & -\,\xi_j^2\sinh\chi \end{vmatrix} \tag{25}$$

(26) $\zeta_j(\varphi,\chi;\rho) = M_1(\varphi,\chi)\zeta_j(\rho) = \xi_j(\varphi,\chi;\rho) + i\,\eta_j(\varphi,\chi;\rho)$,

(27) $\zeta_j(\varphi,\chi;1) = \zeta_j(\varphi,\chi)$,

(28) $\eta^0_j(\varphi,\chi;\rho) = \eta^0_j(\varphi,\chi) = \eta^0_j \cos\varphi + \xi^1_j \sin\varphi$,

and

(29)
$$-(\eta_j(\varphi,\chi;\rho))^2 = -(\eta_j(\varphi,\chi))^2 + (1-\rho^2)(\underset{\sim}{\eta}_j(\varphi,\chi))^2$$
$$\geq -(\eta_j(\varphi,\chi))^2, \quad \text{for} \quad -1 \leq \rho \leq 1 \; .$$

Then, using eqs. (28) and (29) it follows from the condition

$$\eta_j(\varphi,\chi) \in V_+ \quad \text{that} \quad \eta_j(\varphi,\chi;\rho) \in V_+ \quad \text{for} \quad -1 \leq \rho \leq 1 \; .$$

That is to say

(30)
$$\{M_1(\varphi,\chi)\zeta_j\} \in \mathcal{T}_N \Longrightarrow \{M_1(\varphi,\chi)\zeta_j(\rho)\} \in \mathcal{T}_N$$
$$\text{for} \quad -1 \leq \rho \leq 1 \; .$$

The eqs. (24) and (30) establish the Lemma.

Lemma 6.

The set C of the points $\{\zeta_j\}$, which are of the form

(31)
$$\zeta_j = \left|\begin{matrix} 0 \\ \xi^1_j \\ 0 \\ 0 \end{matrix}\right| + i \left|\begin{matrix} \eta^0_j \\ 0 \\ 0 \\ 0 \end{matrix}\right|$$

and have the properties

(32a) $\mathrm{Im}(P(g)\xi_j) = \tilde{\eta}^0_j > 0$

and

(32b)
$$\xi^1_j > 0 \, , \qquad \text{for} \quad \forall j$$

is a connected subset of $\mathcal{T}'_N \cap P(g)\,\mathcal{T}_N$.

P r o o f.

Clearly C is connected (actually it is convex), and it follows from (32a) that $C \subset P(g)\mathcal{T}_N$. To prove that $C \subset \mathcal{T}'_N$ we operate with $M_1(\varphi,\chi)$ on $\{\zeta_j\}$ getting

(33) $$M_1(\varphi,\chi)\zeta_j = \zeta_j(\varphi,\chi) = \begin{vmatrix} 0 \\ \xi_j^1\cos\varphi - \eta_j^0\sin\varphi \\ 0 \\ 0 \end{vmatrix} + i \begin{vmatrix} \eta_j^0\cos\varphi + \xi_j^1\sin\varphi \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

which does not depend on χ. According to (15), (16) and (32a) we have $\eta_j^0 \gtrless 0$ for $j \in Z_\pm(g)$. Defining φ_j such that

(34) $$0 < \varphi_j = \tan^{-1}(-\eta_j^0/\xi_j^1) < \pi,$$

we have that

(35a) $$\pi/2 < \varphi_j < \pi \qquad \text{for } j \in Z_+(g),$$

and

(35b) $$0 < \varphi_j < \pi/2 \qquad \text{for } j \in Z_-(g)$$

according to (32b). Since

$$\eta_j^0(\varphi,\chi) = \eta_j^0\cos\varphi + \xi_j^1\sin\varphi > 0$$

when

(36a) $$\varphi_j - \pi < \varphi < \varphi_j \qquad \text{for } j \in Z_+(g)$$

and

(36b) $$0 < \varphi_j < \varphi \leq \pi \text{ and } -\pi \leq \varphi < \varphi_j - \pi < 0 \qquad \text{for } j \in Z_-(g)$$

we conclude that

(37) $$\mathrm{Im}(M_1(\varphi,\chi)\zeta_j) = \eta_j(\varphi,\chi) \in V_+$$

for

(38) $$\max_{j \in Z_-(g)} \varphi_j < \varphi < \min_{j \in Z_+(g)} \varphi_j \,, \quad \chi \text{ arbitrary}.$$

We note that the domain of (φ, χ) given by eq. (38) is non-empty according to eq. (35). This establishes that $C \subset \mathcal{T}'_N$ and so the Lemma is proved.

Lemma 7.

A point $\{\zeta_j\} \in \mathcal{T}'_N \cap P(g)\mathcal{T}_N$ is connected, inside $\mathcal{T}'_N \cap P(g)\mathcal{T}_N$, to the set C, C being defined in Lemma 6.

Proof.

It is sufficient to consider the point $\{\zeta_j\} \in \mathcal{T}'_N \cap P(g)\mathcal{T}_N$ which satisfies $\{M_1(\varphi, \chi)\zeta_j\} \in \mathcal{T}_N$ for a (φ, χ) according to the explanation given just before Lemma 4, and the Corollary of Lemma 4 (The Lemma is trivial for $g = 1$ and g_I, since $C \subset P(g)\mathcal{T}_N \subset \mathcal{T}'_N$ for $g = 1$ and g_I. Assuming $g \neq 1, g_I$, the above statement is correct).

Thus we can apply Lemma 5: For $\{\zeta_j(\rho)\}$, which is defined by eq. (21), eq. (22) is valid. Changing ρ of $\zeta_j(\rho)$ from 1 to O, we get a continuous curve which connects $\{\zeta_j\} = \{\zeta_j(1)\}$ and $\{\zeta'_j\} = \{\zeta_j(0)\}$ inside $\mathcal{T}'_N \cap P(g)\mathcal{T}_N$, where $\zeta'_j = \xi'_j + i\,\eta'_j$ is of the form (31) and satisfies (32a), and

$$\{M_1(\varphi, \chi)\zeta'_j\} \in \mathcal{T}_N ,$$

i. e. ,

(39) $$\eta^0_j \cos\varphi + \xi^1_j \sin\varphi > 0 \quad \text{for } \forall j \text{ and for a } \varphi .$$

The allowed domain of φ of eq. (39) can be either

(40a) $$0 < \max_{j \in Z_-(g)} \varphi_j < \varphi < \min_{j \in Z_+(g)} \varphi_j < \pi \quad ,$$

or

(40b) $$-\pi < \max_{j \in Z_+(g)} (\varphi_j - \pi) < \varphi < \min_{j \in Z_-(g)} (\varphi_j - \pi) < 0 \quad ,$$

depending on whether

$$\text{(41a)} \qquad \max_{j \in Z_-(g)} \varphi_j < \min_{j \in Z_+(g)} \varphi_j \quad ,$$

or

$$\text{(41b)} \qquad \max_{j \in Z_+(g)} \varphi_j < \min_{j \in Z_-(g)} \varphi_j \quad ,$$

where φ_j is defined by (34). However, the case (41b) can be reduced to the case (41a): Operating with the space rotation $R_3(\pi)$ of angle π, around the third axis, on $\{\zeta_j'\}$, the sign of all ξ_j^1 is inverted (the point being denoted by $\{\bar{\zeta}_j\}$ and φ_j is changed into $\bar{\varphi}_j = \pi - \varphi_j$ which satisfies eq. (41a). Since the space rotation R leaves $\mathcal{T}_N' \cap P(g)\mathcal{T}_N$ invariant, and R is connected, $\{\zeta_j'\}$ and $\{\bar{\zeta}_j\}$ are connected inside $\mathcal{T}_N' \cap P(g)\mathcal{T}_N$. Thus we need to consider only the case (41a).

