Measuring $\Gamma(K_{e3})$ and $|V_{us}|$

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From PDG

	m MeV	Δ Mev	Γ 10^7 s^{-1}	BR($e3$)	$\Gamma(e3)$ 10^6 s^{-1}
$K^{\pm}$	493.677	358.190	8.07	0.0482	3.89
error	-	-	0.19%	1.24%	1.26%
K_L	497.672	357.592	1.93	0.3878	7.50
error	-	-	0.77%	0.72%	1.06%

The above rates for K_{e3} determine, in principle, $|V_{us}|^2$ to 0.8% and $|V_{us}|$ to 0.4%. Yet in PDG

$$|V_{us}| = 0.2196 \pm 1.05\%.$$

The problem is estimating-guessing matrix element corrections due to isospin and $SU(3)_{\text{flavor}}$ symmetry breaking.

KLOE can improve the accuracy in the knowledge of the decay width $\Gamma(e3)$ and can in fact measure the kaon partial width in a more direct way than usually done.

Partial widths for channel j are typically obtained from measurements of the total width, $\Gamma = 1/\tau$ and the branching ratio, BR_j. Then $\Gamma_j = \mathrm{BR}_j \times \Gamma$. This however means that the final result has a fractional error equal to the quadrature of the fractional errors on each measurement:

$$\frac{\delta\Gamma_j}{\Gamma_j} = \sqrt{\left(\frac{\delta\tau}{\tau}\right)^2 + \left(\frac{\delta(BR)}{BR}\right)^2}$$

This is avoidable *in principle* in KLOE. In practice the dependence on $\delta\tau/\tau$ can be reduced by a significant factor, $\sim\times 5$.

STATEMENT OF THE PROBLEM

Consider the decays $A \to B$, $B = \{B_1 \ldots B_j \ldots\}$. Let $N(t)$ be the number of A states and dN_j the decays to B_j in dt. Partial and total widths, Γ_j and Γ, satisfy

$$\mathrm{d}N_j = \Gamma_j \times N(t)\mathrm{d}t,\ \mathrm{d}N = \Gamma \times N(t)\mathrm{d}t \quad \text{with} \quad \Gamma = \sum \Gamma_j$$

from which

$$\mathrm{d}N = -\sum \mathrm{d}N_j = -N\mathrm{d}t \quad \text{and} \quad N(t) = N(t=0)e^{-t\Gamma}.$$

Decay widths or rates are measured by counting the number of decays ΔN_j in a time interval Δt. Then

$$\Gamma_j = (\Delta N_j/\Delta t)/N.$$

For ^{238}U: Γ=4.89×10^{-18} s^{-1} (per nucleus) or 661 decays per second per gram (τ=2.05×10^{17} s).

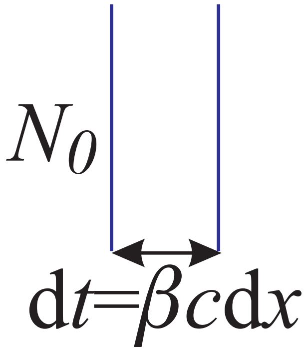
KLOE. By tag, count the number of K_L produced, N_{K_L}. Count the number $N_{\ell 3}$ of semileptonic decays in 1 cm, *i.e.* in 150.7 picoseconds in the K rest frame:

$$\Gamma_{\ell 3} = (N_{\ell 3}/N_{K_L}) \times 6.638 \times 10^9 \text{ s}^{-1}.$$

Γ is still needed because:

