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PER SE DEFINITION OF DIRAC'S δ FUNCTION

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Abstract

We examine critically the various definitions of the Dirac's δ, we note that we are constantly in the presence of a discontinuity at infinity which makes it impossible to give a correct definition of the Dirac's δ per se. Then we give its definition per se according to the standards dictated by Dirac himself: it is a 'function' that is zero everywhere except at the zero point where it has an infinite value and is such that the integral from minus infinity to plus infinity is 1 and that also multiplied by a 'well-behaved function' it gives the value of the function at the point 0. Note also that if we accept the Cesàro summability we arrive quickly, also in this case, at a correct per se definition of the δ itself. We are then reasoning on the relationship between mathematics and physics.

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1. A brief history of theory of distribution or generalised function.

The first (from [8]) [1] [2] to use generalized functions in the explicit and presently accepted form was S. L. Sobolev in 1936 in studying the uniqueness of solutions of the Cauchy problem for linear hyperbolic equations.

From another point the view Bochner's theory of the Fourier transforms of functions increasing as some power of their argument can also bring one to the theory of generalized functions. These Fourier transforms, in Bochner's work the format derivatives of continuous functions, are in essence generalized functions. In 1950-1951 there appeared Laurent Schwartz's monograph Théorie des Distributions. In this book Schwartz systematizes the theory of generalized functions, basing it on the theory of linear topological spaces, relates all the earlier approaches, and obtains many important and far reaching results. Unusually soon after the appearance of Théorie des Distributions, in fact literally within two or three years, generalized functions attained an extremely wide popularity. It is sufficient just to point out the great increase in the number of mathematical works containing the delta function. Physicists have long been using so-called singular functions, although these cannot be properly defined within the framework of classical function theory. The simplest of the singular functions is the delta function $\delta(x - x_0)$. As the physicists define it, this function is "equal to zero everywhere except at x_0 where it is infinite, and its integral is one." It is unnecessary to point out that according to the classical definition of a function and an integral these conditions are inconsistent. One may, however, attempt to analyse the concept of a singular function in order to exhibit its actual content. First of all, we remark that in solving any specific problems of mathematical physics, the delta function (and other singular functions) occur as a rule only in the intermediate stages. *If the singular function occurs at all in the final result, it is only in an integrand where it is multiplied by some other sufficiently well-behaved function. There is therefore no actual necessity for answering the question of just what a singular function is per se; it is sufficient to know what is meant by the integral of a product of a singular function and a "good" function. For instance, rather than answer the question of what a delta function is, it is sufficient for our purposes to point out that for any sufficiently well-behaved function* $\varphi(x)$*we have*

$$\int_{-\infty}^{+\infty} \delta(x - x_0)\varphi(x)\mathrm{d}x \;=\; \varphi(x_0)$$

2. Definition of regular generalized function and singular generalised function

Test Functions

First of all we must define the set of those functions which we have conditionally called " sufficiently good," and on which our functionals will act. As this set we shall choose the set K of all real functions [3] $\varphi(x)$ with continuous derivatives of all orders and with bounded support, [4] which means that the function vanishes outside of some bounded region (which may be different for each of the $\varphi(x)$). We shall call these functions the *test functions*,and we shall call K the *space of test functions.* The test functions can be added and multiplied by real numbers to yield new test functions, so that K is a linear space. Further, we shall say that a sequence $\varphi_1(x), \varphi_2(x), ..., \varphi_n(x), ...$ of test functions *converges to zero in* K if all these functions vanish outside a certain fixed bounded region, the same for al of them, and converge uniformly to zero (in the usual sense) together with their derivatives of any order
As an example of such a function which vanishes for

[1] The reference is the space R^n with the traditional metrics. We will use the integration of Riemann.
[2] see also [7] pag. 225 J. Sebastiao e Silva.
[3] As a rule we shall let $x = (x_1, x_2, ..., x_n)$ denote a point in the n-dimensional space R^n. On first reading the reader may visualize x as a point on the line.
[4] The support of a continuous function $\varphi(x)$ is the closure of the set on which $\varphi(x) \neq 0$.

$$r \;\equiv\; |x| = \sqrt{\Sigma x_i^2} \geq a,$$

consider

$$\varphi(x,a) = \begin{cases} \exp\left(\frac{-a^2}{a^2-r^2}\right) & \text{for } \; r < a, \\ 0 & \text{for } \; r \geq a. \end{cases}$$

