Three lectures on subfactors

Karl-Henning Rehren

Institut für Theoretische Physik, Universität Göttingen

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Plan:

Intro

Lecture I: Subfactors of finite index

Lecture II: Modular theory of subfactors

Lecture III: Subfactors in quantum field theory

Quantum theory = operator algebras

- Quantum theory = self-adjoint operators on a Hilbert space
- Dynamics requires algebraic structure
- Approximations require topology
- Operator algebras = theory of bounded self-adjoint operators

Subalgebras describe subsystems

- In Quantum Physics, subsystems arise in manifold ways.
  - subset of particles
  - localization of fields
  - subset of fields
  - inner degrees of freedom
  - ...

C* algebras vs von Neumann algebras

- C* topology: norm $\|\cdot\|$, $\|a^*a\| = \|a\|^2$,
- $=$ operator norm (spectral radius) on Hilbert space;
- ensures existence of Hilbert space representations via GNS

- Weak topology: convergence of matrix elements
- ...depends on a Hilbert space representation π.
- Weak closure $\overline{A}^{\text{weak}}$ = double commutant $\pi(A)''$.
- A given state may not extend to the weak closure in "the wrong" representation.

- Often (not always!) in quantum theory, "all states of interest" extend to the weak closure w.r.t. some reference state.
- In this case, one may safely work with the von Neumann algebra

$$M = M''.$$

Benefits of von Neumann algebras

- Very robust topology
- Less structure (D. Kastler: "concrete vs sponge")
- Type classification I–III (properties of projections)
- M' has the same type as M.
- Type II possesses tracial states.
- Quantum field theory is type III: $e = tt^*$, $t^*t = \mathbf{1}_M$.

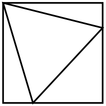
Factors and subfactors

- A factor is a vNA with trivial center $M' \cap M = \mathbb{C} \cdot \mathbf{1}$.
- A subfactor is a pair of factors $N \subset M$ with common unit.
- Many examples of interesting subfactors of type III in QFT.

Lecture I: SUBFACTORS OF FINITE INDEX

Index

- A conditional expectation $E : M \to N$ is a unital positive map such that $E(n_1 m n_2) = n_1 E(m) n_2$.
  A Popa bound for E is an operator estimate (for some $\lambda \geq 1$)
  $$E(mm^*) \geq \lambda^{-1} \cdot mm^* \quad \text{for all} \quad m \in M.$$
- The index λ_E of E is the smallest such λ ($=$ the best bound).
- The index $[M : N]$ of $N \subset M$ is the smallest index λ_E when E ranges over all conditional expectations.
- (There is only one E if $N \subset M$ is irreducible, $N' \cap M = \mathbb{C} \cdot \mathbf{1}$.)
- The index measures the "ratio of sizes" of M and N.
- $[M : N] = 1$ iff $N = M$.
- The index is quantized between 1 and 4: $[M : N] = (2 \cos \frac{\pi}{n})^2$.

- Finite index subfactors are "rigid structures": 
- The (more popular?) theory of type II subfactors of finite index (Jones) is essentially "isomorphic" (with different techniques) to that of type III subfactors.
- "M-N-bimodules" in type II $\Leftrightarrow$ unital homomorphisms $\varphi : N \to M$.
- Here: type III (anticipating its use for QFT).
- While N and M may be isomorphic, a subfactor $N \subset M$ is specified by its embedding homomorphism $\iota : N \to M$ ($\iota(n) = n$).

Homomorphisms

I assume some familiarity with the tensor category of endomorphisms of type III algebra N. Here are the basics.

The objects are endomorphisms $\rho \in \mathrm{End}(N)$, the morphisms $t \in \mathrm{Hom}(\rho, \sigma)$ are intertwiners $t \in N$ such that $t\rho(n) = \sigma(n)t$. Then $\rho \cong \sigma$ if there exists a unitary intertwiner, and $\rho \prec \sigma$ if there exists an isometric intertwiner, so that $\rho(n) = t^*\sigma(n)t$.

Clearly $t^* \in \mathrm{Hom}(\sigma, \rho)$ iff $t \in \mathrm{Hom}(\rho, \sigma)$.

The monoidal product of objects is $\rho \times \sigma \equiv \rho \circ \sigma$ with unit id_N, and the monoidal product of morphisms is $t \times s \equiv t\rho(s) = \sigma(s)t \in \mathrm{Hom}(\rho \times \lambda, \sigma \times \mu)$ whenever $t \in \mathrm{Hom}(\rho, \sigma)$, $s \in \mathrm{Hom}(\lambda, \mu)$, in particular $\mathbf{1}_\rho \times s = \rho(s)$ and $t \times \mathbf{1}_\lambda = t$.

This easily extends to homomorphisms between several algebras, forming a two-category.

Conjugates

$\overline{\varphi} : M \to N$ is conjugate to $\varphi : N \to M$ iff

$$\mathrm{id}_N \prec \overline{\varphi} \circ \varphi \quad \text{and} \quad \mathrm{id}_M \prec \varphi \circ \overline{\varphi},$$

and there exists a pair $[w, v]$ of intertwiners $w \in \mathrm{Hom}(\mathrm{id}_N, \overline{\varphi}\varphi)$ and $v \in \mathrm{Hom}(\mathrm{id}_M, \varphi\overline{\varphi})$, solving the conjugacy (or "zigzag") relations

$$\overline{\varphi}(v^*)w = \mathbf{1}_N \quad , \quad v^*\varphi(w) = \mathbf{1}_M$$

It follows that $m = \varphi(w^*)vm = \varphi(w^*)\varphi\overline{\varphi}(m)v = \varphi(w^*\overline{\varphi}(m))v$, hence

$$M = \varphi(N)v.$$

Conjugates and index

- Because N and M are factors, id_N and id_M are irreducible, hence w and v are multiples of isometries.
- One may WLOG normalize them such that $w^*w = d \cdot \mathbf{1}_N$ and $v^*v = d \cdot \mathbf{1}_M$ for some $d \geq 1$.
- The linear map
$$F(m) := d^{-1} \cdot w^*\overline{\varphi}(m)w$$
is a unital positive map of M onto N such that $F \circ \varphi = \mathrm{id}_N$, and $E := \varphi \circ F$ is a conditional expectation of M onto $\varphi(N)$.
- Because $F(vv^*) = d^{-1} \cdot \mathbf{1}_N$, it follows for $m = \varphi(n)v$: $F(mm^*) = nF(vv^*)n^* = d^{-1} \cdot nn^*$. Because $d^{-1} \cdot vv^*$ is a projection, one gets the Popa bound
$$E(mm^*) = d^{-1} \cdot \varphi(nn^*) \geq d^{-2} \cdot \varphi(n)vv^*\varphi(n^*) = d^{-2} \cdot mm^*.$$
- The bound is optimal for E (choose $n = \mathbf{1}_N$). Thus, $d^2 = \lambda_E$.

Index and dimension

- If φ is irreducible (i.e., $\mathrm{Hom}(\varphi,\varphi) \equiv \varphi(N)' \cap M = \mathbb{C} \cdot \mathbf{1}_M$), then E is the unique conditional expectation, and $[M : \varphi(N)] = d^2$.
- Otherwise, a solution $[w, v]$ that minimizes the index $\lambda_E = d^2$ exists (unique up to unitaries in $\mathrm{Hom}(\varphi,\varphi)$ resp. $\mathrm{Hom}(\overline{\varphi},\overline{\varphi})$).
- The minimal value of d is called the dimension $\dim_\varphi$ of φ.
- The dimension is multiplicative under composition and additive under direct sums of homomorphisms.[a] We have $\dim_{\overline{\varphi}} = \dim_\varphi$ and

$$[M : \varphi(N)] = \dim_\varphi^2 .$$

[a] The direct sum of homomorphisms into a von Neumann algebra M of type III is defined as $\varphi(m) := \sum_i t_i \varphi_1(m) t_i^*$ where t_i are orthonormal isometries such that $\sum_i t_i t_i^* = \mathbf{1}_M$.

All notions here are invariant under unitary equivalence of homomorphisms.

Canonical endomorphisms

We want to characterize an irreducible subfactor $N \subset M$ in terms of properties of the embedding homomorphism $\iota : N \to M$. The index $[M : N] = \dim_\iota^2$ is a first invariant. If the index is 1, then w and v are actually unitaries (by Cauchy-Schwarz), ι is an isomorphism, $\bar\iota$ is unitarily equivalent to ι^{-1}, and $\iota(N) = M$ is the trivial subfactor.