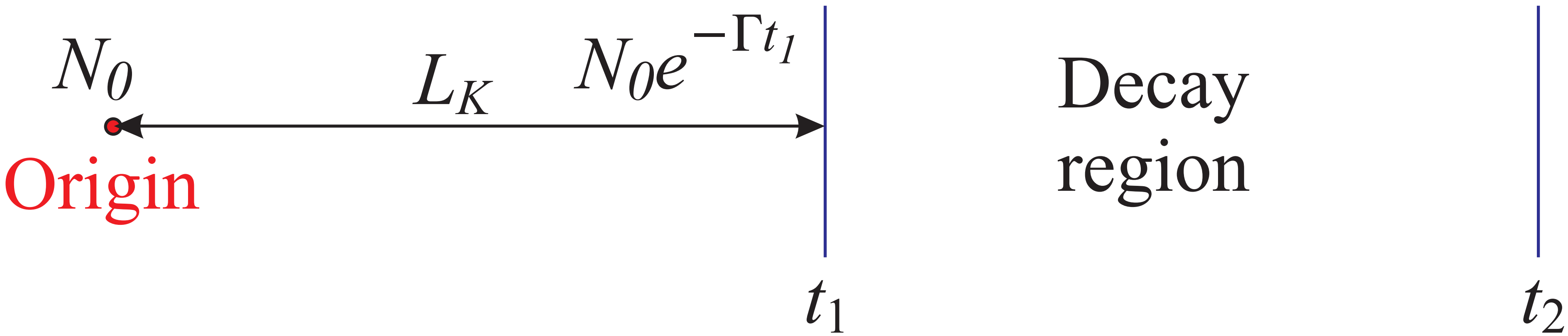
1. Even in one centimeter parent particles are lost to other channels
2. We cannot begin counting at $t = 0$.
3. We need 10^6 $K_{\ell 3}$ decays, *i.e.* many cm of decay path

Ideal case



$$\Gamma = \frac{\mathrm{d}\,N_j}{\mathrm{d}\,t}\frac{1}{N_0}$$

Real case



Corrections for Γ_{K_L} in KLOE.

1. $\delta\Gamma_{\ell 3}/\Gamma_{\ell 3} < l/\lambda \sim 3\times 10^{-3}$ ($\lambda=\gamma\beta c/\Gamma$=343.2 cm)
2. Take $L_K > R = 50$ cm. R/343.2~1/7. Need to know Γ to 0.7%, to reach 0.1% accuracy.
3. Same as above for a decay space of 50 cm.

The exact result is that one needs knowing Γ to ~0.5% for an accuracy on $\Gamma_{\ell 3}$ of 0.14%*. Using $\Gamma_{\ell 3}=\mathrm{BR}(\ell 3)\times\Gamma$ requires an accuracy of 0.1% on Γ.

$$N_j = N_0\Gamma_j\int_{t_1}^{t_2} e^{-\Gamma t}\mathrm{d}t = N_0(\Gamma_j/\Gamma)(e^{-\Gamma t_1}-e^{-\Gamma t_2})$$

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*For equal contributions from Γ and $\Gamma_{\ell 3}$

Expand the exponential up to third order in Δt. To first order there is no dependence on Γ:

$$N_j = N_0 \Gamma_j (t_2 - t_1)\Big(1 - \Gamma\,(t_1 + t_2)/2 + \Gamma^2\,(t_1^2 + t_1 t_2 + t_2^2)/6\Big)$$

The second order correction is -22%, to all orders -19.57%. For 40 cm+40 cm, one gains $\times 6.2$ instead of $\times 5.1$.

Decay rates for $|\, i\,\rangle \to |\, f\,\rangle$ are obtained from the transition probability density $\overline{w}_{fi} = |T_{fi}|^2$ ($S = 1 + iT$):

$$\overline{w}_{fi} = (2\pi)^4 \delta^4(p_i - p_f)(2\pi)^4 \delta^4(0)|\mathfrak{M}|^2$$