Now $\varphi_j = \tan^{-1}(-\eta_j^0/\xi_j^1)$ is a increasing or decreasing function of ξ_j^1 for $j \in Z_+(g)$ or $j \in Z_-(g)$ respectively. Thus for any $\xi_j^{1'} > \xi_j^1$, we have

$$\text{(42)} \qquad \begin{aligned} 0 &< \max_{j \in Z_-(g)} \tan^{-1}(-\eta_j^0/\xi_j^{1'}) < \max_{j \in Z_-(g)} \varphi_j < \min_{j \in Z_+(g)} \varphi_j < \\ &< \min_{j \in Z_+(g)} \tan^{-1}(-\eta_j^0/\xi_j^{1'}) < \pi \quad . \end{aligned}$$

This means that the increasing ξ_j^1 in $\{\zeta_j'\}$ of the form (31), in the case (41a), does not change the property that $\{\zeta_j'\} \in \mathcal{T}_N'$. The condition $\{\zeta_j'\} \in P(g)\mathcal{T}_N$ is invariant under the change of Re ζ_j' since the above condition is relevant only to Im ζ_j'. Therefore the continuous curve which is given by increasing ξ_j^1 can connect $\{\zeta_j'\}$ with the point $\{\zeta_j''\} \in C$, where Re $\zeta_j^{1''} > \max(\text{Re}\, \zeta_j^{1'} = \zeta_j^1,\ 0)$ and Im $\zeta_j^{0''} =$ Im $\zeta_j^{0'} = \eta_j^0$, and is inside $\mathcal{T}_N' \cap P(g)\mathcal{T}_N$. This establishes the Lemma.

Corollary.

$P(g_1)\mathcal{T}_N' \cap P(g_2)\mathcal{T}_N$ is connected.

Proof.

Lemma 6 and 7 show that any point $\{\zeta_j\} \in \mathcal{T}'_N \cap P(g)\mathcal{T}_N$ is connected to the connected set $C \subset \mathcal{T}'_N \cap P(g)\mathcal{T}_N$, inside $\mathcal{T}'_N \cap P(g)\mathcal{T}_N$ which proves the connectedness of $\mathcal{T}'_N \cap P(g)\mathcal{T}_N$. Eq. (13) generalizes it accordingly.

Theorem 2.

$P(g_1)\mathcal{T}'_N \cap P(g_2)\mathcal{T}'_N$ is connected.

Proof.

It follows from Lemma 2 and the last Corollary.

Theorem 3.

The analytic continuation of W-function due to local commutativity is single-valued in the domain $\bigcup_{g \in S \subseteq S_{N+1}} P(g)\mathcal{T}'_N$ where S is an arbitrary subset of the symmetric group S_{N+1}.

Proof.

From Theorem 1, 2 and Lemma 1 it follows that

$$P(g_1)\mathcal{T}'_N \cap P(g_2)\mathcal{T}'_N \qquad \text{for} \qquad \forall g_1, g_2 \in S_{N+1}$$

is a non-empty connected domain, and contains $P(g_1)J_N \cap P(g_2)J_N$ which is non-empty. The latter forms a real environment(13), since the set of Jost points is an open set in the real 4N-dimensional Minkowski space. Thus local commutativity equates (up to a sign) the W-functions at $P(g_1)J_N \cap P(g_2)J_N$ and gives the single-valued analytic continuation in $P(g_1)\mathcal{T}'_N \cap P(g_2)\mathcal{T}'_N$ since $P(g_1)\mathcal{T}'_N \cap P(g_2)\mathcal{T}'_N$ is connected. Applying this process to all the pairs of $\{P(g)\mathcal{T}'_N\}_{g \in S \leq S_{N+1}}$, the statement of the Theorem follows.

3. Simply-connectedness of the union of the extended tubes.

First we prove the simply-connectedness of an extended tube itself. To this end it is convenient to use the covering group (the universal covering group) $\overline{L}_+(C)$ of $L_+(C)$ ($\overline{L}_+(C)$ is isomorphic to

$SL(2,C) \otimes SL(2,C)$ and is simply-connected, where $SL(2,C)$ is the complex unimodular group of 2 dimension. Clearly $\bar{L}_+(C)\mathcal{T}_N = \mathcal{T}'_N = L_+(C)\mathcal{T}_N$, since $\bar{L}_+(C)\mathcal{T}_N$ covers $\mathcal{T}'_N$ twice as does $L_+(C)\mathcal{T}_N$).

Lemma 8.

If a point $\{\zeta'_j\} \in \mathcal{T}'_N$ can be expressed by

(43a) $$\zeta'_j = \bar{\Lambda}_0 (\zeta_j)_0 = \bar{\Lambda}_1 (\zeta_j)_1 \quad ,$$

where

(43b) $$\bar{\Lambda}_\alpha \in \bar{L}_+(C) \quad \text{and} \quad \{(\zeta_j)_\alpha\} \in \mathcal{T}_N \qquad (\alpha = 0, 1),$$

then there exist continuous curves

(44a) $$\bar{\Lambda}(t) \subset \bar{L}_+(C) \quad \text{and} \quad \{\zeta_j(t)\} \subset \mathcal{T}_N \, , \qquad 0 \le t \le 1 \quad ,$$

with

(44b) $$\bar{\Lambda}(\alpha) = \bar{\Lambda}_\alpha \quad \text{and} \quad \zeta_j(\alpha) = (\zeta_j)_\alpha , \qquad (\alpha = 0, 1),$$

such that

(44c) $$\zeta'_j = \bar{\Lambda}(t)\, \zeta_j(t) \quad \text{for} \quad 0 \le t \le 1 \quad .$$

Proof.

From the conditions $(\zeta_j)_1 = (\bar{\Lambda}_1)^{-1} \bar{\Lambda}_0 (\zeta_j)_0$, $\{(\zeta_j)_0\}$, $\{(\zeta_j)_1\} \in \mathcal{T}_N$ and $(\bar{\Lambda}_1)^{-1}\bar{\Lambda}_0 \in \bar{L}_+(C)$, it follows that there exist curves

(45a) $$\bar{\Lambda}'(t) \subset \bar{L}_+(C) \quad \text{and} \quad \{\zeta_j(t)\} \subset \mathcal{T}_N \, , \qquad 0 \le t \le 1 \quad ,$$

with

(45b) $$\bar{\Lambda}'(0) = 1 \, , \qquad \bar{\Lambda}'(1) = (\bar{\Lambda}_1)^{-1} \bar{\Lambda}_0 \, ,$$
$$\zeta_j(0) = (\zeta_j)_0 \, , \qquad \zeta_j(1) = (\zeta_j)_1 \quad ,$$

such that

(45c) $$\zeta_j(t) = \bar{\Lambda}'(t) (\zeta_j)_0 \, , \qquad 0 \le t \le 1 \quad ,$$

according to a Lemma of Hall and Wightman included in Lemma 1 of ref. (4).

