ISTITUTO NAZIONALE DI FISICA NUCLEARE
Laboratori Nazionali di Frascati

**INFN-12-05/LNF**
**April 26, 2012**

# Cesàro's summability vs Dirac's $\delta$

M. Pallotta
*INFN-Laboratori Nazionali di Frascati, via E. Fermi 40, 00044 Frascati, Italy*

**Abstract**
We will see that with the Cesàro's summability we can have a function which is defined on $[0, +\infty)$, it is zero everywhere and $(+\infty)$ at point $0$. Later we will study its properties. In addition we will identify a set of functions that they define the Dirac's $\delta$ too.

*Published by* **SIDS-Pubblicazioni**
*Laboratori Nazionali di Frascati*

## 1. Introduction

Some definitions of Dirac's $\delta$ are:[1]

**Definition 1.** $\delta(x)$ is a function, depending of a parameter, defined on $[0, +\infty)$, which it is zero everywhere and $(+\infty)$ at point 0.

**Definition 2.** $\delta(x)$ is a limit of a continuous functional, of a parameter, defined on $[0, +\infty)$, which it is zero everywhere and $(+\infty)$ at point 0.

$\delta(x)$ is such that have two properties:
first property:

$$\int_0^{+\infty} \delta(x)\mathrm{d}x \;=\; 1. \tag{1.1}$$

second property:

$$\int_0^{+\infty} f(x)\delta(x)\mathrm{d}x \;=\; f(0) \tag{1.2}$$

where $f(x)$ is any continuous function of $x$.
We note that the properties concern a neighbourhood of $x = 0$.
Then we can give

**Definition 3.** $\delta(x)$ is a limit of a continuous functional, of a parameter, defined on $[0, b < +\infty]$, which it is zero everywhere and $(+\infty)$ at point 0.

## 2. A continuous functional which to the limit is 0 everywhere and $+\infty$ at point 0.

It is well known that the Riemann integral [2]

$$f(t) \;=\; \int_0^{\infty} \cos(\omega t)\ \mathrm{d}\omega \quad t \geq 0 \qquad \omega\ (-\infty, +\infty) \tag{2.1}$$

does not exist.
We will see that with Cesàro's summability we can give it a meaning.
The definition of Cesàro's summability of order 1 (C,1) is:

$$F(t) \;=\; \lim_{\lambda \to +\infty} \int_0^{\lambda} (1 - \frac{\omega}{\lambda})\ \cos(\omega t)\ \mathrm{d}\omega \tag{2.2}$$

We put $\omega t = u$ then (1.1) becomes

$$\lim_{\lambda \to +\infty} \frac{1}{t} \int_0^{t\lambda} (1 - \frac{u}{t\lambda})\ \cos(u)\ \mathrm{d}u \qquad t > 0 \tag{2.3}$$

We get undefined integral and apply integration by parts, for $t > 0$:

$$\int (1 - \frac{u}{t\lambda})\ \cos(u)\ \mathrm{d}u \qquad =$$

[1] see also [6] pagg 58-61 or APPENDIX C.
[2] In this article we use Riemann's integration, not Lebesgue's integration. ( see [3] )

$$= \quad (1-\frac{u}{t\lambda})\ \sin(u)\ -\int \frac{-1}{t\lambda}\ \sin(u)\mathrm{d}u \quad =$$

$$= \quad (1-\frac{u}{t\lambda})\ \sin(u)\ -\frac{1}{t\lambda}\cos(u)\mathrm{d}u$$

But from (2.3):

$$\Big[(1-\frac{u}{t\lambda})\ \sin(u)\ -\frac{1}{t\lambda}\cos(u)\Big]\Big|_0^{t\lambda} \quad =$$

$$= \ \Big[\Big(1-\frac{t\lambda}{t\lambda}\Big)\ \sin(t\lambda)\ -\frac{1}{t\lambda}\cos(t\lambda)\Big]\ -\ \Big[\Big(1-\frac{0}{t\lambda}\Big)\ \sin(0)\ -\frac{1}{t\lambda}\cos(0)\Big] \quad =$$

$$= \ \frac{1}{t\lambda}\ -\frac{\cos(t\lambda)}{t\lambda}$$

$$\lim_{\lambda\to+\infty}\ \frac{1}{t}\ \Big[\frac{1}{t\lambda}\ -\frac{\cos(t\lambda)}{t\lambda}\Big]\ =\ 0 \quad \text{for all} \quad t>0$$

$$F(0)\ =\ \lim_{\lambda\to+\infty}\int_0^{\lambda}(1-\frac{0}{\lambda}\ )\ \cos(\omega 0)\ \mathrm{d}\omega \quad =$$

$$= \lim_{\lambda\to+\infty}\int_0^{\lambda}\mathrm{d}\omega \ \ =\ +\infty \qquad (\omega t\ =\ 0)$$

Finally we have:

$$F(t)=\begin{cases}0 & \text{for} \qquad t>0\\ +\infty & \text{for} \qquad t=0 \text{ or } \omega=0\end{cases} \tag{2.4}$$

## 3. Dirac's $\delta$ properties

We introduce now a set of function which it verifies definition of $\delta$.