The sequence $\varphi_\nu(x) = \nu^{-1}\varphi(x,a)(\nu = 1,2,...)$ converges in K. The sequence $\varphi_\nu(x) = \nu^{-1}\varphi(x,a)(\nu = 1,2,...)$ converges to zero uniformly together with all its derivatives, but does not converge to zero in K, since there exists no common bounded region outside which all these functions vanish.
There exist many different kinds of functions in K. For instance, far a given continuous function $f(x)$ with bounded support there always exists a function $\varphi(x)$ in K arbitrarily close to it, i.e., such that far all x and for any $\epsilon > 0$,

$$|f(x) - \varphi(x)| < \epsilon.$$

3. Generalized Functions

We shall say that f is a *continuous linear functional* on K if there exists some rule according to which we can associate with every $\varphi(x)$ in K a real number (f,φ) satisfying the following conditions.
(a) For any two real numbers α_1 and α_2 and any two functions φ_1 and φ_2 in K we have
$(f, \alpha_1\varphi_1 + \alpha_2\varphi_2) \;=\; \alpha_1(f,\varphi_1) + \alpha_2(f,\varphi_2)$ (linearity of f).
(b) If the sequence $\varphi_1, \varphi_2, ..., \varphi_n...$ converges to zero in K, then the sequence $(f,\varphi_1), (f,\varphi_2), ...(f,\varphi_n), ...$ converges to zero (continuity of f).
For instance, let $f(x)$ be absolutely integrable in every bounded region of R^n (we shall call such functions *locally summable*. By means of such a function we can associate every $\varphi(x)$ in K with

$$(f,\varphi) \;=\; \int_{R^n} f(x)\varphi(x)\mathrm{d}x, \qquad (1)$$

where the integral is actually taken only over the bounded region in which $\varphi(x)$ fails to vanish. It is easily verified that conditions (a) and (b) are satisfied for the functional f. Condition [(b) follows, in particular, from the possibility of passing to the limit under the integral sign when the functions in the integrand converge uniformly) in a bounded region.
Equation (1) represents a very special kind of continuous linear functional on K. Other kinds of functionals are easily shown to exist. For instance, the functional which associates with every $\varphi(x)$ its value at $x_0 = 0$ is obviously linear and continuous. It is easily shown, however, that this functional cannot be written in the form of (1) with any locally summable function f(x).
Indeed, let us assume that there exists some locally summable function $f(x)$ such that for every $\varphi(x)$ in K we have

$$\int_{R^n} f(x)\varphi(x)\mathrm{d}x \;=\; \varphi(0)$$

In particular, for the function $\varphi(x,a)$ discussed in the previous section, we have

$$\int_{R^n} f(x)\varphi(x,a)\mathrm{d}x \;=\; \varphi(0,a) \;=\; e^{-1} \qquad (2)$$

But as $a \to 0$ the integral on the left converges to zero, which contradicts Eq. (2).

We shall call the functional we are now discussing the δ function in accordance with the established terminology (although this terminology is inaccurate, since the delta function is not a function in the classical sense of the word), and we shall denote it by $\delta(x)$. We thus write

$$(\delta(x), \varphi(x)) \;=\; \varphi(x).$$

One often has to deal with the "translated" delta function, or the functional $\delta(x - x_0)$ defined by We now define a *generalized function* as any linear continuous functional defined on K. Those functionals which can be given by an equation such as (1) shall be called *regular*, and all others (including them delta function) will be called *singular*. We shall call the regular generalized function f defined by [3]