where

$$\mathfrak{M} = \langle\, f\,|\mathcal{H}|\, i\,\rangle$$

from which

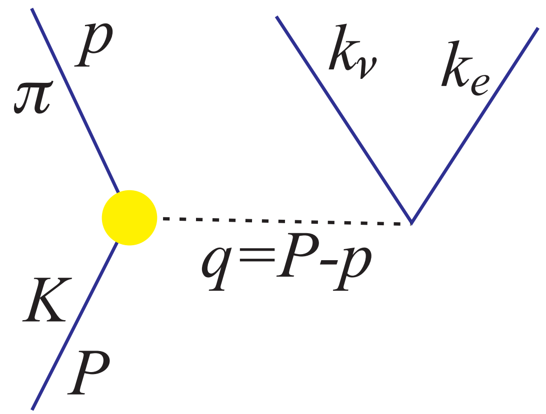
$$\mathrm{d}\Gamma = \frac{1}{8M(2\pi)^3}|\mathfrak{M}|^2 \mathrm{d}E_1 \mathrm{d}E_2.$$

$\Gamma(\ell 3) \propto G_F^2 \times |V_{us}|^2$ but we must deal with a few details.

1. Numerical factors equivalent to an overlap integral between final and initial state. Symmetry breaking corrections, both isospin and $SU(3)_F$.
2. An integral over phase space of $|\mathfrak{M}|^2$.
3. Experiment dependent radiative corrections. Or, bad practice, correct the data.

We are interested in $|V_{us}|$ and we must do something about 1.) through 3.). 2.) is easy, including the q^2 dependence of form factors. Carefull with double counting.

Corrections. We want the amplitude



$$\langle\, \pi\, |J^H_\alpha|\, K\, \rangle$$

with

$$J^H_\alpha = \bar{u}\gamma_\alpha(1-\gamma_5)s.$$

From L-invariance, $\langle\, \pi\, |\bar{u}\gamma_\alpha\gamma_5 s|\, K\, \rangle \equiv 0$

and

$$\langle\, \pi\, |J^H_\alpha|\, K\, \rangle = f_+(q^2)(P+p)_\alpha + f_-(q^2)(P-p)_\alpha$$

with P and p the kaon and pion four momenta and $q = P - p = k(\nu) + k(e)$. Figure above.

For $K \to \pi e \nu$ the f_- term gets an m_e from $(\not{p} - m)u = 0$.

Ignore f_-, error $\sim(m_e/\Delta \sim 1/350)^2 \sim 10^{-5}$.

For $m(u) = m(d) = m(s)$, $\psi(\pi) \equiv \psi(K)$. Then

$$\langle\, \pi^0\, | J^H_\alpha |\, K^+ \,\rangle = \langle\, (u\bar{u} - d\bar{d})/\sqrt{2}\, | u\bar{u} \rangle = 1/\sqrt{2}$$
$$\langle\, \pi^-\, | J^H_\alpha |\, K^0 \,\rangle = \langle\, d\bar{u}\, | d\bar{u} \rangle = 1$$
$$\langle\, \pi^+\, | J^H_\alpha |\, \overline{K^0} \,\rangle = \langle\, \bar{d}u\, | \bar{d}u \rangle = 1$$
$$\langle\, \pi^+\, | J^H_\alpha |\, K_L \,\rangle = -\langle\, \bar{d}u\, | \bar{d}u \rangle/\sqrt{2} = -1/\sqrt{2}$$
$$\langle\, \pi^-\, | J^H_\alpha |\, K_L \,\rangle = \langle\, \bar{d}u\, | \bar{d}u \rangle/\sqrt{2} = 1/\sqrt{2}$$
$$\langle\, \pi^+\, | J^H_\alpha |\, K_S \,\rangle = \langle\, \bar{d}u\, | \bar{d}u \rangle/\sqrt{2} = 1/\sqrt{2}$$
$$\langle\, \pi^-\, | J^H_\alpha |\, K_S \,\rangle = \langle\, \bar{d}u\, | \bar{d}u \rangle/\sqrt{2} = 1/\sqrt{2} \quad (\times f_+ q_\alpha \langle J^L \rangle^\alpha \ldots)$$

Ignoring phase space and form factor differences:

$$\Gamma(K_L \to \pi^\pm e^\mp \bar{\nu}(\nu)) = \Gamma(K_S \to \pi^\pm e^\mp \bar{\nu}(\nu))$$
$$= 2\Gamma(K^\pm \to \pi^0 e^\pm \nu(\bar{\nu}))$$

