Hadronic Vacuum Polarization Contribution to $g-2$ from the Lattice

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- Understanding hadronic contributions to $g_\mu - 2$ from the lattice
  - systematics: finite volume, non-zero lattice spacing
  - dis-connected contribution
  - improved observables

- Adlerfunction and $\alpha_{\text{QED}}$

- A word about light-by-light scattering
Motivation

- have a $\approx 3.5\sigma$ discrepancy
- rather constant over time
- $\tau$-data seem to become consistent
  (Jegerlehner, Szafon) $\rho^0 - \gamma$-mixing
- leading order hadronic contribution very important piece

$\Rightarrow$ challenge for the lattice

$\Rightarrow$ unique opportunity for the lattice
Computer and algorithm development over the years

time estimates for simulating $32^3 \cdot 64$ lattice, 5000 configurations

$\rightarrow O(\text{few months})$ nowadays with a typical collaboration supercomputer contingent
Todays landscape of lattice simulations worldwide
(from C. Hoelbling, Lattice 2010)
### The lattice QCD benchmark calculation: the spectrum

ETMC ($N_f = 2$), BMW ($N_f = 2 + 1$)

<table>
<thead>
<tr>
<th>M (GeV)</th>
<th>K</th>
<th>N</th>
<th>Δ</th>
<th>Λ</th>
<th>Σ</th>
<th>Ξ</th>
<th>Ξ*</th>
<th>Ω</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exp. results</td>
<td>Width</td>
<td>Input</td>
<td>Lattice results</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

#### Baryon Spectrum

- $N_f = 2$
- $N_f = 2 + 1$
Our fermion discretization: twisted mass fermions

(Frezzotti, Grassi, Weisz, Sint; Frezzotti, Rossi)

\[ D_{\text{tm}} = m_q + i\mu\tau_3\gamma_5 + \frac{1}{2}\gamma_\mu \left[ \nabla_\mu + \nabla^*_\mu \right] - a\frac{1}{2}\nabla^*_\mu \nabla_\mu \]

quark mass parameter \( m_q \), twisted mass parameter \( \mu \), lattice spacing \( a \)

- \( m_q = m_{\text{crit}} \leftrightarrow m_{\text{PCAC}} = 0 \rightarrow O(a) \) improvement for hadron masses, matrix elements, form factors, decay constants, \( \cdots \), \( g_\mu - 2 \)

- this means: \( (g_\mu - 2)_{\text{latt}} = (g_\mu - 2)_{\text{cont}} + O(a^2) \)
  \( \star \) based on symmetry arguments

- no need for further operator specific improvement coefficients

- expected to simplify mixing problems for renormalization

- drawback: explicit violation of isospin \( \rightarrow O(a^2) \) effect
  \( \rightarrow \) seems to affect only neutral pion sector
- Cyprus (Nicosia)
- France (Orsay, Grenoble)
- Italy (Rome I,II,III, Trento)
- Netherlands (Groningen)
- Poland (Poznan)
- Spain (Huelva, Madrid, Valencia)
- Switzerland (Bern)
- United Kingdom (Glasgow, Liverpool)
- Germany (Berlin/Zeuthen, Bonn, Frankfurt, Hamburg)
\( N_f = 2 + 1 + 1 \) light quark sector: scaling

\[
\begin{align*}
N_f &= 2 + 1 + 1 \quad r_0 m_{PS} = 0.614 \\
N_f &= 2 + 1 + 1 \quad r_0 m_{PS} = 0.728 \\
N_f &= 2 \quad r_0 m_{PS} = 1.100
\end{align*}
\]

\[
(a/r_0)^2
\]

pseudo scalar decay constant \( f_{PS} \)

nucleon mass
Do we control hadronic vacuum polarisation?
(Xu Feng, Dru Renner, Marcus Petschlies, K.J.; Lattice 2010)

- experiment: $a_{\mu,N_f=2}^{\text{hvp,exp}} = 5.66(05)10^{-8}$
- lattice: $a_{\mu,N_f=2}^{\text{hvp,old}} = 2.95(45)10^{-8}$

(numbers are scaled to $N_f = 4$ in plot)

→ misses the experimental value
→ order of magnitude larger error

- have used different volumes
- have used different values of lattice spacing
it’s not only our problem

- twisted mass: us
- Imp. clover: Mainz
- data are fully consistent
Can it be the dis-connected (singlet) contribution?

