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Application of Fractional Operators in Physics and Engineering

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Abstract

In this paper we present a solution to a fractional integral of order 3/2 with the use of fractional Cauchy-like integral formula. The integral arises during the solution of Biot-Savart equation to find the exact analytical solution for the magnetic field components of a solenoid. The integrals are computed by cutting the branch line in order to have an analytic function inside the integral instead of multi-valued operation.

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1 Introduction

Fractional calculus studies the fractional powers of the differentiation and integration operators both in real and complex domains. The concept was initially brought to the literature by Gottfried Wilhelm Leibniz [1] and then developed by Niels Henrik Abel [2], Riemann-Liouville, Riesz, Caputo (see [3]), Hadamard [4] and Atangana-Baleanu [5]. The fractional derivatives and integrals appear in mathematics, different branches of physics and engineering. Physicists usually avoid confronting fractional integrals and derivatives and they try to find other ways to solve the integrals like using the elliptical integral. The idea recently was introduced to quantum mechanics by Laskin [6] under the name ”fractional quantum mechanics” in which the Lèvy-like paths in the Feynman path integrals are substituted by the Brownian-like quantum paths. The Levy-like paths corresponds to regular operators and Brownian-like paths are fractional. In this paper we report some important equations of fractional calculus, then we will investigate their limitations, and finally propose a novel way to solve a fractional integral which appears in the field of accelerator physics.

It should be noted that the integer derivative of a function f(x) is a local property at a point x. On the other hand only, the fractional derivative of a function f(x) at the point x depends only on values of f close to x. This means that the boundary conditions should be considered by involving information on the function further out.

In the next section we review some fractional derivatives and integrals.

2 Preliminaries: Various forms of Fractional Integrals and Derivatives

In this section we review several known forms of the fractional integrals in mathematics which arise in some physics and engineering problems.

2.1 Riemann-Liouville Fractional Integral

Definition: Let Re $\alpha > 0$ and f be piecewise continuous and integrable on (0, ∞). Then we define

$$ {}_{z_0}D_z^{-\alpha} f(z) = \frac{1}{\Gamma(\alpha)} \int_{z_0}^{z} \frac{f(\tau)}{(z-\tau)^{-\alpha+1}} d\tau \tag{1} $$

when $z = z_0$, this is called Riemann-Liouville fractional integral of the function f of order α.

2.2 Weyl fractional derivatives

The Weyl fractional derivatives are used when z in the Eq. (1) takes on a singular value from $-\infty$ to ∞, and it can be expressed as,

$$D^{-\alpha}_{z^{-\infty}}f(z)=\frac{(-1)^{-\alpha}}{\Gamma(-\alpha)}\int_{z_0}^{\infty}\frac{f^{(n)}(\tau)}{(z-\tau)^{-\alpha-1}}d\tau \tag{2}$$

and

$$D^{-\alpha}_{z^{+\infty}}f(z)=\frac{1}{\Gamma(-\alpha)}\int_{-\infty}^{z_0}\frac{f^{(n)}(\tau)}{(z-\tau)^{-\alpha-1}}d\tau. \tag{3}$$

where Re $\alpha > 0$.

2.3 Caputo fractional derivative

The Caputo fractional derivative [7] is used in order to solve the differential equations without defining the fractional order initial conditions. Caputo's definition is as follows.

$$_{z_0}D^{-\alpha}_{z}f(z)=\frac{1}{\Gamma(n-\alpha)}\int_{z_0}^{z}\frac{f^{(n)}(\tau)}{(z-\tau)^{\alpha+1+n}}d\tau,\ \ n-1<\alpha<n. \tag{4}$$

2.4 Hadamard Fractional Integral

The Hadamard fractional integral is introduced by Jacques Hadamard and is given by the following formula,

$$_{a}D^{-\alpha}_{t}f(z)=\frac{1}{\Gamma(\alpha)}\int_{a}^{t}(log\frac{t}{\tau})^{\alpha-1}f(\tau)\frac{d\tau}{\tau}\quad t>a \tag{5}$$