**Theorem 4.** *Given a continouos function $f(u)>0$ on a closed interval $[0,b]$, with first derivative continue and $\neq 0$. Given a $0\le\eta\le b$.*
*Given a continuous functional of $\eta$*

$$W(\eta)=\begin{cases}\frac{f(u)}{\int_0^{\eta} f(u)\ \mathrm{d}u} & \text{for} \quad 0\le u\le\eta\\ 0 & \text{for} \quad \eta<u\le b\end{cases} \tag{3.1}$$

*then*

$$\lim_{\eta\to 0^+}\ W(\eta)\ =\ +\infty \tag{3.2}$$

*and*

$$\int_0^{\eta} W(\eta)\mathrm{d}u = 1 \tag{3.3}$$

*Proof.* For the Weierstrass theorem $f(u)$ have a maximum $M \neq 0$ and a minimum $m \neq 0$. Then

$$m \lim_{\eta \to 0^+} \frac{1}{\int_0^{\eta} \ f(u)\mathrm{d}u} \leq \lim_{\eta \to 0^+} \ W(\eta) \ \leq \ M \lim_{\eta \to 0^+} \frac{1}{\int_0^{\eta} \ f(u)\mathrm{d}u} \tag{3.4}$$

but

$$m \lim_{\eta \to 0^+} \frac{1}{\int_0^{\eta} \ f(u)\mathrm{d}u} \ = \ +\infty \tag{3.5}$$

$$M \lim_{\eta \to 0^+} \frac{1}{\int_0^{\eta} \ f(u)\mathrm{d}u} \ = \ +\infty \tag{3.6}$$

Then (3.2) is valid.
Now we prove (3.3)

$$\lim_{\eta \to 0^+} \frac{\int_0^{\eta} f(u) \ \mathrm{d}u}{\int_0^{\eta} f(u) \ \mathrm{d}u} = \frac{0}{0}$$

We apply de l'Hôpital theorem

$$\lim_{\eta \to 0^+} \frac{\frac{d}{d\eta} \int_0^{\eta} f(u) \ \mathrm{d}u}{\frac{d}{d\eta} \int_0^{\eta} f(u) \ \mathrm{d}u} \ = \ \lim_{\eta \to 0^+} \frac{f(\eta) - f(0)}{f(\eta) - f(0)} = \frac{0}{0}$$

We apply de l'Hôpital theorem again

$$\lim_{\eta \to 0^+} \frac{\frac{d}{d\eta}[f(\eta) - f(0)]}{\frac{d}{d\eta}[f(\eta) - f(0)]} \ = \lim_{\eta \to 0^+} \ \frac{f^{'}(\eta)}{f^{'}(\eta)} \ = \frac{f^{'}(0)}{f^{'}(0)} = \ 1.$$

$W(\eta)$ is a function which is 0 everywhere on interval $(0, b]$ and $(+\infty)$ at point 0, and verify (1.1 )and (1.2).
So $\delta(\eta) \ = \ W(\eta)$.
Q.E.D.

□

**Theorem 5.** *Given a continuous function $f(u) > 0$ in a closed interval $[0, b]$, with first derivative continue and $\neq 0$.*
*Given a continuous function $\phi(x)$ in the closed interval $[0, b]$. Given a $0 \leq \eta \leq b$.*
*Given a function*

$$Z(u) \;=\; \frac{f(u)\ \phi(u)}{\int_0^{\eta} f(u)\ \mathrm{d}u} \qquad 0 \le u \le b$$

*then*

$$\int_0^{\eta} \ Z(u)\mathrm{d}u \;=\; \phi(0)$$

*Proof.*

$$\int_0^{\eta} \ Z(u)\mathrm{d}u \;= \frac{\int_0^{\eta} \ f(u)\phi(u)\mathrm{d}u}{\int_0^{\eta} \ f(u)\mathrm{d}u}$$

Now we apply the 'ONE DIMENSIONAL MEAN VALUE THEOREM' [3] to numerator and Theorem 4 to function $f(u)$.

$$\lim_{\eta \to 0^+} \frac{\int_0^{\eta} \ f(u)\phi(u)\mathrm{d}u}{\int_0^{\eta} \ f(u)\mathrm{d}u} \;=\; \lim_{\eta \to 0^+} \frac{\phi(x_0) \int_0^{\eta} \ f(u)\mathrm{d}u}{\int_0^{\eta} \ f(u)\mathrm{d}u} \;=\; \phi(0) \qquad 0 \le x_0 \le \eta \le b$$

Q.E.D. □

## 4. Cesàro again

Let $\varepsilon \ge 0$ be, now we introduce the function $G(t)$

$$G(t) = \frac{\cos(\omega t)}{\int_0^{\varepsilon} \ \cos(\omega t)\ \mathrm{d}\omega} \qquad t \ge 0\ ,\ 0 < p \le \omega t \le \varepsilon < \ \pi/2 \tag{4.1}$$

For $t = 0$

$$G(0) \;=\; \lim_{\varepsilon \to 0^+} \frac{\cos(\omega 0)}{\int_0^{\varepsilon} \ \cos(\omega 0)\mathrm{d}\omega} \;=\; +\infty$$

If $t > 0$ then $\cos(\omega t) > 0$, and its derivative $[-t\ \sin(\omega t)]$ is $\neq 0$.
So the conditions of Theorem 4 are verified. Then for Theorem 4 [4]

$$\lim_{\varepsilon \to 0^+} G(t) = \lim_{\varepsilon \to 0^+} \frac{\cos(\omega t)}{\int_0^{\varepsilon} \ \cos(\omega t)\ \mathrm{d}\omega} \;=\; +\infty \tag{4.2}$$

and:

$$\int_0^{\varepsilon} \ G(t)\mathrm{d}\omega \;= 1. \tag{4.3}$$

Is the formula (4.3) valid to the limit $+\infty$?

$$\lim_{\lambda \to +\infty} \int_0^{\lambda} \ G(t)\mathrm{d}\omega \;= 1 \tag{4.4}$$

[3] see APPENDIX A
[4] In this case there is no need resort Cesàro

For $t > 0$ and applying (2.4), with $\lambda \neq 0$:

$$\frac{\int_0^{+\infty}\ (1-\frac{\omega}{\lambda})\ \cos(\omega t)\ \mathrm{d}\omega}{\int_0^{+\infty}\ (1-\frac{\omega}{\lambda})\ \cos(\omega t)\mathrm{d}\omega} \ = \ \frac{0}{0} \tag{4.5}$$