Otherwise, define the canonical endomorphism $\gamma := \iota \circ \bar\iota \in \mathrm{End}(M)$ and the dual canonical endomorphism $\theta := \bar\iota \circ \iota \in \mathrm{End}(N)$.

Remark: ι corresponds to M as an M-N bimodule and $\bar\iota$ to M as an N-M bimodule, where M acts directly and N acts via ι.
θ corresponds to M as an N-N bimodule and γ to M_1 as an M-M bimodule, where $N \subset M \subset M_1$ is Jones' upward basic construction.

One gets a sequence of alternatingly isomorphic subfactors

$$\cdots \subset \gamma^2(M) \subset \theta(N) \subset \gamma(M) \subset N \subset M$$

(the analogue of the downward basic construction). It contains further information through the sequence of relative commutants = the irreducible decomposition of nested embeddings.

If the index is > 1, then the canonical endomorphism γ splits as $\gamma = \mathrm{id}_M + ??$ where the remainder has dimension $\dim_\gamma -1 = [M : N] - 1$, and likewise for the dual canonical endomorphism $\theta = \mathrm{id}_N + ??$.

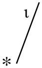
The idea is to look at these remainders.

Principal graph

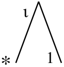
Compose id_M from the right, alternatingly with ι and $\bar{\iota}$. Decompose each time into irreducibles. Draw a bi-partite graph, until it stops.

Example:

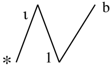
$\mathrm{id}_M \circ \iota = \iota$:



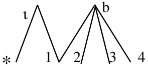
$\iota \circ \bar{\iota} = \mathrm{id}_M + \gamma_1$:



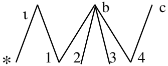
$\gamma_1 \circ \iota = \iota + \iota_b$:



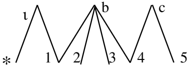
$\iota_b \circ \bar{\iota} = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$:



$\gamma_2 \circ \iota = \gamma_3 \circ \iota = \iota_b$, $\gamma_4 \circ \iota = \iota_b + \iota_c$:



$\iota_c \circ \bar{\iota} = \gamma_4 + \gamma_5$:



and $\gamma_5 \circ \iota = \iota_c$. **STOP**

Edges may have multiplicities. Frobenius reciprocity ensures that all edges go both ways ("the new old stuff is the old new stuff").

Same game, starting with id_M, but composing alternatingly with $\bar{\iota}$ and ι from the left $\Rightarrow$ same graph, but the vertices stand for the conjugates of the respective sectors.

Instead, composing id_N alternatingly with ι and $\bar{\iota}$ from the left, gives the dual principal graph, which has the same odd vertices $\iota_b \prec (\iota\bar{\iota})^n\iota = \iota(\bar{\iota}\iota)^n : N \to M$ as the principal graph, but its even vertices are subsectors $\theta_c \prec (\iota\bar{\iota})^n : N \to N$ for some n.

Again, the conjugate game produces the same graph with vertices = conjugate sectors.

The dual graph is the principal graph of the dual subfactor $\gamma(M) \subset N$ (or $M \subset M_1$). The pair of graphs is another invariant for the subfactor.

Paragroup

Provided the graphs are finite, they can be supplemented to become a complete invariant:

The principal and the dual graph have the same odd vertices. The paths of length 2 within either graph, connecting odd vertices ι_b, ι_c, specify two orthonormal bases of intertwiner spaces $\mathrm{Hom}(\iota_b, \iota\bar{\iota}_c\iota)$.

A pair of such paths, one in either graph, is a "cell" in the juxtaposition of the two graphs. The "weight" of the cell is the scalar product of the corresponding intertwiners.

The complete invariant (called "paragroup") is the collection of all weights, for all ι_b, ι_c.

The paragroup data (two bi-partite graphs together with a collection of weights, subject to certain axioms) can be used to axiomatize subfactors in terms of combinatorial data.

With sufficient knowledge on the subfactor, one can compute the pair of graphs "by hand". Frobenius reciprocity and the balance of dimensions are very restrictive. They imply that the index equals $\|A^*A\|$ where A is the incidence matrix of the graph. Thus for index below 4, the graphs must be of A-D-E type.

In QFT applications ($\to$ Lecture III), the fusion rules of the sectors γ_a (or rather θ_a) may be known.
The graph in the example above (with index $[M:N]=3+\sqrt{3}$) can been computed just from this info; e.g., knowing
$\dim \mathrm{Hom}(\iota\bar\iota\iota,\iota\bar\iota\iota)\equiv\dim \mathrm{Hom}(\gamma,\gamma^2)\equiv\dim \mathrm{Hom}(\theta,\theta^2)=5$
implies that $\dim \mathrm{Hom}(\gamma_1\iota,\gamma_1\iota)=2$, hence $\gamma_1\iota$ must split into two irreducibles, one of which must be ι.

Subfactors of small index can be classified by their graphs.

In contrast, computing the paragroup data is a highly nontrivial nonlinear problem. A pair of graphs may admit several (or none) paragroups = subfactors.

Charge creating operators

$\mathrm{Hom}(id_M, \gamma) \subset \mathrm{Hom}(\gamma\iota, \iota) \equiv \mathrm{Hom}(\iota\theta, \iota)$ implies the commutation relation

$$v \cdot \iota(n) = \iota(\theta(n)) \cdot v.$$

v "creates" the endomorphism θ on the embedded elements of N.

For $\theta_a \prec \theta$ and $0 \neq t_a \in \mathrm{Hom}(\theta_a, \theta)$, let $\psi_a := \iota(t_a^*)v \neq 0$. Then

$$\psi_a \cdot \iota(n) = \iota(\theta_a(n)) \cdot \psi_a.$$

Moreover, one has a unique decomposition of every $m \in M$

$$m = \iota(n)v = \iota(n)\iota\Big(\sum\nolimits_{a,i} t_{a,i}t_{a,i}^*\Big)v = \sum\nolimits_{a,i} \iota(n_{a,i})\psi_{a,i}.$$

The "charge creating operators" $\psi_{a,i}$ form an N-basis of M.

Example: the fixed points $N = M^G$ (G finite, outer action $g \mapsto \alpha_g$).

We want to compute the principal graph. Let $w \in \mathrm{Hom}(\mathrm{id}_N, \theta)$ and $v \in \mathrm{Hom}(\mathrm{id}_M, \gamma)$ be a solution to the conjugation equations. Define $v_g := \alpha_g(v)$ ($g \in G$). $v \in \mathrm{Hom}(\mathrm{id}_M, \gamma)$ implies $v_g \in \mathrm{Hom}(\alpha_g, \alpha_g\gamma)$ $= \mathrm{Hom}(\alpha_g, \gamma)$, because $\gamma(M) \subset \iota(N) = M^G$. Then
$v_g^* v_h \in \mathrm{Hom}(\alpha_h, \alpha_g)$ must vanish unless $g = h$, because the action is outer, hence $e_g := d^{-1} v_g v_g^*$ are mutually orthogonal projections.

The unique conditional expectation $E(m) = d^{-1} \cdot \iota(w^* \bar\iota(m) w)$ onto $\iota(N) \subset M$ coincides with the group average $|G|^{-1} \cdot \sum_g \alpha_g(m)$. Thus,

$$\mathbf{1}_M = d \cdot E(vv^*) = d|G|^{-1} \cdot \sum\nolimits_g v_g v_g^* = d^2 |G|^{-1} \cdot \sum\nolimits_g e_g .$$

This implies $d^2 = |G|$ and $\sum_g e_g = \mathbf{1}_M$.
It follows $\gamma(m) = d \cdot E(vmv^*) = d^{-1} \cdot \sum_g v_g \alpha_g(m) v_g^*$, i.e.,

$$\gamma \cong \bigoplus\nolimits_{g \in G} \alpha_g .$$

Let us now compute the dual graph.

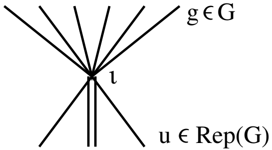
The map $\phi : g \mapsto d \cdot E(vv_g^*) \equiv d \cdot E(v_{g^{-1}}v^*)$ is a homomorphism of G into $\mathrm{Hom}(\theta,\theta)$, and $\iota(\phi(g))v_h = v_{hg^{-1}}$.