This is based on the generalization of the integral

$$\int_{a}^{t}\frac{d\tau_1}{\tau_1}\int_{a}^{\tau_1}\frac{d\tau_2}{\tau_2}...\int_{a}^{\tau_{\alpha-1}}\frac{d\tau_\alpha}{\tau_\alpha}d\tau_\alpha=\frac{1}{\Gamma(\alpha)}\int_{a}^{t}(log\frac{t}{\tau})^{\alpha-1}f(\tau)\frac{d\tau}{\tau}\quad t>a,\ \ \alpha>0 \tag{6}$$

to obtain the above equation, the n-fold integral of the form below is used

$$^{\rho}_{a}D^{\alpha}_{x}f(x)=\int_{a}^{x}\tau_1^{\rho}d\tau_1\int_{a}^{\tau_1}\tau_2^{\rho}d\tau_2...\int_{a}^{\tau_{n-1}}\tau_n^{\rho}f(\tau_n)d\tau_n \tag{7}$$

2.5 Generalized Fractional Integration Operator

The author [8] obtained a generalized fractional integration operator which bounded in the Lebesgue measurable space. The procedure is as follow:

The Lebesgue measurable functions f on [a, b] in the space X^p_c (a,b) (c $\in \mathbb{R}$, $1\leq p\leq\infty$) for which $||f_{X^p_c}||<\infty$, where the norm is defined by

$$||f||_{X^p_c}=\frac{1}{(\int_a^b|t^cf(t)|^p\frac{dt}{t})^p}<\infty\quad(c\in\mathbb{R},1\leq p<\infty) \tag{8}$$

and for the case of p when is equal ∞ we have

$$||f||_{X_c^\infty} = \text{ess sup}_{a \leq t \leq b}[|t^c f(t)|] \quad (c \in \mathbb{R}) \tag{9}$$

where “ess sup”denotes for the “essential supremum”of the function f, representing that value where f is larger or equal than the function values everywhere, when ignoring what the function does at a set of points of “measure zero”(measure zero is a set of points capable of being enclosed in intervals whose total length is arbitrarily small).

By using Dirichlet technique for n-fold integral, the fractional integral of ${}_a^\rho D_x^\alpha$ yields [8],

$${}_a^\rho D_x^\alpha f(x) = \frac{(\rho+1)^{1-\alpha}}{\Gamma(\alpha)} \int_a^x (x^{\rho+1} - \tau^{\rho+1})^{\alpha-1} \tau^\rho f(\tau) d\tau \tag{10}$$

where α and $\rho \neq -1$ are real numbers.

As an additional information, the Dirichlet technique [9] is given:

$$\int_a^x \tau_1^\rho d\tau_1 \int_a^{\tau_1} \tau^\rho f(\tau) d\tau = \int_a^x \tau^\rho f(\tau) d\tau \int_\tau^x \tau_1^\rho d\tau \tag{11}$$

$$= \frac{1}{\rho+1} \int_a^x (x^{\rho+1} - \tau^{\rho+1}) \tau^\rho f(\tau) d\tau \tag{12}$$

2.6 Fractional Atangana-Baleanu derivative

Atangana and Baleanu proposed a new fractional derivative [5] with non-local and no-singular kernel using the Mittag-Leffler function. They started with the fractional ordinary differential equation

$$\frac{d^\alpha y}{dx^\alpha} = ay \quad 0 < \alpha < 1 \tag{13}$$

after some manipulation and using Caputo-Fabrizio derivative they obtained an expression and solved the problem of non-locality.

$$D_t^\alpha f(t) = \frac{N(\alpha)}{1-\alpha} \Sigma_{k=0}^\infty \frac{(-a)^k}{\Gamma(\alpha k+1)} \int_b^t \frac{df(y)}{dy} ((t-y))^{\alpha k} dy. \tag{14}$$

where $a = \alpha(1-\alpha)^{-1}$ and $N(\alpha)$ stands for a normalization function obeying $N(0) = N(1)$=1.