Then we have

(46) $$\zeta'_j = \bar{\Lambda}_0(\zeta_j)_0 = \bar{\Lambda}(t)\,\zeta_j(t)\,, \qquad 0 \leq t \leq 1\,,$$

where we put

(47) $$\bar{\Lambda}(t) = \bar{\Lambda}_0(\bar{\Lambda}'(t))^{-1}\,.$$

It is easy to see that $\bar{\Lambda}(t) \subset \bar{L}_+(C)$ and $\{\zeta_j(t)\} \subset \mathcal{T}_N$, $(0 \leq t \leq 1)$, as defined by (47) and (45c) respectively, are the required curves.

Lemma 9.

For a continuous closed curve

(48a) $$\{\zeta'_j(t)\} \subset \mathcal{T}'_N\,, \qquad 0 \leq t \leq 1\,,$$

with

(48b) $$\zeta'_j(0) = \zeta'_j(1)\,,$$

we can find continuous closed curves

(49a) $$\bar{\Lambda}(s) \subset \bar{L}_+(C) \quad \text{and} \quad \{\zeta_j(s)\} \subset \mathcal{T}_N\,, \qquad 0 \leq s \leq 1\,,$$

with

(49b) $$\bar{\Lambda}(0) = \bar{\Lambda}(1) \quad \text{and} \quad \zeta_j(0) = \zeta_j(1)\,,$$

such that

(50) $$\zeta'_j(t(s)) = \zeta''_j(s) = \bar{\Lambda}(s)\,\zeta_j(s)\,, \qquad 0 \leq s \leq 1\,,$$

with a suitable parametrization by s, where t(s) is a non-decreasing continuous function of s with $t(0) = 0$ and $t(1) = 1$.

Proof.

From the facts that $\mathcal{T}'_N = \bigcup_{\bar{\Lambda} \in L_+(C)} \bar{\Lambda}\,\mathcal{T}_N$, that $\bar{\Lambda}\mathcal{T}_N$ is an open set, and that the curve (48) is an image of the compact set $[0, 1]$ due to a continuous mapping T, it follows[14] that there exists a $\delta > 0$ such that the image of δ-neighbourhood of each point $t \in [0, 1]$ due to

T is found in a suitably chosen $\bar{\Lambda}\mathcal{T}_N$. Therefore we can find a finite number of open sets $\bar{\Lambda}_\alpha \mathcal{T}_N$ ($\alpha = 1, 2, \ldots, n$), which covers the curve (48) entirely in such a way that for a suitable division $0 = t_0 < t_1 < \ldots \ldots < t_n = 1$ of $[0, 1]$ and images $Q'_\alpha = \{\zeta'_j(t_\alpha)\}$ of t_α, ($\alpha = 0, 1, \ldots \ldots, n$), the arc $\widehat{Q'_{\alpha-1} Q'_\alpha}$ of the curve is contained in $\bar{\Lambda}_\alpha \mathcal{T}_N$, ($\alpha = 1, 2, \ldots, n$). Since $\bar{\Lambda}_\alpha \in \bar{L}_+(C)$ is a homoeomorphic mapping of $\mathcal{T}_N$ onto $\bar{\Lambda}_\alpha \mathcal{T}_N$, the mapping $(\Lambda_\alpha)^{-1}$ of $\widehat{Q'_{\alpha-1} Q'_\alpha} \subset \mathcal{T}'_N$ gives a continuous curve.

$$(51a) \qquad Q^{(\alpha)}_{\alpha-1}\, Q^{(\alpha)}_{\alpha} = \left\{\{\zeta^{(\alpha)}_j(t)\}\, ;\; t_{\alpha-1} \le t \le t_\alpha\right\} \subset \mathcal{T}_N$$

with

$$(51b) \qquad Q'_\alpha = \bar{\Lambda}_\alpha Q^{(\alpha)}_\alpha = \bar{\Lambda}_{\alpha+1} Q^{(\alpha+1)}_\alpha , \qquad (\alpha = 1, 2, \ldots, n),$$

where we put $\bar{\Lambda}_{n+1} = \bar{\Lambda}_1$ and $Q^{(n+1)}_n = Q^{(1)}_0$. The relation (51) enables us to apply Lemma 8 to find curves

$$(52a) \qquad \bar{\Lambda}_{(\alpha,\alpha+1)}(\tau_\alpha) \subset \bar{L}_+(C) \quad \text{and} \quad Q^{(\alpha,\alpha+1)}(\tau_\alpha) \subset \mathcal{T}_N , \qquad 0 \le \tau_\alpha \le 1 ,$$

with

$$(52b) \qquad \begin{gathered} \Lambda_{(\alpha,\alpha+1)}(0) = \bar{\Lambda}_\alpha, \quad \Lambda_{(\alpha,\alpha+1)}(1) = \bar{\Lambda}_{\alpha+1} , \\ Q^{(\alpha,\alpha+1)}(0) = Q^{(\alpha)}_\alpha , \quad Q^{(\alpha,\alpha+1)}(1) = Q^{(\alpha+1)}_\alpha , \end{gathered}$$

such that

$$(52c) \qquad Q'_\alpha = \Lambda_{(\alpha,\alpha+1)}(\tau_\alpha)\, Q^{(\alpha,\alpha+1)}(\tau_\alpha) , \quad 0 \le \tau_\alpha \le 1 ,$$

for $\alpha = 1, 2, \ldots, n$. Now we introduce the parameter s, $0 \le s \le 1$, and with the partition

$$(53a) \qquad 0 = s^0 < s_1 < s^1 < s_2 < s^2 < \ldots\ldots < s^{n-1} < s_n < s^n = 1 ,$$

we put

$$(53b) \qquad \begin{gathered} t_{\alpha-1} \le t(s) \le t_\alpha , \qquad \text{for } s^{\alpha-1} \le s \le s_\alpha , \\ t(s) = t_\alpha \quad \text{and } 0 \le \tau_\alpha(s) \le 1 \quad \text{for } s_\alpha \le s \le s^\alpha \\ \alpha = 1, 2, \ldots, n \end{gathered}$$

where the parameters $t(s)$ and $\tau_\alpha(s)$ are non-decreasing continuous functions of s. The continuous closed curves $\{\zeta_j'(t(s))\} = \{\zeta_j''(s)\} \subset \mathcal{T}_N'$ connecting $Q_0', Q_1', \ldots, Q_n' = Q_0'$, $\bar{\Lambda}(s) \subset \bar{L}_+(C)$ connecting $\bar{\Lambda}_1, \bar{\Lambda}_2, \ldots, \bar{\Lambda}_n, \bar{\Lambda}_{n+1} = \bar{\Lambda}_1$ (putting $\bar{\Lambda}(s) = \bar{\Lambda}_\alpha$ for $s^{\alpha-1} \leq s \leq s_\alpha$) and $\{\zeta_j(s)\} \subset \mathcal{T}_N$ connecting $Q_0^{(1)}, Q_1^{(1)}, Q_1^{(2)}, \ldots, Q_n^{(n)}, Q_n^{(n+1)} = Q_0^{(1)}$, which were described above, give the required relation (50).

Corollary.

For a continuous curve (48a), we can find continuous curves (49a) such that (50) is satisfied with a suitable parametrization by s.

Proof.

This follows from Lemma 9 and from the fact that any continuous curve can be considered as a part of a continuous closed curve.