Now we apply the de l'Hôpital theorem, with $\lambda \neq 0$:

$$\lim_{\lambda\to+\infty} \frac{(1-\frac{\omega}{\lambda})\ \cos(\omega t)\Big|_0^{\lambda}}{(1-\frac{\omega}{\lambda})\ \cos(\omega t)\Big|_0^{\lambda}} \tag{4.6}$$

but

$$\lim_{\lambda\to+\infty} \left\{ \left(1-\frac{\omega}{\lambda}\right)\ \cos(\omega t)\Big|_0^{\lambda} \right\} \ = \lim_{\lambda\to+\infty} \left\{ \left(1-\frac{\lambda}{\lambda}\right)\ \cos(\lambda t)\ -\ \left(1-\frac{0}{\lambda}\right)\ \cos(0\ t) \right\} \ = \ 0-1=-1$$

Then (4.4) is valid.

Given now an $\varepsilon \geq 0$ we introduce a function $h(\omega)$ defined in the closed interval $[0,+\epsilon] \geq 0$ and there continue. Now we define the function $H(t)$: [5]

$$H(t) \ = \ \frac{1}{\int_0^{\varepsilon}\ \cos(\omega t)\mathrm{d}\omega} \int_0^{\varepsilon}\ \cos(\omega t)\ h(t-\omega)\ \mathrm{d}\omega \qquad t\geq 0\ ,\ 0<p\leq \omega t\leq \varepsilon <\ \pi/2 \tag{4.7}$$

For (2.4) is necessary to study H(t) on the neighbourhood of 0. Therefore now we apply the 'ONE DIMENSIONAL MEAN VALUE THEOREM' to numerator. [6]

$$\lim_{\varepsilon\to 0} \frac{h(t-\xi)\int_0^{\varepsilon}\ \cos(\omega t)\ \mathrm{d}\omega}{\int_0^{\varepsilon}\ \cos(\omega t)\ \mathrm{d}\omega} \ = \ \lim_{\varepsilon\to 0}\ h(t-\xi) \ = \ h(t) \qquad 0\leq\xi\leq\varepsilon \tag{4.8}$$

---

[5] sifting property (see [5]) pag.61
[6] See APPENDIX A

## CONCLUTIONS

We saw that with the Cesàro's summability we can have a function which is defined on $[0, +\infty]$, it is zero everywhere and $(+\infty)$ at point 0. This function verifies Dirac's $\delta$ properties.
Theorem 4 and Theorem 5 identify a set of functions that they define the Dirac's $\delta$ too.
It will follow a critical review of distribution theory and generalised functions in the light of this article.

# 5. APPENDIXES

## APPENDIX A [7]

Some definitions:
$S_r$ is a $r$ dimensional space

1) *closed interval* with endpoints $A, B$.
Given in $S_r$ two point $A(a_1, ..., a_r)$, $B(b_1, ..., b_r)$ the set of points $P(x_1, ..., x_r)$ whose coordinates verify the conditions,

$$a_1 \ \leq \ x_1 \ \leq \ b_1, ...., a_r \ \leq \ x_r \ \leq \ b_r.$$

it is said to be *closed interval*.
For r=1
a) *closed interval* ⇒ [a,b]
b) *open interval* ⇒ (a,b)
c) *closed interval on the left and open interval on the right* ⇒ [a,b)
d) *open interval on the left and closed interval on the right* ⇒ (a,b]

2) *Circular domain* of center $C$ and radius $\varrho$

Given in $S_r$ a point $C(c_1, c_2, ....., c_r)$ the set of points $P(x_1, ..., x_r)$ such that

$$(x_1 - c_1)^2 + ... + (x_r - c_r)^2 \ \leq \ \varrho^2$$

It is said *circular domain*
for $r = 1$ it is the *closed interval* $[c_1 - \varrho, c_1 + \varrho]$, for $r = 2$ it is a circle, for $r = 3$ it is a sphere.

*Polygonal*

We define *polygonal of vertices* $(P_1, ..., P_n)$ the set of points of $(n-1)$ segments $(P_1P_2, P_2P_3, ...., P_{n-1}P_n)$ which are the *sides* of poligonal. The sum of the lengths of the sides is called the *perimeter* of the polygonal.

# 6.- ACCUMULATION POINTS (LIMIT POINTS, CLUSTER POINTS, CONDENSATION POINTS). CLOSED SETS.

Let $E$ be a set of points of $S_r$. A point $P$ of space of $S_r$ is said to be *accumulation point* (or *limit point* or *cluster point* or *condesation point*) of set $E$ when on *every* circular domain of center $P$ there is almost a point of $E$ which be distinct from $P$.
A point $A$ of set $E$ is an accumulation point or not; in this second case a point is said *isolated point*.

*Bolzano-Weiestrass' theorem*:
On $S_r$ every bounded infinite set on $S_r$ has almost an accumulation point.

For $n = 1$, an infinite subset of a closed bounded set $S_1$ has an accumulation point in $S_1$.
For instance, given a bounded sequence $a_n$, with $-C \leq a_n \leq C$ for all $n$, it must have a monotonic subsequence $a_{n_k}$. The subsequence $a_{n_k}$ must converge because it is monotonic and bounded. Because

[7] These appendixes are used to speed up reading the article

$S_1$ is closed, it contains the limit of $a_{n_k}$.

The Bolzano-Weierstrass' theorem is closely related to the Heine-Borel theorem and Cantor's intersection theorem, each of which can be easily derived from either of the other two.