2.7 Fractional Riesz derivative

The Riesz derivative of function u(x, t) with respect to x is defined by [10]

$$\frac{\partial^\alpha u(x,t)}{\partial |x|^\alpha} = -\frac{1}{2} \sec(\frac{\pi\alpha}{2}) [{}_{RL}D_{-\infty,x}^\alpha +_{RL} D_{X,+\infty}^\alpha] u(x,t) \tag{15}$$

where $D_{-\infty,x}^\alpha$ and ${}_{RL}D_{X,+\infty}^\alpha$ are the left and right Riemann-Liouville derivatives.

2.8 The Cauchy integral theorem

Let $D \subseteq \mathbb{C}$ be an open set. We shall say that D has a piecewise C^1-boundary if the boundary of D (in $\mathbb{C}$) is a closed piecewise C^1-contour such that each point of the contour is also a boundary point of $\mathbb{C} \setminus \bar{D}$.

Let $D \subseteq \mathbb{C}$ be a bounded open set with piecewise C^1-boundary, let E be a Banach space, and let f : $\bar{D} \to E$ be a continuous function which is holomorphic in D. Then [11]

$$f(z_0) = \frac{1}{2\pi i} \int_{\partial D} \frac{f(z)}{z - z_0} dz, \quad z_0 \in D. \tag{16}$$

Any holomorphic function with values in a Banach space is infinitely times complexly differentiable. In particular, it is of class C^∞. Moreover, if D and f are as in the above theorem and if we denote by $f^{(n)}$ the n-th complex derivative of f in D, then [11]

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int_{\partial D} \frac{f(z)}{(z - z_0)^{n+1}} dz, \quad z_0 \in D. \tag{17}$$

3 Statement of the Problem

Starting from the Biot-Savart law, the axial and radial magnetic field components for a coil of negligible thickness with a stationary electric current are given by [12]:

$$B_r = \frac{\mu_0 I M_z}{4\sqrt{2}\pi R} \left(\frac{\xi}{\eta}\right)^{3/2} I_2(\xi) \tag{18}$$

$$B_z = \frac{\mu_0 I}{4\sqrt{2}\pi R} \left(\frac{\xi}{\eta}\right)^{3/2} (I_1(\xi) - \eta\, I_2(\xi)) \tag{19}$$

where $\eta = \frac{r}{R}$, $M_z = \frac{z}{R}$

$$I_1(\xi) = \int_0^\pi \frac{d\psi}{[1 - \xi\ \cos(\psi)]^{3/2}} \tag{20}$$

$$I_2(\xi) = \int_0^\pi \frac{\cos(\psi)}{[1 - \xi\ \cos(\psi)]^{3/2}}\, d\psi \tag{21}$$

and

$$\xi(R, z, \eta) = \frac{2\eta}{1 + \eta^2 + M_z^2} \tag{22}$$

and we have used the following notation:

B_z is the magnetic field component in the direction of the coil axis.

B_r is the radial magnetic field component.

I is the current in the wire.

R is the radius of the current loop.

z is the distance, on axis, from the center of the current loop to the field measurement point.

r is the radial distance from the axis of the current loop to the field measurement point

θ denotes for the angle of the current element

γ stands for the angle of the observer where the magnetic field components are to be calculated

$\psi = \gamma - \theta$

As it can be observed the fractional integrals (20) and (21) can not be solved by Riemann-Liouville Fractional Integral. The reason is that, the denominator in this method becomes a power of 3/2 when one choose $\alpha = -1/2$ and this is not allowed by the theorem which implies $\alpha > 0$ (see Eq. (1)). On the other hand, our fractional integrals can not be solved by Weyl fractional theorem (Eqs. (2, 3)), because the α should be equal -5/2 to turn the denominator's power to 3/2 and this is not allowed by the limitation of this theorem ($\alpha > 0$). Let us now see if it is possible that the integrals (20) and (21) can be solved by Caputo fractional theorem. For $n = 1$, $\alpha = -1/2$ we can cover the integral's denominators (to arrive at power of 3/2) but as $n - 1 < \alpha < n$ and this means $\alpha > 0$ and this is not allowed either. We face the same limitation considering other theorems mentioned in the previous section.