Theorem 4.

The extended tube $\mathcal{T}_N'$ is simply-connected.

Proof.

$\mathcal{T}_N'$ is connected since $\bar{L}_+(C)$ and $\mathcal{T}_N$ are connected and $\mathcal{T}_N'$ is a continuous image of $\bar{L}_+(C) \otimes \mathcal{T}_N$. According to Lemma 9, any continuous closed curve (48) belonging to $\mathcal{T}_N'$ can be expressed by eq. (50) in terms of the continuous closed curves (49) belonging to $\bar{L}_+(C)$ and $\mathcal{T}_N$. Since $\bar{L}_+(C)$ and $\mathcal{T}_N$ are simply-connected (note that $\mathcal{T}_N$ is convex), we can let the curves (49) shrink to points inside each domain. Therefore the curve (48) shrinks to a point inside $\mathcal{T}_N'$. Thus the Theorem is established. (q. e. d.).

For proving the simply-connectedness of $\cup P(g)\mathcal{T}_N'$, we prove the following Lemmas.

Lemma 10.

If the simply-connected domains $D_1, D_2, \ldots, D_n$ have a non-empty common intersection $\bigcap_{j=1}^{n} D_j$ and the intersection $D_k \cap D_l$, $(k, l, =1, \ldots, n)$, of any two is connected, then the union $\bigcup_{j=1}^{n} D_j$ is simply-connected.

Proof.

$\bigcup_{j=1}^{n} D_j$ is connected, since any point of it is connected to $\bigcap_{j=1}^{n} D_j$. Take a curve belonging to $\bigcup_{j=1}^{n} D_j$ which runs through domains $D_{i_1}, D_{i_2}, \ldots, D_{i_m}$ where $(i_1, i_2, \ldots i_m)$ is a set of integers taken from the set of integers $(1, 2, \ldots, n)$ with repetition allowed but $i_k \neq i_{k+1}$, $(k = 1, 2, \ldots, m-1)$. Since the curve necessarily must pass through $D_{i_k} \cap D_{i_{k+1}}$ befors leaving D_{i_k}, we can choose a set of points, $Q_{1,2}, \ldots Q_{m-1,m}$ on the given curve $\widehat{Q_1 Q_m}$ such that $Q_1 \in D_{i_1}$, $Q_{k,k+1} \in D_{i_k} \cap D_{i_{k+1}}$ $(k = 1, \ldots, m-1)$, $Q_m \in D_{i_m}$, and $\widehat{Q_1 Q_{1,2}} \subset D_{i_1}$, $\widehat{Q_{k-1,k} Q_{k,k+1}} \subset D_{i_k}$, $(k = 1, \ldots m-1)$, $\widehat{Q_{m-1,m} Q_m} \subset D_{i_m}$, and where $\widehat{Q_{k-1,k} Q_{k,k+1}}$ is the portion of the curve between $Q_{k-1,k}$ and $Q_{k,k+1}$. Taking a point $O \in \bigcap_{j=1}^{n} D_j$, we can draw the continuous curves joining 0 and Q's in such a way that $\widehat{OQ_1} \subset D_{i_1}$, $\widehat{OQ_{k,k+1}} \subset D_{i_k} \cap D_{i_{k+1}}$, $(k = 1, \ldots, m-1)$, and $\widehat{OQ_m} \subset D_{i_m}$, since $D_k \cap D_l$, $(k, l = 1, \ldots, n)$, is connected. Thus all the curves $\widehat{OQ_{k-1,k}}$, $\widehat{Q_{k-1,k} Q_{k,k+1}}$ and $\widehat{OQ_{k,k+1}}$ are inside D_{i_k}. Using the terminology of equivalence[15] (denoted by $\sim$) for the case where two curves with same ends can be deformed continuously to each other, and multiplica- tion (denoted by $\cdot$) for joining two curves when the end point of the first is the starting point of the second, we have the following equivalence relations, since D_j is simply connected:

$$
(54) \qquad \left\{
\begin{array}{l}
\widehat{Q_1 Q_{1,2}} \sim \widehat{Q_1 O} \cdot \widehat{OQ_{1,2}} \;, \\
\widehat{Q_{1,2} Q_{2,3}} \sim \widehat{Q_{1,2} O} \cdot \widehat{OQ_{2,3}} \;, \\
\widehat{Q_{m-1,m} Q_m} \sim \widehat{Q_{m-1,m} O} \cdot \widehat{OQ_m} \;.
\end{array}
\right.
$$

By multiplying these successively we get

$$
\widehat{Q_1 Q_m} \sim \widehat{Q_1 O} \cdot \widehat{OQ_m} \;,
$$

since[15] $\widehat{AB} \cdot \widehat{BA} \sim 0$ and $\widehat{AB} \cdot 0 \sim \widehat{AB}$ (Here $\widehat{BA}$ represents the same curve as $\widehat{AB}$ but with opposite direction). Thus we have proved that any curves joining Q_1 and Q_m are equivalent to $\widehat{Q_1O} \cdot \widehat{OQ_m}$, and therefore are equivalent to each other. This proves the statement.

Lemma 11.

Let $D_1, D_2, \ldots, D_n$ be simply-connected domains. If any three of them has a non-empty common intersection, and the intersection of any two of them is connected, then the union $\bigcup_{j=1}^{n} D_j$ is simply-connected.

Proof.

$\bigcup_{j=1}^{n} D_j$ is connected since any two points $Q_\alpha \in D_{i_\alpha}$ ($\alpha = 1, 2$), are connected to each other through $D_{i_1} \cap D_{i_2}$. Take a continuous curve contained in $\bigcup_{j=1}^{n} D_j$ which runs through the domains $D_{i_1} \to D_{i_2} \to \ldots$ $\ldots \to D_{i_m}$, and take the points $Q_1, Q_{1,2}, \ldots, Q_{m-1,m}$, Q_m on the curve, defined in the same way as in the proof of Lemma 10. For the case $n \leq 3$ and the case $n > 3$ and $m \leq 3$, the statement is true according to Lemma 10. Assume $n > 3$ and $m > 3$. Since D_{i_1}, D_{i_2}, and D_{i_3} satisfy the conditions of Lemma 10, the curve $\widehat{Q_1 Q_{1,2}} \cdot \widehat{Q_{1,2} Q_{2,3}}$ is equivalent to $\widehat{Q_1 Q_{1,2,3}} \cdot \widehat{Q_{1,2,3} Q_{2,3}}$ where $Q_{1,2,3} \in D_{i_1} \cap D_{i_2} \cap D_{i_3}$ and $\widehat{Q_1 Q_{1,2,3}} \subset D_{i_1}$, $\widehat{Q_{1,2,3} Q_{2,3}} \subset D_{i_3}$. Then the curve $\widehat{Q_1 Q_m}$ is equivalent to the curve $\widehat{Q_1 Q_{1,2,3}} \cdot \widehat{Q_{1,2,3} Q_{2,3}} \cdot \widehat{Q_{2,3} Q_m}$ which runs through $D_{i_1} \to D_{i_3} \to \ldots\ldots \to D_{i_m}$, where the number of domains is reduced by one from that of $\widehat{Q_1 Q_m}$. Thus by induction we arrive at the statement of the Lemma.

Theorem 5.