$$(f, \varphi) = C \int \varphi(x) \mathrm{d}x = \int C\varphi(x) \mathrm{d}x$$

the constant C. For instance, the unit generalized function is defined by

$$(1, \varphi) = \int \varphi(x) \mathrm{d}x.$$

It can be shown (see Volume II, Chapter I, Section 1.5) that if one knows the value of a regular functional on all functions of K, the function $f(x)$ corresponding to it can be established everywhere except on a set of measure zero (almost everywhere). This means that to different functions $f_1(x)$ and $f_2(x)$ correspond different generalized functions (i.e., for some functions in K these functionals have different values). Thus the set of ordinary locally summable functions can be considered a subset of the set of all generalized functions. For this reason, it is sometimes convenient to use the notation $f(x)$ for generalized functions, as in the case of the delta function, although we may no longer speak of the value of a generalized function at a given point (so that, rigorously speaking, the notation $f(x)$ is meaningless for a generalized function). In addition, we shall sometimes denote $f(x)$ by $\int f(x)\varphi(x)\mathrm{d}x$, although according to ordinary analysis such notation is meaningless. For instance, we will sometimes write $\int \delta(x)\varphi(x)\mathrm{d}x$ instead of $(\delta(x), \varphi(x))$. Thus $\int \varphi(x)\mathrm{d}x = 0$. We shall denote the set of all generalized functions by $K^{'}$.

4. Traditional definition of Dirac's δ

Let's try to define the $\delta(x)$ by means of a suitable sequence:
A sequence $\{\delta(x)\}$ of functions defined in $(-\infty, +\infty)$ is said $\delta -$ *sequence* if verifies the following property:
α) there is a sequence of positive numbers $\{\varepsilon_n\}$, decreasing and infinitesimal, such that $\delta_n(x) = \delta_n(-x) > 0$ for $x\epsilon(-\varepsilon_n, \varepsilon_n)$ and $\delta_n(x) = 0$ elsewhere;
β) $\delta_n(x)$ is of class C^∞ (waves must nullify together with all the derived, in the points $(-\varepsilon_n, \varepsilon_n)$;
γ) it is $\int_{-\varepsilon_n}^{+\varepsilon_n}(x)dx = 1$
For example is immediate to verify that:

$$(1) \qquad \delta_n(x) = \begin{cases} 0 & (-\infty < x \leq -\varepsilon_n) \\ \frac{1}{c_n} e^{-\frac{1}{\varepsilon_n^2 - x^2}} & (-\varepsilon_n \leq x < \varepsilon_n) \\ 0 & (-\varepsilon_n \leq x < +\infty) \end{cases}$$

Where c_n is:

$$(1a) \qquad c_n = \int_{-\varepsilon_n}^{\varepsilon_n} e^{-\frac{1}{\varepsilon_n^2 - x^2}} dx.$$

is a $\delta -$ *sequence*

[3] We shall suppress the symbol R^n on the integral sign whenerver the integral is taken over the entire space

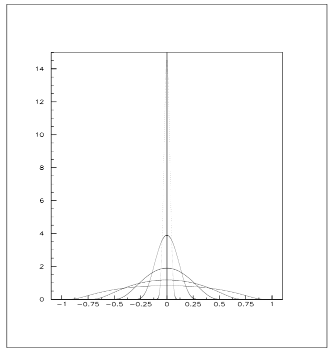

FIGURE 4.1. from (1)-(1a)

Looking at the figure 1.1 we can see that we can consider the sequence as a functional in two ways. In the first way, as the value of the intersection of the various curves with the ordinate axis. In the second way, as the area (or the integral or measure) of the various curves. The question we have to ask is: what happens to infinity?. To answer this question we consider the areas under the curves of the succession as sets of points. The limit of these sets is the semiaxis of the ordinates whose area is 0. Therefore in both cases there is a discontinuity to infinity which makes the δ not well defined. We are thus in the presence of sequences approximating a function which is defined only in words, which is not a definition. In order to search for a definition we must have first a functional that for $\delta = +\infty$ at $x = 0$ and zero for a nonvanishing x. Appropriately using the function $f(t) = cos(\omega t)$ we will see that we have first of all the sought functional.

5. Per se definition of Dirac's δ

DEFINITION. $\delta(x)$ is a limit of a continuous functional, of a parameter, defined on $[0, +\infty)$, which it is zero everywhere and $(+\infty)$ at point 0. first property:

$$\int_0^{+\infty} \delta(x)\mathrm{d}x = 1. \tag{5.1}$$

second property:

$$\int_0^{+\infty} f(x)\delta(x)\mathrm{d}x = f(0) \tag{5.2}$$

where $f(x)$ is any continuous function of x.
We note that the properties concern a neighbourhood of $x = 0$.