An approximate integration gives

$$\Gamma = \frac{G^2 |V_{us}|^2}{768\pi^3} |f_+(0)|^2 M_K^5 (0.57 + 0.004 + 0.14\delta\lambda_+)$$

with $\delta\lambda = \lambda - 0.0288$. Integration over phase space gives a leading term $\propto \Delta^5$, where $\Delta = M_K - \Sigma_f(m)$.
$(\Delta_+^5 - \Delta_0^5)/\Delta^5$=0.008.
From data, $\Gamma_0 = (7.5{\pm}0.08){\times}10^6$, $2\Gamma_+ = (7.78{\pm}0.1){\times}10^6$ and $(2\Gamma_+ - \Gamma_-)/\Gamma = (3.66 \pm 0.06)\%$.
This is quite a big difference, but typical of violation of I-spin invariance.
The slope difference is ∼0.001, thus irrelevant. The big problem remains the $s - u, d$ mass difference. For K^0 the symmetry breaking is $\propto (m_s - \langle m_{u,d}\rangle)^2$ in accordance with A-G. But then $(m_s - \langle m_{u,d}\rangle)^2$ acquires dangerous divergences, from a small mass in the denominator. It is argued that it is not a real problem.

Leutwyler and Roos (1985) deal with all these points and radiative corrections. They are quoted by PDG (Gilman *et al.*, 2000), for the value of $|V_{us}|$. After isospin violation corrections, K^0 and K^+ values agree to 1%, experimental errors being 0.5%, 0.6%. I find the paper very confusing and unclear.

They distinguish G_μ from G_F for K-decays and G'_F for hyperon decays. Thencon top put their radiative corrections, distinguishing old corrections (δ_1) a la Alberto Sirlin form new ones (δ_2) in the e-w interaction. As if the Fermi constant in the books were incorrect. They write (Sirlin)

$$\Gamma(\mu) = \frac{G^2 M^5}{192\pi^3}\left(1 - 8\,(m_e/m_\mu)^{\,2}\right)\left\{1 - \frac{\alpha}{2\pi}\left(\pi^2 - \frac{25}{4}\right)\right\}$$

but G above is called G_μ in that paper.

Be as it may, I summarize.

1. $m_s - m_{u,d}$=?, is it small?
2. $m_K/m_\pi = 3.6$
3. $f_K/f_\pi = 1.27$
4. $2 \times (2\Gamma(K^+_{e3}) - \Gamma(K^0_{e3}))/(\Gamma_+ + \Gamma_-) = 0.0366 \pm 0.0006$
5. $\Gamma(K^0 \to 2\pi)/\Gamma(K^+ \to 2\pi) \sim 655$
6. Error on $\sqrt{\langle\Gamma(K_{e3})\rangle}$ ~0.4%
7. $|V_{us}| = 0.2196 \pm 0.0023$, ~1.05%

$\Gamma(K_{e3})$ can be measured better

Both K^0 and $K^\pm$ must be measured

KLOE CAN DO IT IN A UNIQUE WAY

To convert from eV to s^{-1}, multiply by 1.5192676×10^{15}

From τ_μ=2.19703×10^{-6} s and the Sirlin formula above one gets G_F=1.16637×10^{-5} GeV^{-2} to be compared with the PDG value of $1.16637(1)\times10^{-5}$ GeV^{-2} given by J. Erler and P. Langacker in section 10, Electroweak model and constraints on new physics and $1.16639(1)\times10^{-5}$ in section 1, Physical Constants.

$\delta_1(K^+_{e3}) = -2\%$
$\delta_1(K^+_{e3}) = +0.5\%$
$\delta_2(K^+_{e3}) \sim \alpha/\pi \sim 0.2\%$
$(\alpha/2\pi)\,[\pi^2-(25/4)] = 0.4\%,\ \ (3/5)(m_\mu/M_W)^2\sim10^{-6}$