*Derived set*
We say *derived set* of set $E$ the set of accumulation points of $E$ (if they exist); is to indicated by the symbol $DE$. If $E$ not admits accumulation points we say that the set derived is the empty set.
*closed set*
A set is called *closed* when the whole derived $DE$ is empty or is contained on $E$. If $E$ not admits accumulation points we said that the set derived $DE$ is the empty set.

EXAMPLES:

1) $E$ is the set of points of axis $x$ that have abscissas $1, \frac{1}{2}, \frac{1}{3}$......
$DE$ is formed only from the point 0 which does not belong to $E$.
It is *not closed* because the only point of accumulation does not belong to $E$.

2)$E$ is the set of points on the $xy$ plane with both coordinate expressed by rational numbers, the set derived $DE$ coincides with the whole plan.
It is *not closed* because there are infinitely many *accumulation points* not belonging to E.

3)$E$ is the set of points on the $xy$ plane whose coordinates are integers.
The set derived is empty.
$E$ is *closed* because it has no accumulation points.

4)Let $E$ be a segment or a circular domain, or a interval, or a polygon of any space $S_r$.
The set derived $DE$ coincides with the set $E$ itself.
All regarded sets are closed and even $DE \equiv E$.

5) *perfect sets*
Those particular sets *closed* $E$ for which it happens that the set derived $DE$ coincides with the same set $E$ are called *perfect sets.*
The segments, circular domains, intervals, polygons are *perfect sets.*

*Property of set E.*

Given any set $E$, its derived $DE$ assumed nonempty, is a *closed* set.

7. - Interior, exterior, boundary points. Open sets.

Let $E$ be a set of points of $S_r$. A point $P$ of $S_r$ is called *internal* to set $E$ if you can build a circular domain with center $P$ in which the points there are *entirely* of $E$. It is clearly that $P$ inside $E$ belongs to $E$ and is also a point of accumulation.

A point $P$ of the space $S_r$ is said to be *external* to set $E$ if we can build a circular domain with center $P$ which no contains points of $E$, i.e. it is fully built up of complementary set $EC$. A point outside $E$ is inside $EC$, then it belongs to $EC$ and is its accumulation point.

A point $P$ of the space $S_r$ is called *boundary point* for the set $E$ if it is neither internal nor external to $E$. I.e. if into every circular domain $D$ there are points of $E$ and of the complementary $EC$. One point that is point of frontier for $E$ is clearly also point of frontier for $EC$.

We will call *frontier* of a set $E$ of $S_r$ the set formed by all its points of frontier and we will denoted it by the symbol $FE$. We may doubt that the boundary $FE$ may be an empty set, in certain cases, and now this happens for only one set, i.e. for the entire $S_r$.

*Open set* or *Field*.

A set of points of $E$ of $S_r$ is said to be *open* if every point is an interior point. A set *open* therefore do not contains point of its *frontier*.

*Neighbourhood* of the point P.

Taken any point $P$ of the space $S_r$, we will call *neighbourhood of point* $P$ any open and limited set containing $P$.

*Field connected* or *open set connected*

A *field* $E$ is called *connected*, if we take on it two points $P$, $Q$ is always possible to link them up with a whole polygon in $E$.
A circular or rectangular *field* is a *field connected*; for example: the two dimensional field constitued by internal points to two outer circles tangent to one another or externally are not *field connected*.

8. - DEFINITION OF *CONTINUOUS-SETS* AND *DOMAINS*.
Among the sets *perfect*, characterized by being $E \equiv DE$, are of particular importance the so-called *continuous-sets* and so-called *domains*.

A set $E$ is called *continuous-set* if occurs the following conditions:
1) is perfect,
2) is limited,
3)if we take two points $A$, $B$ and we fix $\varepsilon > 0$, one can always construct a polygonal joining $A$ with $B$ such that all its vertices belong to $E$ and its sides have length smaller than $\varepsilon$.

For example, segments, intervals, circular domains are *continuous-set*.

A set $E$ is called a *domain* if occurs the following conditions:
1) is perfect,
2) is closed,
3) every accumulation point is accumulation point of *internal points*.

Note that this definition implies that a *domain* has interior points.

For example, the closed intervals of $S_r$, the circular domains of $S_r$ are domains; a segment it is only in the case r = 1.

The set which is the sum of a circular domain and of a segment (which has one of the ends into the frontier of circular domain), it is not a domain because the first condition occurs but not the second. In fact a point of the segment is a point of accumulation, but it is not of the interior points.

Theorem: If $A$ is any *open set*, then the set $E = A + F$ is a *domain*.

*Domain internally connected*

$E$ is said *domain internally connected* when *E-FE* is a field connected, i.e. when you take two points $P, Q$ internals to $E$, you can always to join them with a polygon whose points are all *internal*.

A circular or rectangular domain is internally connected, it is not the domain consists of two circles tangent externally.

Theorem: Every *domain E* limited and internally connected is a *continuous-set*.

MEAN VALUE THEOREM: Let $f(P),\ \varphi(P)$ be two continuous functions on the limited and measurable domain A an let always be $\varphi(P) \geq 0$. Then let $m, M$ be minimum and maximum of $f(P)$ on A, there is inequality

$$m \int_A \varphi(P)\ \mathrm{dT}\ \leq \int_{\mathrm{A}} \mathrm{f(P)}\ \varphi(\mathrm{P})\ \ \mathrm{dT}\ \leq\ \mathrm{M} \int_{\mathrm{A}} \varphi(\mathrm{P})\ \mathrm{dT},$$

from wich follows

$$\int_A f(P)\ \varphi(P)\ \ \mathrm{dT}\ =\ \mu \int_{\mathrm{A}} \varphi(\mathrm{P})\ \mathrm{dT}$$

where $\mu$ designates a number between m,M. If the domain A is a continuous-set, then it exists into A almost a point Q for which

$$\int_A f(P)\ \varphi(P)\ \ \mathrm{dT}\ =\ \mathrm{f(Q)} \int_{\mathrm{A}} \varphi(\mathrm{P})\ \mathrm{dT}$$

ONE DIMENSIONAL MEAN VALUE THEOREM.