5.1. A continuous functional which to the limit is 0 everywhere and $+\infty$ at point 0.

It is well known that the Riemann integral [1]

$$f(t) = \int_0^{\infty} \cos(\omega t)\ \mathrm{d}\omega \quad t \geq 0 \quad \omega\ (-\infty, +\infty) \tag{5.3}$$

does not exist.
We will see that with Cesàro's summability we can give it a meaning.
The definition of Cesàro's summability of order 1 (C,1) is:

[1] In this article we use Riemann's integration, not Lebesgue's integration. (see [3])

$$F(t) \;=\; \lim_{\lambda\to+\infty} \int_0^{\lambda} (1-\frac{\omega}{\lambda})\;\cos(\omega t)\;\mathrm{d}\omega \tag{5.4}$$

We put $\omega t = u$ then (5.1) becomes

$$\lim_{\lambda\to+\infty} \frac{1}{t} \int_0^{t\lambda} (1-\frac{u}{t\lambda})\;\cos(u)\;\mathrm{d}u \qquad t>0 \tag{5.5}$$

We get undefined integral and apply integration by parts, for $t>0$:

$$\int (1-\frac{u}{t\lambda})\;\cos(u)\;\mathrm{d}u \qquad =$$

$$= \qquad (1-\frac{u}{t\lambda})\;\sin(u)\;\;-\int \frac{-1}{t\lambda}\;\sin(u)\mathrm{d}u \qquad =$$

$$= \qquad (1-\frac{u}{t\lambda})\;\sin(u)\;\;-\frac{1}{t\lambda}\cos(u)\mathrm{d}u$$

But from (5.5):

$$\Big[(1-\frac{u}{t\lambda})\;\sin(u)\;\;-\frac{1}{t\lambda}\cos(u)\Big]\Big|_0^{t\lambda} \quad =$$

$$= \;\Big[\Big(1-\frac{t\lambda}{t\lambda}\Big)\;\sin(t\lambda)\;\;-\frac{1}{t\lambda}\cos(t\lambda)\Big] \;-\; \Big[\Big(1-\frac{0}{t\lambda}\Big)\;\sin(0)\;\;-\frac{1}{t\lambda}\cos(0)\Big] \quad =$$

$$= \;\frac{1}{t\lambda} \quad -\frac{\cos(t\lambda)}{t\lambda}$$

$$\lim_{\lambda\to+\infty} \;\frac{1}{t}\;\Big[\frac{1}{t\lambda} \quad -\frac{\cos(t\lambda)}{t\lambda}\Big] \;=\; 0 \quad \text{for all } \; t>0$$

$$F(0) \;=\; \lim_{\lambda\to+\infty} \int_0^{\lambda} (1-\frac{0}{\lambda}\;)\;\cos(\omega 0)\;\mathrm{d}\omega \quad =$$

$$= \;\lim_{\lambda\to+\infty} \int_0^{\lambda} \mathrm{d}\omega \quad = \;+\infty \qquad\qquad (\omega t \;=\; 0)$$

Finally we have:

$$F(t) = \begin{cases} 0 & \text{for} \qquad t>0 \\ +\infty & \text{for} \qquad t=0 \text{ or } \omega=0 \end{cases} \tag{5.6}$$

5.2. **Dirac's δ properties.** We introduce now a set of function which it verifies definition of δ.

Theorem 1. *Given a continuous function $f(u) > 0$ on a closed interval $[0, b]$, with first derivative continue and $\neq 0$. Given a $0 \leq \eta \leq b$.*
Given a continuous functional of η

$$W(\eta) = \begin{cases} \frac{f(u)}{\int_0^\eta f(u)\ \mathrm{d}u} & \text{for } \ 0 \leq u \leq \eta \\ 0 & \text{for } \ \eta < u \leq b \end{cases} \tag{5.7}$$

then

$$\lim_{\eta \to 0^+} \ W(\eta) \ = \ +\infty \tag{5.8}$$

and

$$\int_0^\eta W(\eta)\mathrm{d}u = 1 \tag{5.9}$$

Proof. For the Weierstrass theorem $f(u)$ have a maximum $M \neq 0$ and a minimum $m \neq 0$. Then

$$m \lim_{\eta \to 0^+} \frac{1}{\int_0^\eta \ f(u)\mathrm{d}u} \leq \lim_{\eta \to 0^+} \ W(\eta) \ \leq \ M \lim_{\eta \to 0^+} \frac{1}{\int_0^\eta \ f(u)\mathrm{d}u} \tag{5.10}$$

but

$$m \lim_{\eta \to 0^+} \frac{1}{\int_0^\eta \ f(u)\mathrm{d}u} \ = \ +\infty \tag{5.11}$$

$$M \lim_{\eta \to 0^+} \frac{1}{\int_0^\eta \ f(u)\mathrm{d}u} \ = \ +\infty \tag{5.12}$$