ϕ extends linearly to a $*$-homomorphism of the group algebra $\mathbb{C}G$ into $\mathrm{Hom}(\theta,\theta)$, that is injective (by the action of $\iota \circ \phi(g)$ on v_h) and surjective (because $\dim \mathrm{Hom}(\theta,\theta) = \dim \mathrm{Hom}(\gamma,\gamma) = d^2 = |G|$).

By Peter-Weyl, $\mathrm{Hom}(\theta,\theta) \cong \mathbb{C}G$ is a direct sum of matrix rings $\mathrm{Mat}_{\dim(u)}$, where u runs over the irreducible matrix representations of G. This means that the irreducible subsectors of θ are in 1:1 correspondence with the irreps of G and arise with multiplicity $n_u = \dim(u)$:

$$\theta \cong \bigoplus_{u \in \mathrm{Rep}(G)} \dim(u) \cdot \rho_u.$$

The graphs look like this:



The full tensor subcategory of $\mathrm{End}_0(N)$ generated by the endomorphisms $\rho_u \prec \theta$ ("$=$ the even vertices of the dual principal graph") is equivalent to $\mathrm{Rep}(G)$.

A "cell" is now an upper path from ι to ι (through $g \in G$) and a lower path (through u with a multiplicity index both ways). The "cell weights" are just the matrix elements $u(g)_{ij}$.

The dual subfactor is $M \rtimes G$ (with graphs interchanged).

Let $X_{u,ij}$ be matrix units of $\mathbb{C}G$ that reduce the left and right regular representation: $g = \sum_{u,ij} u_{ij}(g) X_{u,ij}$. Write the corresponding matrix units in $\mathrm{Hom}(\theta, \theta)$ as $\phi(X_{u,ij}) = t_{u,i} t^*_{u,j}$ with orthonormal bases $t_{u,i} \in \mathrm{Hom}(\rho_u, \theta)$.

The charged generators $\iota(t^*_{u,i})v \in \mathrm{Hom}(\iota, \iota\rho_u)$ satisfy

$$v^* \iota(t_{u,i} t^*_{u,i}) v = \frac{1}{d} \cdot \iota(w^*) v^* \iota(t_{u,i} t^*_{u,i}) v \iota(w) = \frac{\dim_{\rho_u}}{d} \cdot \mathbf{1}_M.$$

Thus, $\psi_{u,i} := (\frac{d}{\dim_{\rho_u}})^{1/2} \cdot \iota(t^*_{u,i})v$ are isometries in $\mathrm{Hom}(\iota, \iota\rho_u)$. We compute their linear transformation law

$$\alpha_g(\psi_{u,i}) = c \cdot t^*_{u,i} v_g = c \cdot t^*_{u,i} \phi(g^{-1}) v =$$

$$= c \sum\nolimits_{u',i'j} u'_{i'j}(g^{-1}) t^*_{u,i} t_{u',i'} t^*_{u',j} v = \sum\nolimits_j u_{ij}(g^{-1}) \psi_{u,j}.$$

$\psi_{u,i}^*\psi_{u,j} \in \mathrm{Hom}(\iota,\iota)$ are multiples of $\mathbf{1}_M$, and hence invariant under α_g. Because of the linear transformation law, the matrix $c_{u,ij} = \psi_{u,i}^*\psi_{u,j}$ commutes with $u(G)$ and hence $c_{u,ij} = \delta_{ij}$.
On the other hand, the sums $\sum_i \psi_{u,i}\psi_{u,i}^* = \frac{d}{\dim_{\rho_u}} \sum_i \iota(t_{u,i}^*)vv^*\iota(t_{u,i})$ are also invariant under α_g, hence

$$\sum_i \psi_{u,i}\psi_{u,i}^* = E\Big(\sum_i \psi_{u,i}\psi_{u,i}^*\Big) = \frac{d}{\dim_{\rho_u}} \sum_i t_{u,i}^* E(vv^*)t_{u,i} = \frac{\dim(u)}{\dim_{\rho_u}} \cdot \mathbf{1}_M.$$

Because $\psi_{u,i}$ are orthonormal isometries, it follows that $\dim_{\rho_u} = \dim(u)$, and that the orthonormal projections $\psi_{u,i}\psi_{u,i}^*$ (for each u separately) are a partition of $\mathbf{1}_M$ into orthonormal projections.

In particular, with $t_0 = d^{-1/2} \cdot w$ for the trivial representation $\pi_0(g) = 1$, one gets $\psi_0 = \mathbf{1}_M$, and ψ_u is unitary for every one-dimensional representation u. If $\dim(u) > 1$, then $\psi_{u,i}$ are the generators of the Cuntz algebra $O_{\dim(u)}$.

Comparison with the general case

For a general subfactor, $\gamma\iota = \iota\theta$ fails to be a multiple of ι (depth > 2). The decompositions are of the form

$$\gamma = \bigoplus_a m_a \gamma_a, \qquad \theta = \bigoplus_c n_c \theta_c$$

and γ_a or θ_c need not be automorphisms. The multiplicities m_a and $n_c$are bounded by $m_a \leq \dim_{\gamma_a}$ and $n_c \leq \dim_{\theta_c}$. Neither d nor the dimensions $\dim_{\gamma_a}$ or $\dim_{\theta_c}$ need to be integers, but always $d^2 = \dim_\theta = \sum_c n_c \dim_{\theta_c} = \dim_\gamma = \sum_a m_a \dim_{\gamma_a}$.

The charged generators $\psi_{c,i} := (\frac{\dim_\iota}{\dim_{\theta_c}})^{1/2} \cdot \iota(t^*_{c,i})v \in \mathrm{Hom}(\iota, \iota\theta_c)$ are always orthonormal isometries but in general $\sum_i \psi_{c,i}\psi^*_{c,i}$ will fail to be $= \mathbf{1}_M$. (The linear transformation law is only under a "hypergroup".)

Q-systems

A subfactor $N \subset M$ can be characterized in terms of data referring only to N. These data then define M as an extension of N.

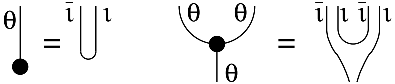
With ι, $\bar{\iota}$ and $[w, v]$ as before, let $x := \bar{\iota}(v)$. Then $x \in \mathrm{Hom}(\theta, \theta^2)$, thus the triple

$$[\theta, w \in \mathrm{Hom}(\mathrm{id}_N, \theta), x \in \mathrm{Hom}(\theta, \theta^2)]$$

refers to N only. The idea is to retain structure info in terms of these data, in such a way that you can recover $N \subset M$ if you "have forgotten" that $\theta = \bar{\iota}\iota$ and $x = \bar{\iota}(v)$.

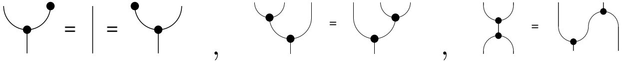
Bert Schroer: "Like a dog who digs his favorite bone, and then he is proud if he is able to find it back."

Forgetting:



Then the triple $[\theta, w, x]$ fulfils the relations

$$w^*x = \theta(w^*)x = \mathbf{1}_N, \quad xx = \theta(x)x, \quad xx^* = \theta(x^*)x,$$



and the normalization $w^*w = x^*x = \dim_\theta^{1/2} \cdot \mathbf{1}_N$. Moreover, $\dim \mathrm{Hom}(\mathrm{id}_N, \theta) = 1$ iff $N \subset M$ is irreducible.

A Q-system is a triple $[\theta, w, x]$ with these properties ($\equiv$ Frobenius algebra in the C* tensor category $\mathrm{End}_0(N)$).

Reconstruction

Theorem: Every Q-system defines an extension $N \subset M$, irreducible iff $\dim \mathrm{Hom}(\mathrm{id}_N, \theta) = 1$.

The reconstruction builds on the fact that, for $N \subset M$,

$$M = \iota(N)v, \quad \iota(n_1)v \cdot \iota(n_2)v = \iota(n_1\theta(n_2)x)v, \quad v^* = \iota(w^*x^*)v.$$

Thus, M can be defined as the *-algebra of elements $[n]$ with relations

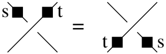
$$[n_1][n_2] = [n_1\theta(n_2)x], \quad [n]^* = [w^*x^*\theta(n^*)],$$

inheriting the topology from N. Then $\iota : n \mapsto [nw^*]$, $\bar\iota : [n] \mapsto \theta(n)x$ are homomorphisms giving back $\bar\iota\iota = \theta$; $w \in \mathrm{Hom}(\mathrm{id}_N, \bar\iota\iota)$ and $v := [1] \in \mathrm{Hom}(\mathrm{id}_M, \iota\bar\iota)$ give back the solution of the conjugacy relations such that $\bar\iota(v) = x$; and one has $[n] = \iota(n)v$.