The union $\bigcup_{g \in S \subseteq S_{N+1}} P(g) \mathcal{T}'_N$ of any number of extended tubes is simply-connected.

Proof.

According to Theorem 1, 2 and 4, any subset of $\{P(g)\mathcal{T}'_N\}_{g \in S_{N+1}}$ satisfies the conditions of Lemma 11. Thus the Theorem is established.

4. Simply-connectedness of the intersection of two extended tubes.

Lemma 12.

$P(g_1)\mathcal{T}'_N \cap P(g_2)\mathcal{T}_N$ is simply-connected.

Proof.

It is sufficient to prove the case $g_1 = 1$ and $g_2 \doteq g$ arbitrary but $g \neq 1, g_I$, as mentioned before (see eq. (13) and Lemma 3). We can prove the Lemma by continuously deforming a continuous closed curve of $\mathcal{T}'_N \cap P(g)\mathcal{T}_N$ into a continuous closed curve of the convex set $C \subset \mathcal{T}'_N \cap P(g)\mathcal{T}_N$, C being defined in Lemma 6. (C is convex because of the definition (31) and (32), and thus is simply-connected).

For a continuous closed curve $\{\zeta''_j(t)\} \subset \mathcal{T}'_N \cap P(g)\mathcal{T}_N$, $0 \leq t \leq 1$, we can find continuous closed curves $\overline{\Lambda}(s) \subset \overline{L}_+(C)$ and $\{\zeta_j(s)\} \subset \mathcal{T}_N$ such that

$$\zeta''_j(t(s)) = \zeta'_j(s) = \overline{\Lambda}(s)\zeta_j(s), \qquad 0 \leq s \leq 1, \tag{55}$$

with a suitable parametrization by s, according to Lemma 9. A corresponding expression to eqs. (17), (18) and the Corollary of Lemma 4 for the case of the covering group $\overline{L}_+(C)$ enable us to express $\overline{\Lambda}(s)$ by[4]

$$(\overline{\Lambda}(s))^{-1} = \overline{L}_1(s)M_1(\varphi(s), \chi(s))\overline{L}_2(s) \qquad 0 \leq s \leq 1, \tag{56}$$

where $\overline{L}_1(s), \overline{L}_2(s) \subset \overline{L}_+^{\uparrow}$, $\overline{L}_+^{\uparrow}$ being the covering group of $L_+^{\uparrow}$; $M_1(\varphi(s), \chi(s)) \subset \overline{L}_+(C)$ is given by eq. (18a), and $\overline{L}_1(s)$, $\overline{L}_2(s)$ and $M_1(\varphi(s), \chi(s))$, $(0 \leq s \leq 1)$, are continuous closed curves. Without loss of generality, we can ignore the $\overline{L}_1(s)$ and $\overline{L}_2(s)$ by a similar reason[16] as described before Lemma 4.

For $\zeta'_j(s) = \xi'_j(s) + i\,\eta'_j(s)$, define a set of continuous closed curves

$$\zeta'_j(s;\rho) = \begin{vmatrix} \rho\,\xi^{0\prime}_j(s) \\ \xi^{1\prime}_j(s) \\ \rho\,\xi^{2\prime}_j(s) \\ \rho\,\xi^{3\prime}_j(s) \end{vmatrix} + i \begin{vmatrix} \eta^{0\prime}_j(s) \\ \rho\,\eta^{1\prime}_j(s) \\ \rho\,\eta^{2\prime}_j(s) \\ \rho\,\eta^{3\prime}_j(s) \end{vmatrix}, \quad 0 \leq s \leq 1, \quad \rho \text{ real}. \tag{57}$$

Since $\{\zeta'_j(s)\} = \{\zeta'_j(s;1)\} \subset \mathcal{T}'_N \cap P(g)\mathcal{T}_N$ and $\{M_1(\varphi(s), \chi(s))\zeta'_j(s)\} \subset \mathcal{T}_N$, we get

(58a) $$\{\zeta'_j(s;\rho)\} \subset \mathcal{T}'_N \cap P(g)\mathcal{T}_N \quad ,$$

and

(58b) $$\{M_1(\varphi(s), \chi(s))\,\zeta'_j(s;\rho)\} \subset \mathcal{T}_N \quad ,$$

for

(58c) $$-1 \leq \rho \leq 1 \quad \text{and} \quad 0 \leq s \leq 1 \quad ,$$

according to Lemma 5. Thus a continuous chenge of ρ from 1 to 0 af fords a continuous deformation of $\{\zeta'_j(s)\}$ into the continuous closed curve $\{\zeta'_j(s;0)\} \subset \mathcal{T}'_N \cap P(g)\mathcal{T}_N$, inside $\mathcal{T}'_N \cap P(g)\mathcal{T}_N$. The latter curve is of the form of eq. (31), and satisfies the relation $\{M_1(\varphi(s), \chi(s))\,\zeta'_j(s;0)\} \subset \mathcal{T}_N$, $0 \leq s \leq 1$. Finally the procedure described in the proof of Lemma 7 affords a continuous deformation of $\{\zeta'_j(s;0)\}$ into a continuous closed curve contained in the convex set $C \subset \mathcal{T}'_N \cap P(g)\mathcal{T}_N$, inside $\mathcal{T}'_N \cap P(g)\mathcal{T}_N$. (A space rotation and increasing of the same amount of the real part ζ^1_j of the each component 4-vector of $\{\zeta'_j(s;0)\}$ as described in Lemma 7 keep the continuous closed curve inside $\mathcal{T}'_N \cap P(g)\mathcal{T}_N$). Since the set C is simply-connected, we can deform the derived continuous closed curve into a point in $C \subset \mathcal{T}'_N \cap P(g)\mathcal{T}_N$. Thus the Lemma is established.

Theorem 6.

$$P(g_1)\mathcal{T}'_N \cap P(g_2)\mathcal{T}'_N \text{ is simply-connected.}$$

Proof.

For a continuous closed curve $\{\zeta''_j(t)\} \subset P(g_1)\mathcal{T}'_N \cap P(g_2)\mathcal{T}'_N$, $0 \leq t \leq 1$, we can find continuous closed curves $\bar{\Lambda}(s) \subset \bar{L}_+(C)$ and $\{\zeta_j(s)\} \subset P(g_1)\mathcal{T}'_N \cap P(g_2)\mathcal{T}_N$ such that the eq. (55) is satisfied with a suitable parametrization by s, according to Lemma 9. From the fact that $\bar{L}_+(C)$ and $P(g_1)\mathcal{T}'_N \cap P(g_2)\mathcal{T}_N$ are simply-connected (Lemma 12), we can conclude the Theorem.

5. Discussion.

5. 1. Theorem 3 follows from Theorem 5, since the analytically continued function is connected. We note, however, that the former has been derived by a smaller number of pieces of knowledge compared with the case of the latter, as seen in the proofs in the text. Similarly Theorem 4 can be considered as a stronger statement than that of the Bargmann-Hall-Wightman Theorem[4].

5. 2. According to Theorem 3, the Ruelle Theorem[17], which states that the holomorphy envelope of $\bigcup_{g \in S_{N+1}} P(g)\mathcal{T}'_N$ contains the totally space-like points[5] S, turns out to be applicable to the quantum field theory which is based on axioms (I)-(IV). The difference of the contents of the Ruelle Theorem and the Dyson Theorem[6] lies in that the former is global while the latter is local in character.