Then (5.7) is valid.
Now we prove (5.8)

$$\lim_{\eta \to 0^+} \frac{\int_0^\eta f(u)\ \mathrm{d}u}{\int_0^\eta f(u)\ \mathrm{d}u} = \frac{0}{0}$$

We apply de l'Hôpital theorem

$$\lim_{\eta \to 0^+} \frac{\frac{d}{d\eta} \int_0^\eta f(u)\ \mathrm{d}u}{\frac{d}{d\eta} \int_0^\eta f(u)\ \mathrm{d}u} \ = \ \lim_{\eta \to 0^+} \frac{f(\eta) - f(0)}{f(\eta) - f(0)} = \frac{0}{0}$$

We apply de l'Hôpital theorem again

$$\lim_{\eta\to 0^+} \frac{\frac{d}{d\eta}[f(\eta)-f(0)]}{\frac{d}{d\eta}[f(\eta)-f(0)]} = \lim_{\eta\to 0^+} \frac{f^{'}(\eta)}{f^{'}(\eta)} = \frac{f^{'}(0)}{f^{'}(0)} = 1.$$

$W(\eta)$ is a function which is 0 everywhere on interval $(0,b]$ and $(+\infty)$ at point 0, and verify (5.1)and (5.2).
So $\delta(\eta) = W(\eta)$.
Q.E.D.

□

Theorem 2. *Given a continuous function $f(u) > 0$ in a closed interval $[0,b]$, with first derivative continue and $\neq 0$.*
Given a continuous function $\phi(x)$ in the closed interval $[0,b]$. Given a $0 \le \eta \le b$.
Given a function

$$Z(u) = \frac{f(u)\ \phi(u)}{\int_0^\eta f(u)\ \mathrm{d}u} \qquad 0 \le u \le b$$

then

$$\int_0^\eta Z(u)\mathrm{d}u = \phi(0)$$

Proof.

$$\int_0^\eta Z(u)\mathrm{d}u = \frac{\int_0^\eta f(u)\phi(u)\mathrm{d}u}{\int_0^\eta f(u)\mathrm{d}u}$$

Now we apply the 'ONE DIMENSIONAL MEAN VALUE THEOREM' to function $f(u)$.

$$\lim_{\eta\to 0^+} \frac{\int_0^\eta f(u)\phi(u)\mathrm{d}u}{\int_0^\eta f(u)\mathrm{d}u} = \lim_{\eta\to 0^+} \frac{\phi(x_0)\int_0^\eta f(u)\mathrm{d}u}{\int_0^\eta f(u)\mathrm{d}u} = \phi(0) \qquad 0 \le x_0 \le \eta \le b$$

Q.E.D. □

5.3. **Cesàro again.** Let $\varepsilon \ge 0$ be, now we introduce the function $G(t)$

$$G(t) = \frac{\cos(\omega t)}{\int_0^\varepsilon \cos(\omega t)\ \mathrm{d}\omega} \qquad t \ge 0\ ,\ 0 < p \le \omega t \le \varepsilon < \pi/2 \tag{5.13}$$

For $t = 0$

$$G(0) = \lim_{\varepsilon\to 0^+} \frac{\cos(\omega 0)}{\int_0^\varepsilon \cos(\omega 0)\mathrm{d}\omega} = +\infty$$

If $t > 0$ then $\cos(\omega t) > 0$, and its derivative $[-t\ \sin(\omega t)]$ is $\neq 0$.
So the conditions of Theorem 1 are verified. Then for Theorem 1 [2]

$$\lim_{\varepsilon\to 0^+} G(t) = \lim_{\varepsilon\to 0^+} \frac{\cos(\omega t)}{\int_0^\varepsilon \cos(\omega t)\ \mathrm{d}\omega} = +\infty \tag{5.14}$$