Let $f(x)$ and $\varphi(x)$ be two continuous and measurable functions on the interval [a,b], and let always be $\varphi(x) \geq 0$. Then let m, M be minimum and maximum value of $f(x)$ in [a,b], there is the inequality

$$m \int_a^b \varphi(x)\ \mathrm{dx}\ \leq \int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f(x)}\ \varphi(\mathrm{x})\ \mathrm{dx}\ \leq\ \mathrm{M} \int_{\mathrm{a}}^{\mathrm{b}} \varphi(\mathrm{x})\ \mathrm{dx},$$

from which follows

$$\int_a^b f(x)\ \varphi(x)\ \ \mathrm{dx}\ =\ \mu \int_{\mathrm{a}}^{\mathrm{b}} \varphi(\mathrm{x})\ \mathrm{dx}$$

where $\mu$ designates a number between m, M. But [a,b] being a continuous-set exists almost a number $x_0$ such that

$$\int_a^b f(x)\ \varphi(x)\ \mathrm{dx}\ =\ \mathrm{f(x_0)} \int_{\mathrm{a}}^{\mathrm{b}} \varphi(\mathrm{x})\ \mathrm{dx}.$$

APPENDIX B.

Alcune definizioni:
$S_r$ è lo spazio a $r$ dimensioni.

1) *intervallo chiuso* di punti estremi $A, B$.
Dati in $S_r$ due punti $A(a_1, ..., a_r)$, $B(b_1, ..., b_r)$ l'insieme dei punti $P(x_1, ..., x_r)$ le cui coordinate verificano le relazioni,

$$a_1 \leq x_1 \leq b_1, ...., a_r \leq x_r \leq b_r.$$

si dice *closed interval*.
For r=1
a) *intervallo chiuso* ⇒ [a,b]
b) *intevallo aperto* ⇒ (a,b)
c) *intervallo chiuso a sinistra e aperto a destra* ⇒ [a,b)
d) *intervallo aperto a sinistra e chiuso a destra* ⇒ (a,b]

2) *Dominio circolare* di centro $C$ e di raggio $\varrho$
Dato in $S_r$ un punto $C(c_1, c_2, ....., c_r)$ l'insieme dei punti $P(x_1, ..., x_r)$ tali che

$$(x_1 - c_1)^2 + ... + (x_r - c_r)^2 \leq \varrho^2$$

Per $r = 1$ un *dominio circolare* è lintervallo chiuso [a,b], per $r = 2$ è un cerchio, per $r = 3$ è una sfera.

*Poligonale*

Chiameremo *poligonale di vertici successivi* $(P_1, ..., P_n)$ l'insieme costituito dai punti degli $n - 1$ segmenti $(P_1P_2, P_2P_3, ...., P_{n-1}P_n)$ che diconsi i *lati* della poligonale. La somma delle lunghezze dei lati dicesi *perimetro* della poligonale.

6.- PUNTI DI ACCUMULAZIONE. INSIEMI CHIUSI.

Sia $E$ un insieme di punti di $S_r$. Un punto $P$ dello spazio $S_r$ si dice *punto di accumulazione* o *punto limite* dell'insieme $E$ quando in *ogni* dominio circolare di centro $P$ esiste almeno un punto di $E$ che sia distinto da $P$.
Un punto $A$ dell'insieme $E$ può essere punto di accumulazione oppure può non esserlo; in questo secondo caso si dice un *punto isolato* di $E$

*teorema di Bolzano-Weiestrass*:
Un insieme limitato di $S_r$, contenente infiniti punti ammette almeno un punto di accumulazione.
Per $n = 1$, un sottoinsieme infinito di un insieme chiuso e limitato $S_1$ ha un punto di accumulazione.
Per esempio, data una successione limitata $a_n$, con $-C \leq a_n \leq C$ per ogni $n$, deve avere una sottosuccessione monotona $a_{n_k}$. La sottosuccessione $a_{n_k}$ deve convergere perchè è monotone e limitata. Perchè $S_1$ è chiuso, ha lo stesso limite di $a_{n_k}$.

Il teorema di Bolzano-Weiestrass è strettamente correlato a quello di Heine-Borel e il teorema dell'intersezione di Cantor, ciascuno dei quali può esserefacilmate derivato dagli altri due.
*Insieme derivato*
Si chama *insieme derivato* di un dato insieme $E$ l'insieme costituito dai punti di accumulazione di $E$ (se esistono); lo si indica col simbolo $DE$. Se $E$ non ammette punti di accumulazione si dice che

l'insieme derivato è l'insieme vuoto.

*Insieme chiuso*
Un insieme si dice *chiuso* quando il suo insieme derivato $DE$ è vuoto oppure è contenuto in $E$ In altre parole che $E$ è chiuso significa che o non ha punti di accumulazione oppure che ha punti di accumulazione i quali appartengono all'insieme stesso $E$.

ESEMPI:

1) $E$ è l'insieme dei punti dell'asse $x$ che hanno le ascisse $1, \frac{1}{2}, \frac{1}{3}$.....
$DE$ è formato dal solo punto 0 che non appartiene ad $E$
*non* è *chiuso* perchè l'unico punto di accumulazione non appartiene ad $E$

2)$E$ è l'insieme dei punti del piano $xy$ aventi entrambe le coordinate espresse da numeri razionali, l'insieme derivato $DE$ coincide con tutto il piano
$E$ *non* è *chiuso* perchè ci sono infiniti punti di accumulazione che non appartengono ad $E$

3 Sia $E$ l'insieme dei punti del piano $xy$ le cui coordinate sono numeri interi
l'insieme derivato è vuoto.
$E$ è chiuso perchè non ha punti di accumulazione.