In order to solve the integral we use Cauchy's Integral Formula with some modifications. Recalling Cauchy's Integral Formula

$$D^n f(z) = \frac{\Gamma(\alpha+1)}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz \tag{23}$$

we observe that n is an integer number, and this implies that there is one or more singularities in the function. On the other hand, when we have n as a non-integer, the singularity turn to the branch lines and f become a non-local property. It should be noted that integer derivative of a function f is a local property at a point z. We have already observed that only for non-integer power derivatives, the fractional derivative of a function f at the point z depends only on values of f very near z. In order to use this theorem for non-integer n, one should change the multi-valued operation (function) in Eqs. (20) and (21) and turn it into the analytic function. This procedure is called Branch Cut. By branch cut, our multi-valued function becomes an analytic function with a local property and the branch point turns to be a singularity point. Now we can use the Cauchy's Integral Formula for solving the fractional integral. We call this theorem Fractional Cauchy-like Integral Formula:

$${}_{\gamma(z,z^+)}D^{\alpha}_{z-z_0} f(z) = \frac{sin(\pi\alpha)\,\Gamma(\alpha+1)}{\pi} \int_{z_0}^{z} \frac{f(z)}{(z-z_0)^{\alpha+1}} dz. \tag{24}$$

In the next section we will first see how we obtain the above equation and then we will apply that to solve our fractional integrals.

4 Fractional Cauchy-like Integral Formula and the Final Solution

It should be noted that this method has been studied by the authors of [13] and [14]. The procedure is as follows:

Let us recall the Cauchy's integral formula

$$D^n f(z) = \frac{\Gamma(n+1)}{2\pi i} \oint_C \frac{f(z)}{(z-z_0)^{n+1}} dz. \tag{25}$$

Let the contour of integration be $\gamma(z_0, z^+)$. The branch line for $(z-z_0)^{-\alpha-1}$ starts from the position z and ends at the fixed point z_0. The above equation is equivalent to the Riemann-Liouville fractional integral when Re $(\alpha) < 0$. We divide the contour $\gamma(z_0, z^+)$ into three contours (see Fig. 1),

$$\gamma(z_0, z^+) = \gamma_1(z_0 \to z) \cup \gamma_2(O) \cup \gamma_3(z \to z_0) \tag{26}$$

where,
$\gamma_1(z \to z_0)$: line segment from z to z_0;
$\gamma_2(O)$: small circle centered at z_0;
$\gamma_3(z_0 \to z)$: line segment from z_0 to z.

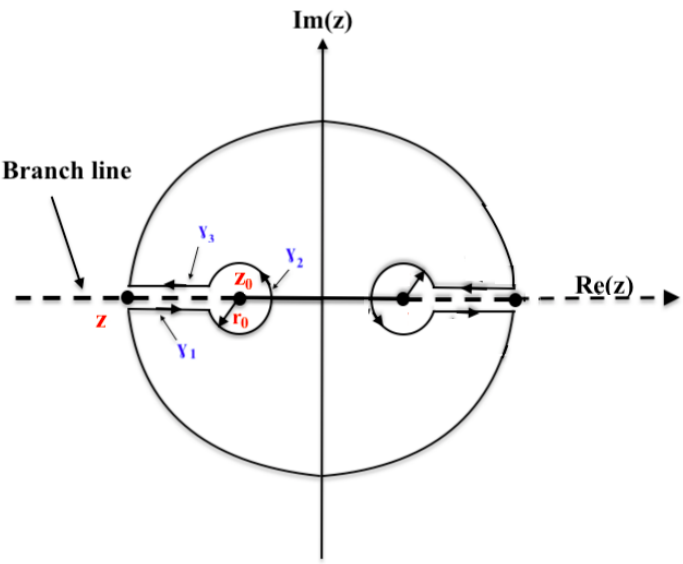

Figure 1: contour of integration

Then the Cauchy's integral formula becomes:

$$D^n f(z) = \frac{\Gamma(n+1)}{2\pi i} \int_{\gamma(z_0,z)} \frac{f(z)}{(z-z_0)^{n+1}} dz = I_{\gamma_1} + I_{\gamma_2} + I_{\gamma_3} \tag{27}$$