Braiding

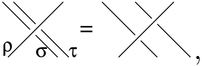
A braided subcategory of $\mathrm{End}_0(M)$ is a full tensor subcategory $\mathcal{C}$ equipped with a unitary braiding $\varepsilon_{\rho,\sigma} \in \mathrm{Hom}(\rho\sigma, \sigma\rho)$ for all $\rho, \sigma \in \mathcal{C}$ such that

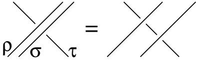
$$(s \times t)\varepsilon_{\rho_1,\sigma_1} = \varepsilon_{\rho_2,\sigma_2}(t \times s)$$



for $t \in \mathrm{Hom}(\rho_1, \rho_2)$, $s \in \mathrm{Hom}(\sigma_1, \sigma_2)$, and

$$\varepsilon_{\rho,\sigma\tau} = \sigma(\varepsilon_{\rho,\tau})\varepsilon_{\rho,\sigma}, \quad \varepsilon_{\rho\sigma,\tau} = \varepsilon_{\rho,\tau}\rho(\varepsilon_{\sigma,\tau}).$$





(Not every subcategory of $\mathrm{End}_0(N)$ admits a braiding, because the braiding implies a unitary equivalence between $\rho\sigma$ and $\sigma\rho$.)

The properties of a braiding imply the relations of the braid group.

A braided subfactor is a subfactor $N \subset M$ such that $\theta = \bar{\iota}\iota$, and hence all sub-endos $\theta_a \prec \theta^n$ belong to a braided category.

The index of braided subfactors is quantized below $d = 6$.

A rather trivial example is $N = M^G \subset M$: The endomorphisms $\theta_u \prec \theta$ ($u \in \mathrm{Rep}(G)$) are equipped with the braiding

$$\varepsilon_{\theta_u,\theta_{u'}} = \sum\nolimits_{i,j} \psi'_i \psi_j \psi'^*_i \psi^*_j$$

where the r.h.s. is in N because it is invariant under G. It extends to direct sums and products by the defining relations. This braiding is in fact a permutation symmetry: $\varepsilon_{\rho,\sigma}\varepsilon_{\sigma,\rho} = 1$.

Non-degeneracy

Many nontrivial examples of braided categories and braided subfactors arise naturally in QFT in two spacetime dimensions (Lecture III).

Special interest: the braiding is non-degenerate iff $\varepsilon_{\rho,\sigma}\varepsilon_{\sigma,\rho} = 1$ for all σ is only possible for ρ a multiple of id. In this case, the numerical values of the intertwiners $\in \mathrm{Hom}(\mathrm{id}, \mathrm{id})$

define unitary matrices T (diagonal) and S satisfying the relations $S^4 = (ST)^3 = \mathbf{1}$, $(S^2T)^2 = T^2$ that define the group $SL(2, \mathbb{Z})$.

A non-degenerate braided tensor category is called modular.

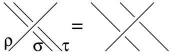
Quantum double

A fusion category $\mathcal{C}$ is a tensor category with finitely many irr objects of finite dimension. Its "global dimension" is $\dim(\mathcal{C}) := \sum_\rho \dim_\rho^2$.

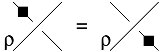
Examples: the full tensor subcategories of $\mathrm{End}_0(N)$ generated by $\theta_c \prec \theta$ of a subfactor with finite principal graph.

Even if $\mathcal{C}$ does not admit a braiding, it always has half-braidings:
A half-braiding for an object $\rho \in \mathcal{C}$ (not necessarily irreducible) is a family of unitaries $e_\rho(\tau) \in \mathrm{Hom}(\rho\tau, \tau\rho)$ for all $\tau \in \mathcal{C}$, such that

$$e_\rho(\sigma\tau) = \sigma(e_\rho(\tau))e_\rho(\sigma)$$



$$te_\rho(\tau_1) = e_\rho(\tau_2)\rho(t) \quad (t \in \mathrm{Hom}(\tau_1, \tau_2))$$



The pairs $[\rho, e_\rho]$ (with e_ρ a half-braiding for ρ) form the objects of a tensor category $D(\mathcal{C})$ (the "center" or "quantum double" or "Drinfel'd double" of $\mathcal{C}$).

The morphisms in $\mathrm{Hom}([\rho, e_\rho], [\sigma, e_\sigma])$ are intertwiners $s \in \mathrm{Hom}(\rho, \sigma)$ such that $\tau(s)e_\rho(\tau) = e_\sigma(\tau)s$, and the monoidal product is $[\rho, e_\rho] \times [\sigma, e_\sigma] := [\rho\sigma, e_{\rho\sigma}]$ with $e_{\rho\sigma}(\tau) := e_\rho(\tau)\rho(e_\sigma(\tau))$.

Theorem:

$$\varepsilon_{[\rho,e_\rho],[\sigma,e_\sigma]} := e_\rho(\sigma)$$

is a non-degerate braiding for $D(\mathcal{C})$, i.e. $D(\mathcal{C})$ is a modular tensor category, and $\dim(D(\mathcal{C})) = \dim(\mathcal{C})^2$.

Lecture II: MODULAR THEORY OF SUBFACTORS

The general setting

Pairs $[M, \Phi]$ where M is given on a Hilbert space H and $\Phi \in H$, cyclic and separating for M. The vector is then also cyclic and separating for M'.

(Equivalently, $\varphi = (\Phi, \cdot\Phi)$ is faithful and Φ its GNS vector.)

The unbounded antilinear map $S : m\Phi \mapsto m^*\Phi$ $(m \in M)$ is densely defined and closable, and its closure admits a polar decomposition

$$S = J\Delta^{\frac{1}{2}}.$$

This defines the "modular data" $[\Delta, J]$ of the pair $[M, \Phi]$.

Main Theorem

(i) J is anti-unitary, $\Delta \geq 0$, and $J\Phi = \Delta\Phi = \Phi$.

(ii) $J^2 = \mathbf{1}$ and $J\Delta = \Delta^{-1}J$, hence $J\Delta^{it} = \Delta^{it}J$.

(iii) $\Delta^{it}M\Delta^{-it} = M$, defining an automorphism group $\sigma_t := \mathrm{Ad}_{\Delta^{it}}$ of M (the modular group).

(iv) $JMJ = M'$, defining an anti-isomorphism $j = \mathrm{Ad}_J$ of M with its commutant (the modular conjugation).

(v) For any fixed $a, b \in M$, the function

$$\mathbb{R} \ni t \mapsto f(t) := \varphi(\sigma_t(a)b) \equiv (\Phi, a\Delta^{-it}b\Phi)$$

extends to a bounded analytic function in the strip $0 < Im(z) < 1$ and satisfies the KMS relation

$$f(t+i) = \varphi(b\sigma_t(a)) \equiv (\Phi, b\Delta^{it}a\Phi).$$

Comment

While the analyticity and boundedness in the strip are nontrivial, the KMS relation itself is an easy computation, using $\Delta^{\frac{1}{2}} = JS$:

$$f(t+i) = (a^*\Phi, \Delta^{-it+1} b\Phi) = (\Delta^{it} a^*\Phi, \Delta b\Phi) =$$

$$= (\Delta^{\frac{1}{2}}\sigma_t(a^*)\Phi, \Delta^{\frac{1}{2}} b\Phi) = (Sb\Phi, S\sigma_t(a^*)\Phi) = (b^*\Phi, \sigma_t(a)\Phi).$$

It controls the failure of φ to be a tracial state: If φ is tracial, then the r.h.s. $= f(t)$, hence f is periodic with period i, hence bounded analytic in $\mathbb{C}$, hence constant, hence σ_t is trivial.

On a commutative algebra, every state is tracial: modular theory is void for Classical Physics.

Relation to Quantum Thermodynamics

The KMS property (v) is well known in Quantum Statistical Mechanics.