[2] In this case there is no need resort Cesàro

and:

$$\int_0^{\varepsilon} G(t)\mathrm{d}\omega = 1. \tag{5.15}$$

Is the formula (5.15) valid to the limit $+\infty$?

$$\lim_{\lambda\to+\infty}\int_0^{\lambda} G(t)\mathrm{d}\omega = 1 \tag{5.16}$$

For $t > 0$ and applying (5.6), with $\lambda \neq 0$:

$$\frac{\int_0^{+\infty} (1-\frac{\omega}{\lambda})\ \cos(\omega t)\ \mathrm{d}\omega}{\int_0^{+\infty} (1-\frac{\omega}{\lambda})\ \cos(\omega t)\mathrm{d}\omega} = \frac{0}{0} \tag{5.17}$$

Now we apply the de l'Hôpital theorem, with $\lambda \neq 0$:

$$\lim_{\lambda\to+\infty}\frac{(1-\frac{\omega}{\lambda})\ \cos(\omega t)\Big|_0^{\lambda}}{(1-\frac{\omega}{\lambda})\ \cos(\omega t)\Big|_0^{\lambda}} \tag{5.18}$$

but

$$\lim_{\lambda\to+\infty}\left\{\left(1-\frac{\omega}{\lambda}\right)\ \cos(\omega t)\Bigg|_0^{\lambda}\right\} = \lim_{\lambda\to+\infty}\left\{\left(1-\frac{\lambda}{\lambda}\right)\ \cos(\lambda t) - \left(1-\frac{0}{\lambda}\right)\ \cos(0\ t)\right\} = 0-1=-1$$

Then (5.16) is valid.

Given now an $\varepsilon \geq 0$ we introduce a function $h(\omega)$ defined in the closed interval $[0,+\epsilon] \geq 0$ and there continue. Now we define the function $H(t)$: [3]

$$H(t) = \frac{1}{\int_0^{\varepsilon} \cos(\omega t)\mathrm{d}\omega}\int_0^{\varepsilon} \cos(\omega t)\ h(t-\omega)\ \mathrm{d}\omega \qquad t \geq 0\ ,\ 0 < p \leq \omega t \leq \varepsilon < \ \pi/2 \tag{5.19}$$

[3] sifting property (see [5]) pag.61

For (5.6) is necessary to study H(t) on the neighbourhood of 0. Therefore now we apply the 'ONE DIMENSIONAL MEAN VALUE THEOREM' to numerator.

$$\lim_{\varepsilon\to 0}\frac{h(t-\xi)\int_0^{\varepsilon}\ \cos(\omega t)\ \mathrm{d}\omega}{\int_0^{\varepsilon}\ \cos(\omega t)\ \mathrm{d}\omega}\ =\ \lim_{\varepsilon\to 0}\ h(t-\xi)\ =\ h(t)\qquad 0\leq\xi\leq\varepsilon \tag{5.20}$$

CONCLUTIONS

We saw that with the Cesàro's summability we can have a function which is defined on $[0,+\infty]$, it is zero everywhere and $(+\infty)$ at point 0. This function verifies Dirac's δ properties.
Theorem 1 and Theorem 2 identify a set of functions that they define the Dirac's δ too.

6. Open issues and some processing

6.1. **Cesàro's summability as Cohen's forcing?** In the mathematical discipline of set theory, forcing is a technique invented by Paul Cohen for proving consistency and independence results. It was first used, in 1963, to prove the independence of the axiom of choice and the continuum hypothesis from Zermelo-Fraenkel set theory. Forcing was considerably reworked and simplified in the 1960s, and has proven to be an extremely powerful technique both within set theory and in areas of mathematical logic such as recursion theory.
Descriptive set theory uses both the notion of forcing from recursion theory as well as set theoretic forcing. Forcing has also been used in model theory but it is common in model theory to define genericity directly without mention of forcing.

The Cesàro summability is part of the analysis. It is such that all the properties and the axioms which the analysis is based on are still valid, but it allows us for example to give full meaning to integrals that would be meaningless using the classical analysis. Therefore it is worth trying to assimilate the extension that determines the summability in the analysis, to the technique of forcing. This is an open problem.