$I_{\gamma_1}, I_{\gamma_2}, I_{\gamma_3}$ denote the integrals over the mentioned contours $\gamma_1, \gamma_2, \gamma_3$. Then, the line in which the branch occurs can be written as

$$\frac{1}{(z-z_0)^{\alpha+1}} = e^{(-\alpha-1)(ln|z-z_0|+i(\theta-\pi))} \quad on \;\; \gamma_1 \tag{28}$$

$$\frac{1}{(z-z_0)^{\alpha+1}} = 0 \quad on \;\; \gamma_2 \tag{29}$$

$$\frac{1}{(z-z_0)^{\alpha+1}} = e^{(-\alpha-1)(ln|z-z_0|+i(\theta+\pi))} \quad on \;\; \gamma_3 \tag{30}$$

It should be noted that the integral tends to zero on γ_2 as the contour's radius r_0 goes to zero. Substituting the above equations inside of Eq. (25) we obtain

$$^{\gamma(z,z^+)}D^{\alpha}_{z-z_0}f(z)) = \frac{(e^{i\pi\alpha}+e^{-i\pi\alpha})\Gamma(\alpha+1)}{2\pi i}\int_{z_0}^{z}\frac{f(z)}{(z-z_0)^{\alpha+1}}dz \tag{31}$$

or

$$^{\gamma(z,z^+)}D^{\alpha}_{z-z_0}f(z) = \frac{sin(\pi\alpha)\,\Gamma(\alpha+1)}{\pi}\int_{z_0}^{z}\frac{f(z)}{(z-z_0)^{\alpha+1}}dz. \tag{32}$$

Notice that above equation is valid for all values of α. By Weierstrass M-test we can show that an infinite series of functions converges. First we show that $\frac{f(z)}{z-z_0}$ is an infinite series:

$$^{\gamma(z,z^+)}D^{\alpha}_{z-z_0}f(z_0) = \frac{sin(\pi\alpha)\,\Gamma(\alpha+1)}{\pi}\int_{z_0}^{z}\frac{f(z)}{(z-z_0)^{\alpha+1}}dz \tag{33}$$

$$= \frac{sin(\pi\alpha)\,\Gamma(\alpha+1)}{\pi}\int_{z_0}^{z}\frac{f(z)}{(z-r_0)^{\alpha}}\cdot\frac{1}{[1-(z_0-r_0)/(z-r_0)]^{\alpha}}dz \tag{34}$$

$$= \frac{sin(\pi\alpha)\,\Gamma(\alpha+1)}{\pi}\int_{z_0}^{z}\frac{f(z)}{(z-r_0)^{\alpha}}\;\Sigma_{n=0}^{\infty}(\frac{z_0-r_0}{z-r_0})^{n+\alpha}dz \tag{35}$$

$$= \Sigma_{n=0}^{\infty}\frac{sin(\pi\alpha)\,\Gamma(\alpha+1)}{\pi}\int_{z_0}^{z}\frac{(z_0-r_0)^{n+\alpha}}{(z-r_0)^{n+2\alpha}}f(z). \tag{36}$$

As $\frac{f(z)}{(z-r_0)^{\alpha}}$ is bounded on γ, when the contour's radius goes to zero, by some positive number M, and $|\frac{z_0-r_0}{z-r_0}| \;\le r < 1$, then we have

$$|\frac{(z_0-r_0)^{n+\alpha}}{(z-r_0)^{n+2\alpha}}f(z)| \;\le Mr^n \tag{37}$$

this means that the series converges on γ.