It obviously holds for Gibbs states $\mathrm{Tr}\,(e^{-\beta H}\,\cdot\,)/\mathrm{Tr}\,(e^{-\beta H})$ w.r.t. the rescaled time evolution $\alpha_t(a) = e^{-i\beta tH} a e^{i\beta tH}$; and it is known to entail the same thermodynamical properties ("Second Law") as one would expect from thermal equilibrium states.

Thus, every faithful normal state of a vN algebra is a KMS state w.r.t. some adapted dynamics with "Hamiltonian" $-\log\Delta$.

Identifying the modular group with some known automorphism group, will be most fruitful in QFT applications (Lecture III).

One has trivial relations $\Delta_{[\mathrm{Ad}_U M, U\Phi]} = U\Delta_{[M,\Phi]}U^*$ and $\Delta_{[M',\Phi]} = \Delta^{-1}_{[M,\Phi]}$, and likewise for J.
There is a deep comparison theory for the modular data of M as the state varies. Spectral properties of $\Delta_{[M,\Phi]}$ varying over all Φ allow a finer classification of type III factors.

Let us see what Modular Theory says about "two algebras and one state", in particular, subfactors.

Takesaki's Theorem

Theorem. $N \subset M$ a pair of von Neumann algebras, Φ cyclic and separating for M, and σ_t the modular group of $[M, \Phi]$.
Then the following are equivalent:

(i) N is globally invariant under the modular group of $[M, \Phi]$, i.e., $\sigma_t(N) = N$.

(ii) There exists a conditional expectation $E : M \to N$ which preserves the state $\varphi = (\Phi, \cdot\, \Phi)$, i.e., $\varphi(E(m)) = \varphi(m)$.

In this case, E is implemented by the projection P onto the cyclic subspace $\overline{N\Phi}$, i.e., $PmP = E(m)P$. Moreover, the modular data of $[N, \Phi]$ on the cyclic subspace coincide with the restrictions of the modular data of $[M, \Phi]$:

$$P\Delta_{[M,\Phi]} = \Delta_{[M,\Phi]}P = \Delta_{[N,P\Phi]}, \quad PJ_{[M,\Phi]} = J_{[M,\Phi]}P = J_{[N,P\Phi]}.$$

Consequences for subfactors

If $N \subset M$ is a proper subalgebra, preserved by the modular group of M, then $P \neq \mathbf{1}$ (because otherweise $E = \mathrm{id}$), hence Φ cannot be cyclic for N.

An irreducible subfactor $N \subset M$ of finite index (as in Lecture I) has a unique conditional expectation $E : M \to N$. By the theorem, the modular group of M w.r.t. every E-invariant state $\varphi = \varphi \circ E$ preserves the embedded algebra $\iota(N)$.

If φ is any faithful state on M, then $\omega := \varphi \circ E$ is invariant, and $\iota(N)$ is invariant under the modular group w.r.t. ω.

A modular formula for the canonical endomorphism

The other extreme, Φ cyclic and separating for both N and M, hence $P = \mathbf{1}$, is also of interest. The theorem implies that N cannot be invariant under the modular group of M, and the conjugations J_N and J_M are different. One has

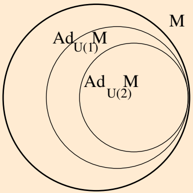
$$\mathrm{Ad}_{J_N}(\mathrm{Ad}_{J_M}(M)) = \mathrm{Ad}_{J_N}(M') \subset \mathrm{Ad}_{J_N}(N') = N,$$

and $\gamma := \mathrm{Ad}_{J_N J_M}$ is a useful formula for the canonical endomorphism of M, from which one may recover a conjugate $\bar{\iota} : M \to N$ via $\bar{\iota} := \iota^{-1} \circ \gamma$. This is well-defined even in the case of infinite index.

The Jones extension (basic construction) $M \subset M_1$ of $N \subset M$ can be defined as $M_1 := \mathrm{Ad}_{J_M J_N}(N) = \mathrm{Ad}_{J_M}(N') \supset \mathrm{Ad}_{J_M}(M') = M$. Then the subfactor $M \subset M_1$ is isomorphic to $\gamma(M) \subset N$.

Borchers' theorem

Theorem. M a von Neumann algebra on a Hilbert space H with cyclic and separating vector Φ, and $U(s)$ a unitary one-parameter group that shifts M into itself: $\mathrm{Ad}_{U(s)} M \subset M$ for $s \geq 0$.



Then any two of the following properties imply the third:

(i) $U(s) = e^{isP}$ has positive generator $P \geq 0$.

(ii) $U(s)\Phi = \Phi$.

(iii) The modular unitary group Δ^{it} and the modular conjugation J of M satisfy the commutation relations with $U(s)$.

$$\Delta^{it} U(s) \Delta^{-it} = U(e^{-2\pi t} s) \quad \text{and} \quad JU(s)J = U(-s).$$

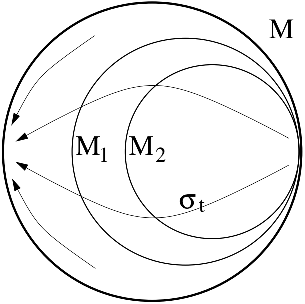
Comment

Notice the one-sided condition $\mathrm{Ad}_{U(s)}M \subset M$ for $s \geq 0$.

If it were true for all s, hence $= M$, and if $U(s)\Phi = \Phi$, then $U(s)$ commutes with the Δ^{it}. (This includes the case $U(s) = \Delta^{it}$.) Then the conclusion (iii) is obviously wrong, hence (i) cannot hold. Thus the generator P (in particular $\log \Delta$) must have two-sided spectrum.

The most remarkable part is (i),(ii) $\Rightarrow$ (iii) (due to Borchers). It establishes an (anti-)unitary representation of the two-parameter translation-dilation group, including the reflection.

Moreover, it fixes how σ_t acts on the subalgebras $M_s = \mathrm{Ad}_{U(s)} M$, namely $\sigma_t(M_s) = M_{e^{-2\pi ts}}$.



A streamlined proof (due to Florig) considers the function

$$\mathbb{R} \ni t \mapsto f(t) = (a'\Phi, \Delta^{it} U(e^{2\pi t} s) \Delta^{-it} a\Phi)$$

for any fixed $a \in M$, $a' \in M'$ and $s > 0$. One shows that by the properties (i)–(iii), f extends to a bounded analytic function on the strip $0 < Im(z) < \frac{i}{2}$.
(Precisely in this strip, $\Delta^{-iz} a\Phi$ and $\Delta^{-i\overline{z}} a'\Phi$ are bounded by the main theorem, and $\| U(e^{2\pi z}) \| \leq 1$ because $Im(e^{2\pi z} s) > 0$ and the generator is positive.)
Then one has

$$f(t + \frac{i}{2}) = (a'\Phi, \Delta^{it} \Delta^{-\frac{1}{2}} U(-e^{2\pi t} s) \Delta^{-it} \Delta^{\frac{1}{2}} a\Phi).$$

Now use $\Delta^{\frac{1}{2}} = JS$ and $\Delta^{-\frac{1}{2}} = SJ$, giving

$f(t + \frac{i}{2}) =$

$$(a'\Phi, \Delta^{it} SJU(-e^{2\pi t}s)\Delta^{-it} JSa\Phi) = (a'\Phi, \Delta^{it} SV(e^{2\pi t}s)\Delta^{-it} a^*\Phi),$$

where $V(s) := JU(-s)J$. Now, $V(s)$ has the same properties as $U(s)$ (e.g., $V(s) = Je^{-isP}J = e^{+isJPJ}$ has generator $JPJ \geq 0$, etc), hence $\mathrm{Ad}_{V(e^{2\pi t}s)}\sigma_{-t}(a^*) \in M$, and one gets

$$f(t + \frac{i}{2}) = (a'\Phi, \Delta^{it} V(e^{2\pi t}s)\Delta^{-it} a\Phi) =: g(t).$$

Interchanging the roles of U and V, one has in the same way that g extends to the strip $0 < Im(z) < \frac{i}{2}$, and $g(t + \frac{i}{2}) = f(t)$.

By glueing one strip to the next, one obtains a bounded analytic function on all of $\mathbb{C}$, which must be constant. Then $f(t) = f(0)$ establishes the first, and $g(0) = f(\frac{i}{2}) = f(0)$ establishes the second of the commutation relations (iii).

The direction (ii),(iii) $\Rightarrow$ (i) (due to Wiesbrock) is very interesting, because it establishes the positivity of a self-adjoint operator (which in QFT applications is related to the Hamiltonian, Lecture III).