5. 3. A continuous mapping[1, 18]

(59) $$\{\zeta_j\} \longrightarrow \{\zeta_j \cdot \zeta_k\}, \qquad j, k = 1, 2, \ldots, N,$$

mapps the domain $\mathcal{T}'_N$ (or $\mathcal{T}_N$) into a space composed of a symmetric complex $N \times N$ matrix of rank ≤ 4, the image of the mapping being denoted by $\mathcal{M}_N$. Since the mapping (59) is such that an inside point of $\mathcal{T}'_N$ is mapped to an inside point of $\mathcal{M}_N$ and a boundary point to a boundary point, all the Theorem 1-6 of this article are valid if we replace $\mathcal{T}'_N$ by $\mathcal{M}_N$.

5. 4. The results of this article seem to clarify the statements about local commutativity given in the paper of ref. (1). This is due to the reason that in constructing a quantum field theory from a set of analytic functions, following Wightman, we need knowledge of what are the domains of analyticity of these analytic functions.

5. 5. Streater[19] has extended the discussion of the analytic properties of the W-function to that of an arbitrary matrix element of the product of field operators, getting the same analyticity domain as

that of the W-function. Our results then equally applicable to the case of Streater's treatment. (Actually he seems to have assumed Theorem 3 of our text in his statement).

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Appendix A - Some properties of the Jost points and the extended tube[20]

Define the sets of the real points, $K_N^{[\alpha]}$ and $K_N^{[\alpha]\pm}$ as follows:

$$K_N^{[\alpha]} = \left\{ \left\{ \xi_j \right\} ; \xi_j^{\mu} = 0 \quad \text{for} \quad \mu \neq \alpha \right\}, \qquad (\alpha = 1, 2, 3) \tag{A. 1}$$

and

$$K_N^{[\alpha]\pm} = \left\{ \left\{ \xi_j \right\} ; \ \xi_j^{\alpha} \gtrless 0 \ , \ \xi_j^{\mu} = 0 \quad \text{for} \ \mu \neq \alpha \right\}, \quad (\alpha = 1, 2, 3) . \tag{A. 2}$$

Clearly $K_N^{[\alpha]}$ is invariant under the operation $P(g)$,

$$P(g) K_N^{[\alpha]} (\left\{ \xi_j \right\}) = K_N^{[\alpha]} (P(g) \left\{ \xi_j \right\}) = K_N^{[\alpha]} (\left\{ \xi_j \right\}) \ , \tag{A. 3}$$

and

$$\begin{aligned} P(g) K_N^{[\alpha]\pm} (\left\{ \xi_j \right\}) &= K_N^{[\alpha]\pm} (P(g) \left\{ \xi_j \right\}) = \\ &= \left\{ \left\{ \xi_j \right\} ; P(g) \xi_j^{\alpha} = \tilde{\xi}_j^{\alpha} \gtrless 0 \ , \ \xi_j^{\mu} = 0 \quad \text{for} \ \mu \neq \alpha \right\} . \end{aligned} \tag{A. 4}$$

If we write as

$$K_N^{[\alpha]} (\left\{ \xi_j \right\}) = K_1^{[\alpha]} (\xi_1) \otimes K_1^{[\alpha]} (\xi_2) \otimes \ldots \otimes K_1^{[\alpha]} (\xi_N) \ , \tag{A. 5}$$

$K_1^{[\alpha]}(\xi)$ is the α-th coordinate axis ($\alpha = 1, 2, 3$) of the space component in the real Monkowski space. The similarly defined $K_1^{[\alpha]+}(\xi)$ and $K_1^{[\alpha]-}(\xi)$ stand for the α-th positive and negative coordinate axes. The set $Q(g_1, g_2, g_3)$ defined by (7) in the text can be expressed in terms of $K_N^{[\alpha]\pm}$ as follows

$$Q(g_1, g_2, g_3) = \bigotimes_{\alpha = 1,2,3} (P(g_\alpha) K_N^{[\alpha]+} + P(g_\alpha) K_N^{[\alpha]-}) \ . \tag{A. 6}$$

Lemma A_1.

$$K_N^{[\alpha]} \cap P(g) J_N = P(g) K_N^{[\alpha]+} + P(g) K_N^{[\alpha]-} \ . \tag{A. 7}$$

Proof.

It is sufficient to prove the case $g = 1$, i. e.,

$$K_N^{[\alpha]} \cap J_N = K_N^{[\alpha]+} + K_N^{[\alpha]-} \ , \tag{A. 7'}$$

since (A. 7) follows from (A. 7') by operating with $P(g)$ and the proper ty (A. 3). First, we have

$$K_N^{[\alpha]+} + K_N^{[\alpha]-} \subset J_N \quad ,$$

since

$$\left(\sum_{j=1}^{N} \lambda_j \xi_j\right)^2 = \left(\sum_{j=1}^{N} \lambda_j \xi_j^{\alpha}\right)^2 > 0$$

for a point $\{\xi_j\} \in K_N^{[\alpha]\pm}$ and for $\{\lambda_j\}$ satisfying eq. (5). For a point $\{\xi_j\} \in (K_N^{[\alpha]} - \sum_{s=\pm} P(g) K_N^{[\alpha]s})$, either at least one component 4-vector $\xi_j = 0$ or one such pair $(\xi_{j1}^{\alpha}, \xi_{j2}^{\alpha})$ have opposite signs. Thus for both cases, we can find a $\{\lambda_j\}$ satisfying (5) which gives $\sum_{j=1}^{N} \lambda_j \xi_j = 0$, and so

$$(K_N^{[\alpha]} - \sum_{s=\pm} K_N^{[\alpha]s}) \cap J_N = \phi \ .$$

This completes the proof. (q. e. d.)

Define

(A. 8) $$K_N^{[\alpha,\beta]} = K_N^{[\alpha]} \otimes K_N^{[\beta]} = \left\{\{\xi_j\};\ \xi_j^{\mu} = 0 \quad \text{for } \mu \neq \alpha, \beta\right\} \quad (\alpha \neq \beta),$$

where $K_1^{[\alpha,\beta]}$ is the (α, β) coordinate plane in the space part of the Minkowski space.

Lemma A_2.

(A. 9) $$K_N^{[\alpha,\beta]} \subset \overline{P(g) J_N}$$

where the right hand side stands for the closure of the set $P(g)J_N$.

Proof.

Take the case $\alpha = 1$, $\beta = 2$ for simplicity. Clearly

(A. 10) $$\left\{\{\xi_j\};\ \xi_j^0 = 0\ ,\ \ P(g)\xi_j^3 > 0 \quad \text{for } \forall j \ \text{ or } < 0 \ \text{ for } \forall j\right\} \subset P(g) J_N$$

according to Lemma A_1, and such a point can be found in any neighbourhood of any point of $K_N^{[1, 2]}$.

Lemma A_3.

The extended tube is concave at any point belonging to $\partial K_N^{[\alpha]\pm}$, the boundary of $K_N^{[\alpha]\pm}$, which is contained in $\partial \mathcal{T}_N' \cap \partial \mathcal{T}_N$.

Proof.