6.2. **Note on the theory of integration.** In his text on the theory of integration Lebesgue rewinded the history of the theory and pointed out that it is subject to the evolution of the theory of functions. At first the integral (considered as an area) concerned the continuous functions. Then with the step-functions, up to the Dirichelet function, the concept of the integral evolved accordingly. However, in order to integrate the Dirichelet function, one had to resort to the use of countable covers i.e. the systematic use of the actual infinity. This poses a problem of principle. In fact, starting from the Greeks, until that time, mathematicians had refused it. Moreover, once the the problem for Dirichelet?s function is solved, Peano and Vitali found two curves, both of them being not Lebesgue integrable. This process can be generalized. The question that one should ask as a physicist is as follows: why to usually use the integration of Lebesgue?. This is justified only if one uses functions such as that of Dirichelet.

6.3. **Note on the relationship between mathematics and physics.** To address this problem you first need to remember that in the foundations of mathematics, the Russell's antinomy, discovered by Bertrand Russell in 1901, showed that the naive set theory created by Georg Cantor leads to a contradiction.
The Russell antinomy is:
According to naive set theory, any definable collection is a set. Let R be the set of all sets that are not members of themselves. If R is not a member of itself, then its definition dictates that it must contain itself, and if it contains itself, then it contradicts its own definition as the set of all sets that are not members of themselves. This contradiction is Russell's antinomy. Symbolically:
Let $R = \ x|\ x \notin x\ $, then $R \in R \Longleftrightarrow R \notin R$

The first incompleteness theorem states that no consistent system of axioms whose theorems can be listed by an "effective procedure" (e.g., a computer program, but it could be any sort of algorithm) is capable of proving all truths about the relations of the natural numbers (arithmetic). For any such system, there will always be statements about the natural numbers that are true, but that are unprovable within the system.

The second incompleteness theorem, an extension of the first, shows that such a system cannot demonstrate its own consistency.

7. Tarski

Tarski [4] : Truth and Proof

"The quetion now arises whether the notion of truth can be precisely defined, and thus a consistent and adequate usage of this notion can be established at least for the semantically restricted languages of scientific discourse. Under certain conditions the answer to this question proves to be affirmative. The main conditions imposed on the language are that its full vocabulary should be available and its syntactical rules concerning the formation of sentences and other meaningful expressions from words listed in the vocabulary should be precisely formulated. Furthermore, the syntactical rules should be purely formal, that is, they should refer exclusively to the form (the shape) of expressions; the function and the meaning of an expression should depend exclusively on its form. In particular, looking at an expression, one should be able in each case to decide whether or not the expression is a sentence. It should never happen that an expression functions as a sentence at one place while an expression of the same form does not function so ar some other place, or that a sentence can be asserted in one context while a sentence of the same form can be denied in another. (Hence it follows, in particular, that demonstrative pronouns and adverbs such as "this" and "here" should not occur in the vocabulary of the language.)
Languages that satisfy these conditions are referred to as formalized languages. When discussing a formalized language there is no need to distinguish between expressions of the same form which have been written or uttered in different places; one often speaks of them as if they were one and me same expression. The reader may have noticed we sometimes use this way of speaking even when discussing a natural language, that is, one which is not formalized; we do so for the sake of simplicity, and only in those cases in which there seems to be no danger of confusion.
Formalized languages are fully adequate for the presentation of logical and mathematical theories; I see no essential reasons why they cannot be adapted for use in other scientific disciplines and in particular to the development of theoretical parts of empirical sciences. I should like to emphasize that, when using the term "formalized languages", I do not refer exclusively to linguistic systems that are formulated entirely in symbols, and I do not have in mind anything essentially opposed to natural languages. On the contrary, the only formalized languages that seem to be of real interest are those which are fragments of natural languages (fragments provided with complete vocabularies and precise syntactical rules) or those which can ar least be adequately translated into natural languages.