The positivity of P also allows to show that the assumptions of Borchers' theorem determine the type of a factor M, which can only be type III_1 (unless $P = 0$).

Halfsided modular inclusions

The situation described in Borchers' theorem implies that the algebra $N := \mathrm{Ad}_{U(1)}(M) \subset M$ is shifted into itself by σ_t (namely $\sigma_t(N) \subset N$) precisely for $t < 0$, because $\sigma_t(N) = \Delta^{it}\mathrm{Ad}_{U(1)}(M)\Delta^{-it} = \mathrm{Ad}_{U(e^{-2\pi t})}(M) \subset \mathrm{Ad}_{U(1)}(M)$ whenever $e^{-2\pi t} > 1$.

An inclusion $N \subset M$ such that $\sigma_{\pm t}(N) \subset N$ for $t > 0$, and Φ also cyclic for N, is called $\pm$half-sided modular ($\pm$hsm). One has the converse

Theorem: Every $-$hsm inclusion $N \subset M$ defines a unitary one-parameter group $U(s)$ such that $N = \mathrm{Ad}_{U(1)}(M)$, with the properties as in Borchers' theorem. Namely, the generator is $P = \frac{1}{2\pi}[\log(\Delta_N) - \log(\Delta_M)]$. In particular, $P \geq 0$.

Modular origin of symmetry groups

There are several extensions of the Borchers and hsm results, involving more than two vN algebras with a joint cyclic and separating vector. "Suitable modular positions" of three algebra w.r.t. each other can guarantee that their modular groups generate unitary representations of the Poincaré group in two dimensions, the Lorentz group in three dimensions, or the Möbius group.

A "suitable modular position" for the last instance is the following: M_1, M_2, M_3 are three commuting algebras with a joint cyclic and separating vector Ω, such that $M_1 \subset M_2'$, $M_2 \subset M_3'$, $M_3 \subset M_1'$ are all $-$hsm.

The resulting representation of the Möbius group $SL(2,\mathbb{R})/\mathbb{Z}_2$ has positive generator for its translation and rotation subgroups. The relevance for QFT will become clear in Lecture III.

Lecture III: SUBFACTORS IN QUANTUM FIELD THEORY

Quantum field theory

QFT describes systems with continuously many degrees of freedom, so they can be decomposed into subsystems in manifold ways, and thus become a source of many interesting subfactors.

The quantum observables in an open region O of spacetime form a C* algebra $A(O)$. It is assumed that

$$A(O_1) \subset A(O_2) \quad \text{if} \quad O_1 \subset O_2$$

(turning the collection of local algebras into a "net").

Einstein causality (= locality) asserts that $A(O_1)$ commutes with $A(O_2)$ (within $A(O)$ containing both) whenever O_1 and O_2 are spacelike separated. Thus, in every representation, one has inclusions

$$\pi(A(O)) \subset \pi(A(O'))'.$$

Representations

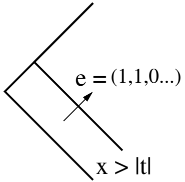
A positive-energy representation is a simultaneous Hilbert space representation of these algebras, compatible with the inclusions and covariant under a unitary representation U_π of the Poincaré group

$$U_\pi(a,\Lambda)\pi(A(O))U_\pi(a,\Lambda)^* = \pi(A(a+\Lambda O)),$$

in which the Hamiltonian (= generator of time translations) has nonnegative spectrum.

This gives rise to the scenario in Borchers' theorem:

Lightlike translations shift wedge regions into themselves. Their unitary representatives $U(s) = U_\pi(s\cdot \underline{e}, 1)$ shift $\pi(A(W))$ into itself, and they have positive generator.



Vacuum representation

The vacuum representation is an irreducible positive-energy representation π_0 with a unique vector Ω invariant under U_{π_0}.

We make some simplifying assumptions: There is a unique vacuum representation, and it is faithful for the local algebras of regions O with a nontrivial causal complement O'. It may then be then taken as the definition of the theory, i.e., $\pi_0(a) \equiv a$. In this case, no information is lost if $A(O) \equiv \pi_0(A(O))$ is replaced by its weak closure $A(O)'' \equiv \pi_0(A(O))''$, hence $A(O)$ may be considered as von Neumann algebras from the outset, and they are factors.

By taking the inductive limit of the net of vNAs $A(O)$ as O runs over increasing doublecones (= intersections of a future and a past lightcone), one gets the C* algebra A of quasilocal observables.[1]

---

[1] One should not take its weak closure of the quasilocal algebra A because this is just $B(H)$ if the vacuum representation is irreducible.

Modular theory applies

Using Poincaré covariance, positivity of the energy and locality, one proves in relativistic QFT the Reeh-Schlieder theorem: the vacuum vector is cyclic for the local algebra of every open region O. [2]

Then, if O has a nontrivial causal complement, Ω is also cyclic for $A(O)' \supset A(O')$, hence also separating for $A(O)$, and modular theory (Lecture II) applies for $[M, \Phi] = [A(O), \Omega]$.

[2] In particular, the Hilbert space does *not* tensor factorize for subsystems in disjoint spacetime or space regions. In this respect relativistic QFT is *very* unlike QM or lattice spin systems.

Borchers' theorem applies

We have already seen that wedges and lightlike translations give

$$U(s)A(W)U(s)^* \subset A(W) \qquad (s \geq 0),$$

and U has positive generator. Moreover Ω is cyclic and separating for $A(W)$ and invariant under $U(s)$.

Borchers' theorem gives then the commutation relation between the modular group of the wedge algebra and the lightlike translations:

$$\Delta^{it} U(s) \Delta^{-it} = U(e^{-2\pi t} s) \quad \text{and} \quad JU(s)J = U(-s).$$

Bisognano-Wichmann property

Thus, the modular group Δ^{it} of the wedge acts on the lightlike translations of the wedge exactly as the Lorentz boosts $U(\Lambda_{-2\pi t})$ in the direction of the wedge. One would like to show that actually

$$\Delta^{it} = U(\Lambda_{-2\pi t}).$$

This relation was proven in the Wightman field setting by Bisognano and Wichmann, but in AQFT it has been established only with various further input.

Because the boosts preserve W and Ω, they commute with Δ^{is}. Therefore, $z(t) := \Delta^{it} U(\Lambda_{2\pi t})$ is a unitary one-parameter group commuting with the 2D Poincaré group of W. The task is to show that such a group must be trivial (see below).

Consequences

If the Bisognano-Wichmann property $\Delta^{it} = U(\Lambda_{-2\pi t})$ is true, then $A(W')$ is a subalgebra of $A(W)'$ invariant under the modular group of the latter, for which Ω is cyclic, hence by Takesaki's theorem (Lecture II), duality follows:

$$A(W)' = A(W').$$

If the BW property is taken as a "first principle" (i.e., not derived from positivity of the energy via Borchers' theorem, (i),(ii) $\Rightarrow$ (iii)), it also implies positivity of the energy and type III_1 for $A(W)$ (again Borchers' theorem, (ii),(iii) $\Rightarrow$ (i)).

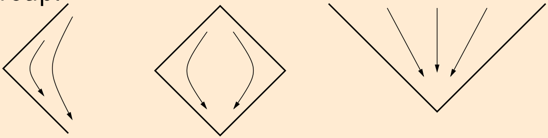
Another interesting case is $M = A(V_+)$ where V_+ is the future lightcone.

The causal complement of V_+ is empty, so the Reeh-Schlieder argument doesn't apply to show that the vacuum is separating for M. But in conformal QFT in even spacetime dimension, the Huygens principle holds: the commutator is non-vanishing only at lightlike distance, hence $A(V_+)$ commutes with $A(V_-)$, and Ω *is* cyclic and separating.

By the Huygens principle, Borchers' theorem applies to *all timelike* translations. It follows that Δ^{it} scales *all* translations and commutes with all Lorentz transformations, and can be used to extend the Poincaré group by scale transformations.

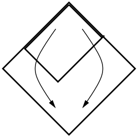
Brunetti-Guido-Longo theorem

Theorem: In a conformal QFT, the Bisognano-Wichmann theorem holds not only for wedges but also for lightcones and doublecones, i.e., one can identify the modular group with a subgroup of the conformal group.



To prove this, one needs the net and the whole conformal group.