4) Sia $E$ un segmento, oppure un dominio circolare, oppure un intervallo,oppure una poligonale di un qualsiasi spazio $S_r$
l'insieme derivato $DE$ coincide con l'insieme $E$ stesso.
Gli insiemi considerati sono tutti chiusi e addirittura $DE \equiv E$.
*insiemi perfetti*
Quei particolari insiemi *chiusi* $E$ per i quali accade che l'insieme derivato $DE$ coincide con l'insieme $E$ stesso si chiamano *insiemi perfetti.*
I segmenti, i domini circolari, gli intervalli,le poligonali sono *insiemi perfetti.*
Proprietà dell'insieme $E$.
Dato un qualsiasi insiene $E$, il suo derivato $DE$,supposto non vuoto, è un insieme chiuso.

*Proprietà dell'insieme $E$.*
Dato un insieme $E$, il suo derivato $DE$ assunto non vuoto, è un insieme chiuso.

7.- Punti interni, esterni, di frontiera. Insiemi aperti.

Sia $E$ un insieme di punti di $S_r$ . Un punto $P$ si dice *interno* all'insieme $E$ se è possibile costruire un dominio circolare di centro $P$ in quale sia *interamente* costituito di punti di $E$. È evidente che un punto $P$ interno ad $E$ appartiene ad $E$ stesso ed è anche punto di accumulazione.

Un punto $P$ dello spazio $S_r$ si dice *esterno* all'insieme $E$ se è possibile costruire un dominio circolare di centro $P$ il quale non contenga alcun punto di $E$, vale a dire sia interamente costituito dall'insieme complementare $CE$. Un punto esterno ad $E$ risulta interno a $CE$; esso appartiene dunque a $CE$ ed è suo punto di accumulazione.

Un punto $P$ dicesi *punto di frontiera* per l'insieme $E$ se non è né interno né esterno ad $E$, vale a dire se in *ogni* dominio circolare di centro $P$ cadono sia punti di $E$ sia punti del complemetare $CE$. Un punto che sia di frontiera per $E$ è evidentemente anche punto di frontiera per $CE$.

Chiameremo *frontiera* di un insieme $E$ di $S_r$ l'insieme formato da tutti i suoi punti di frontiera e la indicheremo col simbolo $FE$. Può sorgere il dubbio che la frontiera $FE$ possa, in certi casi, essere un insieme vuoto; ora ciò accade per un solo insieme e cioè per l'intero $S_r$.

*Insieme aperto* o *Campo.*

Un insieme $E$ di punti di $S_r$ si dice *aperto* se ogni suo punto è punto interno. Un insieme *aperto* non contintiene dunque alcun punto della sua frontiera.

*Intorno del punto P.*

Preso un qualsiasi punto $P$ dello spazio $S_r$, chiameremo *intorno del punto* $P$ ogni insieme aperto e limitato contenente $P$.
*Campo connesso*

Un campo $E$ si dice *connesso* quando, comunque si prenda in esso due punti $P$, $Q$ è sempre possibile congiungerli con una poligonale tutta contenuta in $E$.
Un campo circolare o rettangolare è un *campo connesso*; non lo è invece, per esempio, il campo piano costituito dai punti interni a due cerchi esterni l'uno all'altro oppure tangenti esternamente.

8.- DEFINIZIONE DI CONTINUO E DI DOMINIO
Fra gli insiemi *perfetti*, caratterizzati dall'essere $E \equiv DE$. hanno particolare importanza i cosiddetti *continui* e i cosiddetti *domini.*

Un insieme $E$ dicesi *continuo* se verifica le seguenti condizioni:
1) è perfetto,
2) è limitato,
3) comunque si prendano in esso due punti $A$,$B$ e si fissi un numero $\varepsilon > 0$, si può sempre costruire una poligonale , congiungente $A$ con $B$ tale che tutti i suoi vertici appartengano ad $E$ ed i suoi lati abbiano tutti lunghezza minore di $\varepsilon$.

Per esempio: i segmenti, gli intervalli, i domini circolari, sono *continui.*

Un insieme $E$ dicesi un *dominio* se verifica le seguenti condizioni:
1) è perfetto,
1) è chiuso,
2) ogni suo punto di accumulazione lo è di *di punti interni.*

Si noti che questa definizione implica che un *dominio* è dotato di punti interni.

Per esempio, gli intervalli (chiusi), i domini circolari di $S_r$ sono dei dominii; un segmento lo è soltanto nel caso r=1.
L'insieme piano somma di un dominio circolare e di un segmento (che ha come uno degli estremi un punto della frontiera), non è un dominio perchè verifica la prima condizione ma non la seconda. Infatti un punto del segmento è punto di accumulazione, ma non lo è di punti interni.

TEOREMA: Se $A$ è un qualsiasi insieme aperto, allora l'insieme $E = A + FA$ è un *dominio.*
*dominio internamente connesso*

Un *dominio E* si dice *internamente connesso* quando *E-FE* è un campo connesso, vale a dire quando, si prendano due punti $P, Q$ interni ad $E$,è sempre possibile congiungerli con una poligonale i cui punti sono tutti *interni* ad $E$.
Un dominio rettangolare o circolare è internamente connesso; non lo è il dominio formato da due cerchi tangenti esternamente.