Clearly $\partial K_N^{[\alpha]\pm} \subset \partial \mathcal{T}'_N \cap \partial \mathcal{T}_N$, according to Lemma A_1. First we prove the concavity of $\mathcal{T}'_N$ at the origin $\{0\} \in \partial K_N^{[\alpha]\pm}$. Take a hyperplane passing through the origin, which is expressed as

$$\text{(A. 11)} \qquad \sum_{j=1}^{N} \sum_{\nu=0}^{3} (a_j^\nu \xi_j^\nu + b_j^\nu \eta_j^\nu) = 0 ,$$

where a_j^ν and b_j^ν are real. Then it can be shown that for any choice of a_j^ν and b_j^ν, we can find a point belonging to $J_N \subset \mathcal{T}'_N$, which satisfies the eq. (A. 11), in any neighbourhood of the origin. To prove this, take a real point $\{\xi_j\} \in J_N$ such that

$$\xi_j^0 = 0 \qquad \text{for } \forall j .$$

$$\text{(A. 12)} \qquad \xi_j^\alpha > 0 \ (\text{or} < 0) \ \text{for } \forall j \quad \begin{cases} \text{(a) either when } \exists a_j^\beta \neq 0 \quad (\beta \neq \alpha) \\ \qquad (\alpha, \beta = 1, 2, 3) \\ \text{(b) or when } a_j^\beta = 0 \\ \qquad (\beta = 1, 2, 3 \text{ and for } \forall j) . \end{cases}$$

The point (A. 12) can satisfy eq. (A. 11) easily by adjusting ξ_j^β for the case (a) and evidently for the case (b). Moreover, if a $\{\xi_j\}$ of (A. 12) satisfies eq. (A. 11), then all $\{l\xi_j\}$ does, l being arbitrary real number. This proves the above statement.

Next we consider a point $\{\bar{\xi}_j\} \in \partial K_N^{[\alpha]\pm}$. For simplicity take the case $\alpha = 1$, + sign in the r. h. s., i. e., $\bar{\xi}_j$ satisfies the conditions $\bar{\xi}_j^1 \geq 0$ and $\bar{\xi}_j^\mu = 0$ $(\mu \neq 1)$. Equation for a hyperplane passing through the point $\{\bar{\xi}_j\}$ is

$$\text{(A. 11')} \qquad \sum_{j=1}^{N} \sum_{\nu=0}^{3} \left\{ a_j^\nu (\xi_j^\nu - \bar{\xi}_j^\nu) + b_j^\nu \eta_j^\nu \right\} = 0 .$$

Take $\xi_j = \bar{\xi}_j$ for j such that $\bar{\xi}_j^1 > 0$, and ξ_j of the type (A. 12) (the case, > 0) for j such that $\bar{\xi}_j = 0$. It is easy to see that this is a Jost point and that we can find a solution of (A. 11') from such points in any neighbourhood of the point $\{\bar{\xi}_j\}$. Thus the Lemma is established.

Appendix B - Proof of the irreducibility of the representation P(g).

Lemma B

The set $\{\xi_j\}$, ($\xi_j = x_j - x_{j-1}$, $j = 1, 2, \ldots, N$) forms a basis of the irreducibile representation, of the symmetric group S_{N+1} operating on the suffix of $(x_0, x_1, \ldots\ldots, x_N)$, which corresponds to the partition $(\lambda) = (N, 1)$.

Proof.

Permutation $g = \begin{pmatrix} 0, 1, \ldots\ldots, N \\ i_0, i_1, \ldots, i_N \end{pmatrix}$ induces the transformation:

$$\text{(B. 1)} \qquad \begin{vmatrix} \tilde{x}_0 \\ \tilde{x}_1 \\ \cdot \\ \cdot \\ \tilde{x}_N \end{vmatrix} = A(g) \begin{vmatrix} x_0 \\ x_1 \\ \cdot \\ \cdot \\ x_N \end{vmatrix}, \qquad \text{or for short} \quad \{\tilde{x}_j\} = A(g)\{x_j\}, \quad j = 0, 1, \ldots\ldots, N,$$

where $A(g)$ is a representation of S_{N+1}, and thus $\det (A(g)) \neq 0$. However $A(g)$ is not irreducible, since $\xi_0 = \frac{1}{N+1}\sum_{j=0}^{N} x_j$ is invariant under S_{N+1}.

Make the following change of variables

$$\text{(B. 2)} \qquad \{\xi_j\} = B\{x_j\}, \qquad j = 0, 1, \ldots\ldots, N,$$

where

$$\text{(B. 3)} \qquad B = \begin{vmatrix} \frac{1}{N+1} & \frac{1}{N+1} & \frac{1}{N+1} & \ldots & \frac{1}{N+1} \\ -1 & 1 & 0 & \ldots & 0 \\ 0 & -1 & 1 & \ldots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \ldots & 1 \end{vmatrix}, \qquad \det B = 1 .$$

Writing

$$\text{(B. 4)} \qquad B^{-1} = \begin{vmatrix} 1 & b_{01} & \ldots & b_{0N} \\ 1 & b_{11} & \ldots & b_{1N} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & b_{N1} & \ldots & b_{NN} \end{vmatrix},$$

we have the relations

(B. 5) $$\sum_{j=0}^{N} b_{jk} = 0 \ , \qquad (k = 1, \ldots, N)$$

from $B B^{-1} = 1$.

Now the change of basis, (B. 2), leads to

(B. 6) $$\{\tilde{\xi}_j\} = A'(g)\{\xi_j\} \ , \qquad j = 0, 1, \ldots, N \ ,$$

where, using eqs. (B. 1) - (B. 5), we get

$$A'(g) = B\,A(g)B^{-1} = B \begin{vmatrix} 1 & b_{i_0 1} & \cdots & b_{i_0 N} \\ 1 & b_{i_1 1} & \cdots & b_{i_1 N} \\ \cdots & \cdots & \cdots & \cdots \\ 1 & b_{i_N 1} & \cdots & b_{i_N N} \end{vmatrix} =$$

(B. 7)

$$= \left| \begin{array}{c|ccc} 1 & 0 & \cdots & 0 \\ \hline 0 & & & \\ \cdot & & & \\ \cdot & & P(g) & \\ \cdot & & & \\ 0 & & & \end{array} \right|$$

Thus we get a representation $P(g)$ of S_{N+1}, which basis are $\{\xi_j\}$, $(j = 1, \ldots, N)$, and $\det(P(g)) \neq 0$.

Using the relation

$$tr(A'(g)) = tr(A(g)) = \text{number of } x_j \text{ which is not changed by the permutation } g \in S_{N+1} \ ,$$

we can calculate the character $\chi^{P}_{(1^\alpha, 2^\beta, \ldots)}$, for the representation $P(g)$, of the class corresponding to a partition in cycles $(1^\alpha, 2^\beta, \ldots)$ (for notation see ref. 21). Since $\chi^{A'}_{(1^\alpha, 2^\beta, \ldots)} = \alpha$, we have

(B. 8) $$\chi^{P}_{(1^\alpha, 2^\beta, \ldots)} = \alpha - 1 = \chi^{(N,1)}_{(1^\alpha, 2^\beta, \ldots)} \ ,$$

where $\chi^{(N,1)}_{(1^\alpha, 2^\beta, \ldots)}$ is the character of the class $(1^\alpha, 2^\beta, \ldots)$ for the irreducible representation corresponding to the partition $(N, 1)$, and the second equality can be readily obtained by the graphical method[22].