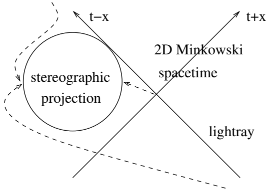
In all these cases, when the modular group has a geometric interpretation in terms of spacetime symmetries, there are lots of "geometric" halfsided modular subfactors, like this:



Chiral QFT

Chiral QFTs appear within conformal QFT in two spacetime dimensions, as subtheories invariant under one of the two chiral subgroups of the 2D conformal group.

Their "regions" are intervals on the circle S^1 (that arises by a one-point compactification of a lightlike axis), and their spacetime symmetry group is the Möbius group Möb $= SL(2,\mathbb{R})/\mathbb{Z}_2$.



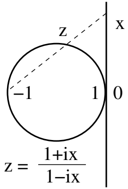
The Möbius group

The Möbius group $SL(2,\mathbb{R})/\mathbb{Z}_2$ acts by fractional linear transformations

$$x \mapsto \frac{ax+b}{cx+d}$$

on $x \in \mathbb{R} \cup \{\infty\}$. Under a stereographic projection $\mathbb{R} \to S^1$, it becomes the group $SU(1,1)/\mathbb{Z}_2$ mapping

$$S^1 \ni z \to \frac{\alpha z+\beta}{\overline{\beta} z+\overline{\alpha}}.$$



The subgroups

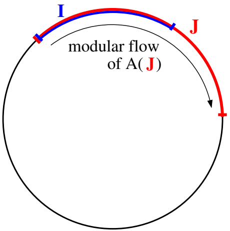
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}, \quad \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\sin\varphi & \cos\varphi \end{pmatrix}$$

are, respectively, the translation and dilation subgroups of $\mathbb{R}_+$ ($=$ upper half circle) and the rotations of the circle.

For an interval J (e.g., $\mathbb{R}_+$ = upper half circle) and a subgroup τ_s that leaves exactly one of its endpoints fixed (e.g., the translations), one gets a one-sided embedding $\mathrm{Ad}_{U(\tau_s)}A(J) \subset A(J)$ for $s > 0$.

Borchers' theorem entails that $z(t) = \Delta_J^{it} U(\delta_{2\pi t})$ commutes with $U(\tau_s)$, where $\delta(t)$ is the subgroup that leaves both endpoints of J fixed (e.g., the scale transformations of $\mathbb{R}_+$), and (using Möbius covariance) one can establish the Bisognano-Wichmann property $\Delta_J^{it} = U(\delta_{-2\pi t})$, and Haag duality $A(I)' = A(I')$ follows.

Then $A(I) \subset A(J)$ is $-$hsm
for $I = \tau_1(J)$ and $A(I) = \mathrm{Ad}_{U(\tau_1)}A(J)$.



Guido-Longo-Wiesbrock theorem

Every splitting of the circle into three intervals gives rise to three commuting algebras $M_i = A(I_i)$ for which the vacuum is a joint cyclic and separating vector, and $M_i \subset M'_{i+1\,(mod 3)}$ (in the appropriate cyclic order) is a $-$hsm subfactor.

Theorem. Three vNAs with $M_i \subset M'_{i+1\,(mod 3)}$ all $-$hsm w.r.t. a joint cyclic and separating vector Φ define a chiral conformal QFT by taking Φ as the vacuum vector, the positive-energy representation U of Möb generated by the three modular groups as the covariance (cf. Lecture II), and $A(I_i) := M_i$. The local algebras for other intervals are consistently defined by transporting $A(I_i)$ with $U(g)$.

A full QFT is encoded in three factors in suitable modular position.

Haag duality

Along with the Bisognano-Wichmann property, "Haag duality for wedges" holds in the vacuum representation: $A(W) = A(W')'$.

"Haag duality" is the same relation required to hold also for doublecones:

$$A(O) = A(O')'.$$

This a priori stronger property can be derived only with some further assumptions; it is automatic in conformal QFT because conformal transformations can take wedges to doublecones.

Doplicher-Haag-Roberts theory

Assume Haag duality in the vacuum representation. In any other representation, it may (and will) fail, and one has a subfactor

$$\pi(A(O)) \subset \pi(A(O'))'.$$

An important class of representations are "indistinguishable from the vacuum representation in the causal complement of every doublecone" (with the physical idea, that some "charge", that may be localized in O, cannot be detected by measurements in O').

Requiring unitary equivalence

$$\pi(a) = VaV^* \quad \text{for} \quad a \in A(O')$$

(where the unitary operator $V : H_0 \to H_\pi$ will depend on the choice of O), one defines for some fixed O

$$\rho(a) := V^*\pi(a)V \quad \text{for all} \quad a \in A.$$

A priori, ρ is just an operator representation on H_0 equivalent to π.

But $\rho(a)$ commutes with $A(O_2')$ whenever $a \in A(O_1)$ and O_2 contains both O and O_1, hence, by Haag duality, $\rho(a)$ is actually in $A(O_2)$. In particular, ρ is an endomorphism of the C* algebra A of quasilocal observables (DHR endomorphism); and $\rho|_{A(O')} = \mathrm{id}$ and

$$\rho(A(O)) \subset A(O).$$

The latter subfactor is isomorphic to $\pi(A(O)) \subset \pi(A(O'))'$ because $V^*\pi(A(O))V = \rho(A(O))$ while
$V^*\pi(A(O)')'V = (V^*\pi(A(O'))V)' = \pi_0(A(O'))' = A(O)$ by Haag duality.

Thus, ρ viewed "locally" as an endomorphism of $N = A(O)$, carries information about the failure of Haag duality of the "global" representation of the net associated with ρ.

DHR category

DHR endomorphisms are the objects of a unitary braided tensor category DHR(A). The existence of the braiding is another consequence of locality, Haag duality, and covariance.

There is a Spin-Statistics theorem relating properties of the braiding to quantum numbers of the representation of the spacetime symmetry group.

In 4D, because the braiding is actually a permutation symmetry, DHR(A) is always equivalent to Rep(G) for some compact group, and a canonical local or graded local field algebra F can be reconstructed such that $A(O) = F(O)^G$.

These facts represent a wealth of structural links, mediated by subfactor theory, between locality and duality, inner and outer symmetries, charges, and statistics.

Chiral conformal QFT admits DHR representations of a very different nature than those related to Rep(G) of some gauge group.

There exist many constructions of models of chiral QFT.
Examples:

- Loop groups: A unitary representation of a loop group LG gives rise to local algebras $A(I) := U(L_I G)''$, where $L_I G$ is the subgroup of loops $\ell : S^1 \to G$ supported in an interval I).
- Lattices: The exponentiation of the chiral current of the massless Klein-Gordon field in 2D gives rise to local "vertex operators" provided the exponential weights are taken from an even lattice.
- ...

These chiral QFTs possess very rich DHR categories (and are one of the main tools to find modular tensor categories).

Even if $A(I) = B(I)^G$ for some other local CFT B, apart from the sectors corresponding to Rep(G), there arise additional "twisted" sectors that can be associated with the Drinfel'd double $D(\mathrm{Rep}(G))$.

Kawahigashi-Longo-Müger theorem

The "two-interval subfactor" (also known as Jones-Wassermann subfactor) is

$$A(E) \subset A(E')'$$

where E is the union of two non-touching intervals of the circle, so that E' is also the union of two non-touching intervals.

Theorem. If a chiral QFT A is strongly additive and split, assume that the index μ of the two-interval subfactor is finite. Then

(i) $\mu = \sum_\rho \dim_\rho^2$, where the sum runs over the irreducible DHR sectors of A.

(ii) The theory possesses finitely many inequivalent irreducible DHR representations (sectors), all with finite dimension $\dim_\rho < \infty$.

(iii) The braiding is non-degenerate, hence the category DHR(A) is modular.

Comments.

Property (i) states that all DHR sectors (= charged representations) can be "detected" by looking at the two-interval subfactor in the vacuum representation.

Property (iii) raises the question (presently much debated) whether every modular tensor category, such as Drinfel'd doubles, arises as DHR(A) for some chiral QFT A.

In large classes of models, the partition functions Tr $e^{-\beta U_\pi(L_0)}$ (L_0 is the generator of the Möbius rotations) are known in all DHR sectors, and they transform into each other under $SL(2,\mathbb{Z})$ transformations $\beta \mapsto \beta - 2\pi i$ and $\beta \mapsto \frac{4\pi^2}{\beta}$ of the inverse temperature β. The linear transformation matrices are the unitary matrices T and S coming along with property (iii) (Lecture I). It is conjectured that this is true in general.