TEOREMA: Ogni *dominio E* limitato ed internamento connesso è un *continuo.*

TEOREMA DELLA MEDIA: [8] Siano $f(P),\ \varphi(P)$ due funzioni continue nel dominio A limitato e misurabile e sia sempre $\varphi(P) \geq 0$. Allora, detti $m,\ M$ il minimo e il massimo valore della $f(P)$ in A, sussiste la disuguaglianza

$$m \int_A \varphi(P)\ \mathrm{dT}\ \leq \int_{\mathrm{A}} \mathrm{f(P)}\ \varphi(\mathrm{P})\ \mathrm{dT}\ \leq\ \mathrm{M} \int_{\mathrm{A}} \varphi(\mathrm{P})\ \mathrm{dT},$$

dalla quale discende

$$\int_A f(P)\ \varphi(P)\ \mathrm{dT}\ =\ \mu \int_{\mathrm{A}} \varphi(\mathrm{P})\ \mathrm{dT}$$

ove $\mu$ designa un opportuno numero compreso fra m,M. Se il dominio A è un continuo, allora esiste in A almeno un punto Q per cui risulta

$$\int_A f(p)\ \varphi(P)\ \mathrm{dT}\ =\ \mathrm{f(Q)} \int_{\mathrm{A}} \varphi(\mathrm{P})\ \mathrm{dT}$$

TEOREMA DELLA MEDIA IN UNA DIMENSIONE

Siano $f(x),\ \varphi(x)$ due funzioni continue nell'intervallo limitato e misurabile [a,b]e sia sempre $\varphi(P) \geq 0$. Allora, detti $m,\ M$ il minimo e il massimo valore della $f(x)$ in [a,b], sussiste la disuguaglianza

$$m \int_a^b \varphi(x)\ \mathrm{dx}\ \leq \int_{\mathrm{a}}^{\mathrm{b}} \mathrm{f(x)}\ \varphi(\mathrm{x})\mathrm{rmdx}\ \leq\ \mathrm{M} \int_{\mathrm{a}}^{\mathrm{b}} \varphi(\mathrm{x})\ \mathrm{dx},$$

dalla quale discende

$$\int_a^b f(x)\ \varphi(x)\ \ \mathrm{dx}\ =\ \mu \int_{\mathrm{a}}^{\mathrm{b}} \varphi(\mathrm{x})\ \mathrm{dx}$$

ove $\mu$ designa un opportuno numero compreso fra m,M. Essendo però [a,b] un continuo esiste almeno un punto $x_0$ per cui risulta

$$\int_a^b f(x)\ \varphi(x)\ \ \mathrm{dx}\ =\ \mathrm{f(x_0)} \int_{\mathrm{a}}^{\mathrm{b}} \varphi(\mathrm{x})\ \mathrm{dx}$$

[8]( see [3] pag.61)

APPENDIX C.

see [6] pagg 58-61.

"...The $\delta$ function. Our work ...... led us to consider quantities involving a certain kind of infinity. To get a precise notation for dealing with these infinities, we introduce a quantity $\delta(x)$ depending on a parameter x satisfying the conditions

$$\int_{-\infty}^{+\infty} \delta(x)\ dx \ = \ 1 \tag{5.1}$$

$$\delta(x) \ = \ 0 \quad for \quad x \neq 0. \tag{5.2}$$

To get a picture of $\delta(x)$, take a function of the real variable x which vanishes everywhere except inside a small domain, of length $\epsilon$ say, surrounding the origin $x = 0$, and which is so large inside this domain that its integral over this domain is unity. The exact shape of the function inside this domain does not matter, provided there are no unnecessarily wild variations (for example provided the function is always of order $\epsilon^{-1}$. Then in the limit $\epsilon \to 0$ this function will go over into $\delta(x)$. $\delta(x)$ is not a function of x according to the usual mathematical definition of a function, which requires a function to have a definite value for each point in its domain, but is something more general, which we may call an 'improper function' to show up its difference from a function defined by the usual definition. Thus $\delta(x)$ is not a quantity which can be generally used in mathematical analysis like an ordinary function, but its use must be confined to certain simple types of expression for which it is obvious that no inconsistency can arise. The most important property of $\delta(x)$ is exemplified by the following equation,

$$\int_{-\infty}^{+\infty} f(x)\delta(x)\ dx \ = \ f(0), \tag{5.3}$$

where f(x) is any continuous function of x. We can easily see the validity of this equation from the above picture of $\delta(x)$. The left-handside of (5.3) can depend only on the values of $f(x)$ very close to the origin, so that we may replace $f(x)$ by its value at the origin, $f(0)$, without essential error. Equation (5.3) then follows from the first of equations (5.2). By making a change of origin in (5.3), we can deduce the formula

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)\ dx \ = \ f(a), \tag{5.4}$$

where a is any real number. Thus *the process of multiplying a function of $x$ by $\delta(x-a)$ and integrating aver all $x$ is equivalent to the process of substituting $a$ for $x$.* This general result holds also if the function of x is not a numerical one, but is a vector or linear operator depending on x. The range of integration in (5.3) and (5.4) ne ed not be from $-\infty$ to $+\infty$, but, may be over any domain surrounding the critical point at which the $\delta$ function does not vanish. In future the limits of integration will usually be omitted in such equations, it being understood that the domain of integration is a suitable one.
Equations (5.3) and (5.4) show that, although an improper function does not itself have a well-defined value, when it occurs as a factor in an integrand the integral has a well-defined value. In quantum theory, whenever an improper function appears, it will be something which is to be used ultimately in an integrand. Therefore it should be possible to rewrite the theory in a form in which the improper functions appear all through only in integrands. One could then eliminate the improper functions altogether. The use of improper functions thus does not involve any lack of rigour in the theory, but is merely a convenient notation, enabling us to express in a concise form certain relations which we could, if necessary, rewrite in a form not involving improper functions, but only in a cumbersome way which would tend to obscure the argument.