Recalling the Eq. (32)

$$^{\gamma(z,z^+)}D^{\alpha}_{z-z_0}f(z_0) = \frac{sin(\pi\alpha)\,\Gamma(\alpha+1)}{\pi}\int_{z_0}^{z}\frac{f(z)}{(z-z_0)^{\alpha+1}}dz. \tag{38}$$

and rearranging that we have

$$\int_{z_0}^{z} \frac{f(z)}{(z-z_0)^{\alpha+1}} dz = \frac{\pi}{sin(\pi\alpha)\,\Gamma(\alpha+1)} \;\; {}^{\gamma(z,z^+)}D^{\alpha}_{z-z_0} f(z_0). \tag{39}$$

By substituting f(x)=1, $\alpha = 1/2$ and changing the interval to [0, π], the above integral becomes our fractional integral of Eq. (20). Recalling Eq. (20),

$$\int_0^{2\pi} \frac{1}{[1-\xi\ \cos(\psi)]^{3/2}}\, d\psi \tag{40}$$

and writing the variable $\cos(\psi)$ in the complex plane as $\cos\psi = \frac{z+z^{-1}}{2}$, and replacing into the above equation and using the so called modified Cauchy's residue theorem for the fractional integrals we obtain

$$\int_0^{2\pi} \frac{1}{[1-\xi\ \cos(\psi)]^{3/2}}\, d\psi = \frac{2^{3/2}\pi}{\Gamma(3/2)} lim_{z\to z_0} D_z^{1/2} (z-z_0)^{-3/2} f(z) \tag{41}$$

where $z_{01} = \frac{1+\sqrt{1-\xi^2}}{\xi}$ and $z_{02} = \frac{1-\sqrt{1-\xi^2}}{\xi}$ are the branch points of the integral in which the residues should be computed.

We observe that the above equation is similar to the Residue theorem (sometimes called Cauchy's residue theorem) which is used in complex analysis but unlike the Cauchy's residue theorem, the denominator in our case has the 3/2 power. Therefore, we can not use directly this theorem. The reason is that, the residue theorem can be used to evaluate the line integrals of analytic functions over closed curves, but in our case, the integral of 20 and 21 are the line integrals of multi-valued functions. To Cauchy's residue theorem to be used for our case, we must "Branch Cut" in order to change the multi-valued functions to analytic functions.

As there is a symmetry in the integral, it is not necessary to branch cut both the branch lines. For this reason we will take the interval [0,π] where one of the branch line is located,

$$\int_0^{\pi} \frac{1}{[1-\xi\ \cos(\theta)]^{3/2}}\, d\psi = 2^{5/2} \frac{\pi}{\Gamma(3/2)} [lim_{z\to\frac{1-\sqrt{1-\xi^2}}{\xi}} D_z^{1/2} (z-z_1)^{3/2} f(z)] \tag{42}$$

$$= 2^{5/2} \frac{\pi}{\Gamma(3/2)} [D^{1/2}_{z-z_1} \frac{z^{1/2}}{(z-(\frac{1-\sqrt{1-\xi^2}}{\xi}))^{3/2}}] \tag{43}$$

where $D_z^{1/2}$ is the fractional derivative of the order $1/2$. Applying $D_z^{1/2}$ to the function we obtain,

$$D_z^{1/2}\,(\frac{z^{1/2}}{(z-z_0)^{3/2}}) = \frac{\Gamma(3/2)}{(z-z_0)^{3/2}} + \frac{z\,\Gamma(5/2)}{2(z-z_0)^{5/2}} - \frac{z^2\,\Gamma(7/2)}{16(z-z_0)^{7/2}} + ... \tag{44}$$

Finally substituting the Eq. (44) inside the (42) we obtain:

$$\int_0^{\pi} \frac{1}{[1-\xi\ \cos(\theta)]^{3/2}}\, d\theta = 2^{5/2}\frac{\pi}{\Gamma(3/2)}[\frac{\Gamma(3/2)}{(z-z_0)^{3/2}} + \frac{z\,\Gamma(5/2)}{2(z-z_0)^{5/2}} - \frac{z^2\,\Gamma(7/2)}{16(z-z_0)^{7/2}} + ...] \tag{45}$$

$$= \frac{\pi}{\Gamma(3/2)}[\frac{\Gamma(3/2)}{(1-\xi^2)^{3/4}} + \frac{(1-\sqrt{1-\xi^2})\,\Gamma(5/2)}{4(1-\xi^2)^{5/4}} - \frac{(1-\sqrt{1-\xi^2})^2\,\Gamma(7/2)}{64(1-\xi^2)^{7/4}} + ...] \tag{46}$$