The two-interval subfactor has many further remarkable properties:

(iv) It does not depend (up to equivalence) on the pair of intervals, hence it is isomorphic to its commutant $A(E') \subset A(E)'$, hence anti-isomorphic to its dual.

(v) It is an instance of a "Longo-Rehren subfactor" whose dual canonical endomorphism has the structure $\Theta = \bigoplus_\rho \rho \otimes \overline{\rho}$, where the isomorphism $A(E) \equiv A(I) \vee A(J) \cong A(I) \otimes A(J)$ for $E = I \cup J$ (split property) is appealed to.

(vi) It is isomorphic to a canonical local extension

$$A_2(O) \subset B_2(O)$$

of two-dimensional CFTs, where $A_2(O) = A_+(I) \otimes A_-(J)$ for $O = I \times J$ is the tensor product of two isomorphic chiral subtheories of B_2. ("Extensions" are discussed next.)

Extensions

A local extension of a QFT A is another QFT B such that $A(O) \subset B(O)$ for all spacetime regions O, which admits a conditional expectation $E : B \to A$ such that $E(B(O)) = A(O)$ and the vacuum state is invariant, $\omega_B \circ E = \omega_A$.
Obvious example: B with a global gauge symmetry and an invariant vacuum state, extends its invariant subtheory $A = B^G$.

If B is not local, but relatively local w.r.t. A (i.e., $A(O_1)$ commutes with $B(O_2)$ if O_1 and O_2 are spacelike separated), we call it a non-local extension.
Example: A fermionic (graded local) theory B, and A its bosonic (even) subtheory.

Chiral and two-dimensional conformal QFTs admit "much more interesting" local and non-local extensions.

Characterization by Q-systems

Longo and myself have established the basic facts: the index $[B(O) : A(O)]$ is independent of O, and a single local subfactor $N = A(O_0) \subset M = B(O_0)$ encodes all the structure of the extension.

Thus, the theory of finite index subfactors applies: $A(O_0) \subset B(O_0)$ can be characterized by its Q-system $[\theta, w, x]$.

As a "memory" of the presence of an entire local net of algebras, $\theta \in \mathrm{End}(A(O_0))$ is a DHR endomorphism of A, localized in O_0 and restricted to O_0. In fact, as a representation of A, θ is the vacuum rep of B, restricted to A.

The even vertices of the dual graph ($=$ subsectors of θ^n) are DHR endomorphisms of A. On the other hand, the even vertices of the principal graph ($=$ subsectors of γ^n) are endomorphisms of B without localization properties, such as the global group automorphisms in the case of $A = B^G$.

Reconstruction

The reconstruction uses only the Q-system $[\theta, w, x]$ with $\theta \in DHR(A)$ localized in some O_0.

It works "from local to global": First recover $B(O_0) = A(O_0)v_0$ from the Q-system and $A(O_0)$. Then recover the whole net B via $B(O) = A(O)v$ where $v = uv_0$ with a unitary $u \in \mathrm{Hom}(\theta, \theta^{(O)}) \subset A$ that takes θ to an equivalent DHR endo localized in O.

Theorem. An extension $A \subset B$ defined by a Q-system in DHR(A) is local iff the Q-system is "commutative" w.r.t. the DHR braiding:

$$\varepsilon_{\theta,\theta} \circ x = x.$$

Classifications

Theorem. If A is completely rational, then DHR(A) admits only finitely many irreducible Q-systems, hence A admits only finitely many irreducible extensions.

For some classes of chiral theories, the local and non-local extensions have been classified. Namely, for given DHR(A), there are only few candidates for θ, and the defining relations of a Q-system $[\theta, w, x]$ allow to "solve for x" in the finite-dimensional space $\mathrm{Hom}(\theta, \theta^2)$.

If $A_2 = A_+ \otimes A_-$ is the tensor product of two chiral theories, two-dimensional extensions $A_2 \subset B_2$ can be characterized in the same way with the product of two DHR categories.

Boundary conditions in AQFT

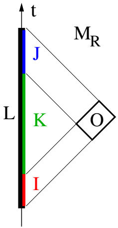
In QFT, boundary conditions cannot be simply *imposed* (like "Dirichlet") as operator relations, but are subject to algebraic consistency conditions. (E.g., something that should satisfy canonical commutation relations, cannot be asked to be zero. Causality imposes further restrictions.)
This is why a "classification of boundary conditions" is the outcome rather than an input of the subsequent analysis.

Hard boundaries

In 2D conformal QFT on a halfspace $M_R = \{(t,x) : x > 0\}$, the boundary at $x = 0$ implies an identification of the two chiral subtheories. Thus, a boundary theory is defined in a representation space of a single chiral theory A, rather than a tensor product $A_+ \otimes A_-$.

If A is completely rational, the local boundary theories with chiral subtheory A can be classified:

The local algebras $B_R(O)$ ($O \subset M_R$) are intermediate between $A(I) \vee A(J)$ and $B(K)' \cap B(L)$. Here, B is any relatively local chiral extension of A.



In the case $B = A$, the subfactor $A(I) \vee A(J) \subset A(K)' \cap A(L)$ is again the two-interval subfactor.

Transparent boundaries

A transparent boundary between a QFT B^L on the left halfspace and a QFT B^R on the right halfspace by definition preserves energy and momentum. In the 2D conformal case this implies equality of the stress-energy tensors of the two theories, hence both theories extend to the full Minkowski spacetime and share a common subtheory $A = A_+ \otimes A_-$ containing the stress-energy tensor.

However, locality for the observables “in their own halfspace” only requires that $B^L(O_1)$ commutes with $B^R(O_2)$ when O_1 is in the *left* causal complement of O_2 (“left locality”). A net defined by local algebras $C(O) := B^L(O) \vee B^R(O)$ is in general non-local.

Conformal QFTs with a transparent boundary are then realizations of diagrams of subfactors for every O

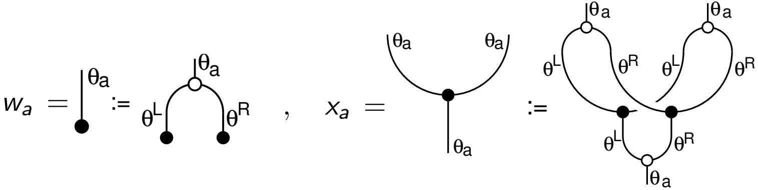
$$\begin{tikzcd} & B^L(O) \arrow[dr, hook, "\jmath^L"] & \\ A(O) \arrow[ur, hook, "\iota^L"] \arrow[dr, hook, "\iota^R"'] & & C(O) \\ & B^R(O) \arrow[ur, hook, "\jmath^R"'] & \end{tikzcd}$$

such $\jmath^L \circ \iota^L = \jmath^R \circ \iota^R$ (common subalgebra A) and that $\jmath^L(B^L)$ is left-local w.r.t. $\jmath^R(B^R)$.

DHR theory is powerful enough that this kind of problems (i.e., also the inclusions $\jmath^X : B^X \to C$ in the diagram) can be controlled in terms of data in DHR(A).

We (Bischoff-Kawahigashi-Longo-KHR) have shown: If both B^L, B^R are local, then the irreducible realizations of such commuting diagrams are classified by certain elements of $\mathrm{Hom}(\theta^R, \theta^L)$.

These correspond (by Frobenius reciprocity) to irreducible sub-homomorphisms γ_a^{LR} of $\iota^R\bar{\iota}^L$. The Q-system for $A \subset C$ is given by $\theta_a = \bar{\iota}^L\gamma_a^{LR}\bar{\iota}^R \prec \theta^L\theta^R \in DHR(A)$ along with the pair of intertwiners



In the standard case $B^L = B^R$, $\iota^L = \iota^R$, this leads to a new physical interpretation of subsectors γ_a of $\gamma = \iota\bar{\iota}$: they determine certain quadratic algebraic relations between the embedded charged generators $\jmath^L(\psi_\rho) \in C$ of $\jmath^L(B)$ and $\jmath^R(\psi_\rho) \in C$ of $\jmath^R(B)$, that constitute the boundary conditions. In special cases, the quadratic relations may reduce to linear relations, generalizing the gauge transformations of charged generators $\psi_{u,i}$ of $A = B^G$.