[4][[Scientific American, June 1969,63-70,775-77]]

There are some further conditions on which the realization of our program depends. *We should make a strict distinction between the language which is the object of our discussion and for which in particular we intend to construct the definition of truth, and the language in which the definition is to be formulated and its implications are to be studied. The latter is referred to as the metalanguage and the former as the object-language. The metalanguage must be sufficiently rich; in particular, it must include the object-language as a part.* In fact, according to our stipulations, an adequate definition of truth will imply as consequences all partial definitions of this notion, that is, all equivalences of form (3):

"p" is true if and only if p,

where "p" is to be replaced (on both sides of the equivalence) by an arbitrary sentence of the object-language. Since all these consequences are formulated in the metalanguage, we conclude that every sentence of the object-language must also be a sentence of the metalanguage. Furthermore, the metalanguage must contain names for sentences (and other expressions) of the object-language, since these names occur on the left sides of the above equivalences. le must also contain some further terms that are needed for the discussion of the object-language, in fact terms denoting certain special sets of expressions, relations between expressions, and operations on expressions; for instance, we must be able to speak of the set of all sentences or of the operation of juxtaposition, by means of which, putting one of two given expressions immediately after the other, we form a new expression. Finally, by defining truth, we show that semantic terms (expressing relations between sentences of the object-language and objects referred to by these sentences) can be introduced in the metalanguage by means of definitions. *Hence we conclude that the metalanguage which provides sufficient means for defining truth must be essentially richer than the object-language; it cannot coincide with or be translatable into the latter, since otherwise both languages would turn out to be semantically universal, and the antinomy of the liar could be reconstructed in both of them....*

We shall return to this question in the last section of this article. If all the above conditions are satisfied, the construction of the desired definition of truth presents no essential difficulties. Technically, however, it is too involved to be explained here in detail. For any given sentence of the object-language one can easily formulate the corresponding partial definition of form (3). Since, however, the set of all sentences in the object-language is as a rule infinite, whereas every sentence of the metalanguage is a finite string of signs, we cannot arrive at a general definition [[69]] simply by forming the logical conjunction of all partial definitions. Nevertheless, what we eventually obtain is in some intuitive sense equivalent to the imaginary infinite conjunction. Very roughly speaking, we

proceed as follows. First, we consider the simplest sentences,
which do not include any other sentences as parts; for these
simplest sentences we manage to define truth directly (using
the same idea that leads to partial definitions). Then, making
use of syntactical rules which concern the formation of
more complicated sentences from simpler ones, we extend
the definition to arbitrary compound sentences; we apply
here the method known in mathematics as definition by recursion.
(This is merely a rough approximation of the actual
procedure. For some technical reasons the method of recursion
is actually applied to define, not the notion of truth,
bur the related semantic notion of satisfaction. Truth is then
easily defined in terms of satisfaction.)
On the basis of the definition thus constructed we can develop
the entire theory of truth. In particular, we can derive
from it, in addition to all equivalences of form (3), some
consequences of a general nature, such as the famous laws of
contradiction and of excluded middle. By the first of these
laws, no two sentences one of which is the negation of the
other can both be true; by the second law, no two such sentences
can both be false."

Remember now the sixth Hilbert's problem: Can physics be axiomatized?
We generally think that mathematics is the language of physics. So when we try to axiomatize physics we axiomatized mathematics that we use to explain physical phenomena. We then generalize the problem posed above regarding the use Lebesgue's integration. In the case of Lesbegue's integration we accept implicitly the actual infinitive. In the other case, in which we attempt to axiomatize phisics, we ignore the problems arising from the crisis of the foundations of mathematics.

The introduction by Tarski of the concept of metalanguage, as we have seen, begin by the following assumption: (1) "the snow is white" is true if and only if the snow is white.
(1') "the snow is white" is false if and only if the snow is not white.
In general
"p" is true if and only if p
"p" is false if and only if not p
where "p" is to be replaced (on both sides of the equivalence) by an arbitrary sentence of the object-language.

These statements indicate that there is an objective reality and about it we can make judgements of truth or falsehood on the basis of observations. In this way, in the experimental sciences we separate language, with all its semantic problems, from processing experimental data.

Employing the concept of metalanguage we can create the following scheme:
1) the axioms of physics are the laws derived from processing experimental data.
2) mathematics is the metalanguage of physics.

The metalanguage which provides sufficient means for defining truth must be essentially richer than the object-language; it cannot coincide with or be translatable into the latter, since otherwise both languages would turn out to be semantically universal, and the antinomy of the liar could be reconstructed in both of them.

7.1. **Conjecture.** If the statements of section 5, 6 and 7 are all verified there will exist a mathematics that satisfies the sixth Hilbert's problem.

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