Therefore $P(g) = (\!(p_{jk}(g))\!)$ is the irreducible representation which corresponds to the partition (N, 1). Incidentaly the dimension n^P of the representation is

$$n^P = \chi^P_{(1^{N+1})} = N$$

which is, of course, consistent with the number of $\{\xi_j\}$.

Footnotes and References

(1) - A. S. Wightman, Phys. Rev. 101, 860 (1956)

(2) - For detail discussion of these axioms and for further references, see
A. S. Wightman in "Les Problèmes Mathématiques de la Théorie Quantique des Champs" (Colloque Internationaux du CNRS, Lille, 1957) p. 1;
A. S. Wightman in "Cours de la Faculté des Sciences de l'Université de Paris (1957-1958);
A. S. Wightman, Nuovo Cimento Suppl. 14, 192 (1959);
A. S. Wightman, J. Indian Math. Soc. 24, 625 (1960);
R. Haag, Nuovo Cimento Suppl. 14, 131 (1959);
G. Källen in "Relations de Dispersion et Particules Elémentaires" (Hermann, Paris, 1960) p. 387;
R. Jost in "Theoretical Physics in the Twentieth Century" (Interscience Pub. Inc., New York, 1960) p. 107;
R. Jost in "Lectures on Field Theory and the Many Body Problem" (Academic Press, New York, 1961) p. 127;
H. Araki, Suppl. Prog. Theor. Phys. (Kyoto) n. 18, 83 (1961);
R. Haag and B. Schroer, J. Math. Phys. 3, 248 (1962);
W. Schmidt and K. Baumann, Nuovo Cimento 4, 860 (1956).

(3) - Here "proper" means "connected component with unit element". Sometimes it is called "restricted" or "proper orthochronous".

(4) - D. Hall and A. S. Wightman, Kgl. Danske Videnskab. Selskab. Mat-fys. Medd. 31, n. 5 (1957);
R. Jost, loc. cit.

(5) - It follows from the Dyson Theorem(6) that the analytic continuation of the W-function due to local commutativity is single-valued at least in the small neighbourhood of the totally space-like points $S = \{\{\xi_j\}; \xi_j = x_j - x_{j-1},\ x_k - x_l \in V_s,\ (k, l = 0, 1, \ldots, N;\ k \neq l)\}$. Araki has concluded(7) the connectedness of $\bigcup_{g \in S_{N+1}} P(g)\mathcal{T}'_N$ by showing that $\bigcup_{g \in S_{N+1}} P(g)J_N$ is connected, $P(g)J_N \subset P(g)\mathcal{T}'_N$ being the set of Jost points (for definition, see Sect. 2) which is connected. The fact that $\bigcup_{g \in S_{N+1}} P(g)J_N$ is connected follows from Araki's Lemma(7), which states that any point of the connected set $K = \{\{\xi_j\};\ \xi^0_j = 0,\ \underset{\sim}{\xi}_j = \underset{\sim}{x}_j - \underset{\sim}{x}_{j-1}$ are real, $(\underset{\sim}{x}_0, \underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_N)$ are all distinct$\}$ is contained in $\bigcup_{g \in S_{N+1}} P(g)J_N$ and from the fact that $K \cap P(g)J_N$ is non-empty for $\forall g \in S_{N+1}$. (It also follows from the existence(8) of the non-empty intersection $J_N \cap P((k-1, k))J_N$, $g = (k-1, k)$ being a neighbouring transposition belonging to S_{N+1}, which seems

to be not a direct consequence of that Lemma contrary to Araki's statement). Then Araki has stated that the analytic continuation of W-function due to local commutativity is single-valued in a small neighbourhood of $\bigcup_{g\in S_{N+1}} P(g)J_N$, which is the consequence of the connectedness of $\bigcup_{g\in S_{N+1}} P(g)J_N$. This is, of course, a weaker conclusion than that of the Dyson Theorem, since $\bigcup_{g\in S_{N+1}} P(g)J_N \subset S$.

For the analysis of the analyticity domain of the three-point function (N = 2), see
G. Källen and A. S. Wightman, Mat. Fys. Skr. Kgl. Danske Videnskab. Selskab. 1, n. 6 (1958);
D. Ruelle, Helv. Phys. Acta 34, 587 (1961).

(6) - F. J. Dyson, Phys. Rev. 110, 579 (1958).

(7) - H. Araki, loc. cit. (see Sect. 5);
H. Araki, Ann. Phys. 11, 260 (1960) Lemma 1.

(8) - A. S. Wightman, J. Indian Math. Soc., loc. cit., p. 660 ;
Y. Tomozawa, Nuovo Cimento 27, 543 (1963) Lemma 2.

(9) - R. Jost, Helv. Phys. Acta 30, 409 (1957).

(10) - We use the metric given by $x^2 = -x_0^2 + \underset{\sim}{x}^2$.

(11) - Actually it is the irreducible representation corresponding to the partition $(\lambda) = (N, 1)$. (See Appendix B).

(12) - Apart from the real Lorentz transformation belonging to L_+. (See the discussion given before Lemma 4).

(13) - S. Bochner and W. T. Martin, Several Complex Variables (Princeton University Press, 1948) Chap. II, § 2.

(14) - H. Seifert and W. Threlfall, Lehrbuch der Topologie (B. G. Teubner Verlag, 1934) Kap. 2, § 7 Satz IV ; or
F. Hausdorff, Set Theory (Chelsea Pub. Comp., New York, 1957) Chap. VI, 26 Theorem III (The Borel Covering Theorem for separable spaces).

(15) - L. Pontrjagin, Topological Groups (Princeton University Press, 1958) Chap. VIII, Sect. 46.

(16) - The continuous closed curve $\{\zeta_j'(s)\} \subset \mathcal{T}_N' \cap P(g)\mathcal{T}_N$ can be deformed continuously into the continuous closed curve $\{\bar{L}_2(s)\,\zeta_j'(s)\} \subset \mathcal{T}_N' \cap P(g)\mathcal{T}_N$, inside $\mathcal{T}_N' \cap P(g)\mathcal{T}_N$, since $\bar{L}_+^{\uparrow}$ leaves $\mathcal{T}_N' \cap P(g)\mathcal{T}_N$ invariant, and since it follows from the simply-connectedness of $\bar{L}_+^{\uparrow}$ that we can deform continuously the continuous closed curve $\bar{L}_2(s) \subset \bar{L}_+^{\uparrow}$ into the unit element of $\bar{L}_+^{\uparrow}$, inside $\bar{L}_+^{\uparrow}$. For the continuous closed curve $\{\zeta_j(s)\} \subset \mathcal{T}_N$, the continuous closed curve $\{L_1^{-1}(s)\,\zeta_j(s)\}$ is contained in $\mathcal{T}_N$.

(17) - D. Ruelle, Helv. Phys. Acta 32, 135 (1959).

(18) - A. S. Wightman, J. Indian Math. Soc. , loc. cit. , p. 640 .

(19) - R. F. Streater, J. Math. Phys. 3, 256 (1962).

(20) - These properties were not used in the text, but it might help in discussing the structure of the extended tube. For systematic analysis of the boundary, see [illegible]
A. S. Wightman, J. Indian Math. Soc. , loc. cit.

(21) - H. Hamermesh, Group Theory and its Applications to Physical Problems (Addison-Wesley Publ. Comp. Inc. , Reading, 1962) Chap. 7 .

(22) - H. Hamermesh, loc. cit. , p. 206 (Problem (3a)).