Where the integral's solution is a hypergeometric function and it can be written in a compact form as

$$I_1(\xi) = \int_0^{\pi} \frac{d\psi}{[1-\xi\ cos(\psi)]^{3/2}} = \frac{\pi}{(1+\xi)^{3/2}}\ {}_2F_1(\frac{1}{2},\frac{3}{2};1;\frac{2\xi}{1+\xi}). \tag{47}$$

By performing the same process using the so called Cauchy-like integral formula, Eq. (21), we solved the integral (21)

$$I_2(\xi) = \int_0^{\pi} \frac{\cos(\psi)}{[1-\xi\ \cos(\psi)]^{3/2}}\, d\psi = \frac{\pi}{(1+\xi)^{3/2}}\ [\,{}_2F_1(\frac{3}{2},\frac{3}{2};2;\frac{2\xi}{1+\xi}) - {}_2F_1(\frac{1}{2},\frac{3}{2};1;\frac{2\xi}{1+\xi})]. \tag{48}$$

Conclusions

In this paper, we used the Branch Cut method to change the non-local property of the fractional derivative to a local property in order to be used the Cauchy's integral formula. The new method can be called the Cauchy-like integral formula for fractional integrals. By Weierstrass M-test we have shown that the integral converges. This method helps to solve important problems in which there is branch line. We applied the method to solved the axial and radial magnetic field components for a coil of negligible thickness with a stationary electric current. At the end, we have shown that the solutions can be expressed by means of the hypergeometric functions ${}_2F_1(a,b;c;z)$.

References

[1] Leibniz, Gottfried Wilhelm. "Gottfried Wilhelm Leibniz."Technology 4 (2002): 1.

[2] Abel, N. H. "Oplosning af et par opgaver ved hjelp af bestemte integraler."Magazin for naturvidenskaberne 2.55 (1823): 2.

[3] Miller, Kenneth S., and Bertram Ross. "An introduction to the fractional calculus and fractional differential equations."Wiley, 1993.

[4] Hadamard, Jacques. "Essai sur l'etude des fonctions, donnees par leur developpement de Taylor."Gauthier-Villars, 1892.

[5] Atangana, Abdon, and Dumitru Baleanu. “New fractional derivatives with nonlocal and non-singular kernel: theory and application to heat transfer model.”arXiv: 1602.03408 (2016).

[6] Laskin, Nick. “Fractional Schrödinger equation.”Physical Review E 66.5 (2002): 056108.

[7] Caputo, Michele. “Linear models of dissipation whose Q is almost frequency independent-II.”Geophysical Journal International 13.5 (1967): 529-539.

[8] Katugampola, Udita N. “New approach to a generalized fractional integral.”Applied Mathematics and Computation 218.3 (2011): 860-865.

[9] Podlubny, Igor. “Fractional differential equations, vol. 198 of Mathematics in Science and Engineering.”(1999).

[10] Samko, Stefan G., Anatoly A. Kilbas, and Oleg I. Marichev. “Fractional integrals and derivatives.”Vol. 1. Yverdon-les-Bains, Switzerland: Gordon and Breach Science Publishers, Yverdon, 1993.

[11] Gohberg, Israel, and Jurgen Leiterer. “Holomorphic operator functions of one variable and applications: methods from complex analysis in several variables.”Vol. 192. Springer Science & Business Media, 2009.

[12] Behtouei, M., Faillace, L., Spataro, B., Variola, A., & Migliorati, M. (2020). “A Novel Exact Analytical Expression for the Magnetic Field of a Solenoid.”arXiv preprint arXiv:2002.09444.

[13] Lovoie, J. L., Thomas J. Osler, and R. Tremblay. “Fractional derivatives and special functions.”SIAM review 18.2 (1976): 240-268.

[14] Osler, Thomas J. “The fractional derivative of a composite function.”SIAM Journal on Mathematical Analysis 1.2 (1970): 288-293.