An alternative way of defining the $\delta$ function is as the differential coefficient $\epsilon^{'}(x)$ of the function $\epsilon(x)$ given by

$$\epsilon(x) \;=\; 0 \;\; (x < 0) \quad ; \quad \epsilon(\mathrm{x}) \;=\; 1 \;\; (\mathrm{x} > 0) \tag{5.5}$$

We may verify that this is equivalent to the previous definition by substituting $\epsilon^{'}(x)$ for $\delta(x)$ in the left-hand side of (5.3) and integrating by parts. We find, for $g_1$ and $g_2$ two positive numbers,

$$\int_{-g_2}^{g_1} f(x)\epsilon^{'}(x)dx \;=\; [f(x)\epsilon(x)]_{-g_2}^{g_1} \;-\; \int_{-g_2}^{g_1} f^{'}(x)\epsilon(x)dx$$
$$= \; f(g_1) \;-\; \int_0^{g_1} f^{'}(x)dx$$
$$= \; f(0),$$

in agreement with (5.3). The $\delta$ function appears whenever one differentiates a discontinuous function. There are a number of elementary equations which one can write down about $\delta$ functions. These equations are essentially rules of manipulation for algebraic work involving $\delta$ functions. The meaning of any of these equations is that its two sides give equivalent results as factors in an integrand. Examples of such equations are

$$\delta(-x) \;=\; \delta(x) \tag{5.6}$$

$$x\delta(x) \;=\; 0, \tag{5.7}$$

$$\delta(ax) \;=\; a^{-1}\delta(x) \quad (a > 0), \tag{5.8}$$

$$\delta(x^2 - a^2) \;=\; \frac{1}{2}a^{-1}\,\{\delta(x-a) + \delta(x+a)\} \qquad (a > 0), \tag{5.9}$$

$$\int \delta(a-x)dx\;\delta(x-b) \;=\; \delta(a-b), \tag{5.10}$$

$$f(x)\delta(x-a) \;=\; f(a)\delta(x-a), \tag{5.11}$$

Equation (5.6), which merely states that $\delta(x)$ is an even function of its variable $x$ is trivial. To verify (5.7) take any continuous function of $x, f(x)$. Then

$$\int f(x)x\delta(x) \;=\; 0,$$

from (5.3). Thus $x\delta(x)$ as a factor in an integrand is equivalent to zero. which is just the meaning of (5.7). (5.8) and (5.9) may be verified by similar elementary arguments. To verify (5.10) take any continuous function of $a, f(a)$. Then

$$\int f(a)da \int \delta(a-x)dx\;\delta(x-b) \;=\; \int \delta(x-b)dx \int f(a)da\;\delta(a-x)$$
$$\int \delta(x-b)dx\;f(x) \;=\; \int f(a)da\;\delta(a-b).$$

Thus the two sides of (5.10) are equivalent as factors in an integrand with $a$ as variable of integration. It may be shown in thc same way that they are equivalent also as factors in an integrand with $b$ as variable of integration, so that equation (5.10) is justified from either of these points of view. Equation (5.11) is also easily justified, with the help of (5.4), from two points of view. Equation (10) would be given by an application of (5.4) with $f(x) \;=\; \delta(x-b)$. We have here an illustration of the fact that we may often use an improper function as though it were an ordinary continuous function. without getting a wrong result.

Equation (5.7) shows that, whenever one divides both sides of an equation by a variable $x$ which can take on the value zero, one should add on to one side an arbitrary multiple of $\delta(x)$, i.e. from an equation

$$A \;=\; B \tag{5.12}$$

one cannot infer

$$A/x = B/x, \tag{5.13}$$

$$A/x = B/x + c\delta(x), \tag{5.14}$$

where c is unknown.
As an illustration of work with the $\delta$ function, we may consider the differentiation or $log\ x$. The usual formula

$$\frac{d}{dx}\ log(x) = \frac{1}{x}, \tag{5.15}$$

requires examination far the neighbourhood of $x = 0$. In order to make the reciprocal function $1/x$ well defined in the neighbourhood of $x = 0$ (in the sense or an improper function) we must impose on it an extra condition, such as that its integral from $-\epsilon$ to $\epsilon$ vanishes. With this extra condition, the integral of the right-hand side of (5.15) from $-\epsilon$ to $\epsilon$ vanishes, while that of the left-hand side of (14) equals $log(-1)$, so that(5.15) is not a correct equation. To correct it, we must remember that, taking principal values, $log\ x$ has a pure imaginary term $i\pi$ for negative values of $x$. As $x$ passes through the value zero this pure imaginary term vanishes discontinuously. The differentiation
of this pure imaginary term gives us the result $-i\pi\delta(x)$, so that (5.15) should read

$$\frac{d}{dx}\ log(x) = \frac{1}{x} - i\pi\delta(x). \tag{5.16}$$

The particular combination of reciprocal function and $\delta$ function appearing in (5.16) plays an important part in the quantum theory of collision processes..."

## References

[1] A. Ghizzetti
*Lezioni di Analisi Matematica*
vol. I edizione 1959
Libreria Eredi Virgilio Veschi

[2] A. Ghizzetti
*Complementi ed Esercizi di Analisi Matematica*
vol. I edizione 1960
Libreria Eredi Virgilio Veschi

[3] A. Ghizzetti
*Lezioni di Analisi Matematica*
vol. II edizione 1960
Libreria Eredi Virgilio Veschi

[4] A. Ghizzetti
*Complementi ed Esercizi di Analisi Matematica*
vol. II edizione 1961
Libreria Eredi Virgilio

[5] Balth. van der Pol and H.Bremmer
*OPERATIONAL CALCULUS based on the two-sides Laplace integral*
Chelsea Publishing Company (New York), N.Y. 3nd Edition 1987.

[6] P.A.M. DIRAC *The Principles of Quantum Mechanics Fourth Edition*
The Internetional Series of Mponograph on Phisics - 27
Oxford Science Publication