(Xu Feng, Dru Renner, Marcus Petschlies, K.J.)

- dedicated effort
- have included dis-connected contributions for first time
- smallness consistent with chiral perturbation theory (Della Morte, Jüttner)
Different extrapolation to the physical point

lattice: simulations at unphysical quark masses, demand only

\[ \lim_{m_{PS} \to m_\pi} a_l^{\text{hvp,latt}} = a_l^{\text{hvp,phys}} \]

\[ \Rightarrow \text{flexibility to define } a_l^{\text{hvp,latt}} \]

standard definitions in the continuum

\[ a_l^{\text{hvp}} = \alpha^2 \int_0^\infty dQ^2 \frac{1}{Q^2} \omega(r) \Pi_R(Q^2) \]

\[ \Pi_R(Q^2) = \Pi(Q^2) - \Pi(0) \]

\[ \omega(r) = \frac{64}{r^2 (1+\sqrt{1+4/r})^4 \sqrt{1+4/r}} \]

with \( r = Q^2/m_l^2 \)
Redefinition of $a_l^{\text{hvp,latt}}$

redefinition of $r$ for lattice computations

$$r_{\text{latt}} = Q^2 \cdot \frac{H_{\text{phys}}}{H}$$

choices

- $r_1$: $H = 1; \ H_{\text{phys}} = 1/m_l^2$
- $r_2$: $H = m_V^2(m_{\text{PS}}); \ H_{\text{phys}} = m_{\rho}^2/m_l^2$
- $r_3$: $H = f_V^2(m_{\text{PS}}); \ H_{\text{phys}} = f_{\rho}^2/m_l^2$

each definition of $r$ will show a different dependence on $m_{\text{PS}}$ but agree by construction at the physical point

remark: strategy often used in continuum limit extrapolations, e.g. charm quark mass determination
comparison using $r_1, r_2, r_3$
Some numbers

- experimental value: $a_{\mu,N_f=2}^{\text{hvp,exp}} = 5.66(05)10^{-8}$
- from our old analysis: $a_{\mu,N_f=2}^{\text{hvp,old}} = 2.95(45)10^{-8}$

→ misses the experimental value
→ order of magnitude larger error

- from our new analysis: $a_{\mu,N_f=2}^{\text{hvp,new}} = 5.66(11)10^{-8}$

→ error (including systematics) almost matching experiment

- different volumes
- different values of lattice spacing
- included dis-connected contributions
Anomalous magnetic moments, a check

\[ a_{e}^{\text{hlo}} = 1.513 (43) \cdot 10^{-12} \quad \text{LQCD} \]
\[ a_{e}^{\text{hlo}} = 1.527 (12) \cdot 10^{-12} \quad \text{PHENO} \]

\[ a_{\mu}^{\text{hlo}} = 5.72 (16) \cdot 10^{-8} \quad \text{LQCD} \]
\[ a_{\mu}^{\text{hlo}} = 5.67 (05) \cdot 10^{-8} \quad \text{PHENO} \]

\[ a_{\tau}^{\text{hlo}} = 2.650 (54) \cdot 10^{-6} \quad \text{LQCD} \]
\[ a_{\tau}^{\text{hlo}} = 2.638 (88) \cdot 10^{-6} \quad \text{PHENO} \]
Why it works: fitting the $Q^2$ dependence

Fit function

$$\Pi_{M,N}(Q^2) = -\frac{5}{9} \sum_{i=1}^{M} \frac{m_i^2}{Q^2 + m_i^2} + \sum_{n=0}^{N} a_n(Q^2)^n$$

$i = 1$: $\rho$-meson $\rightarrow$ dominant contribution $\propto 5.010^{-8}$

$i = 2$: $\omega$-meson $\propto 3.710^{-9}$

$i = 3$: $\phi$-meson $\propto 3.410^{-9}$
Why it works

- $m_V$ consistent with resonance analysis (Feng, Renner, K.J.)
- strong dependence on $m_{PS}$
  ⇒ modified definition takes out the $\rho$-meson dependence
anomalous magnetic moment of muon including strange quark

\[ a_{\mu} = 10^{-8} \]

Pheno, Nf=3
Asqtad, Nf=2+1
DWF, Nf=2+1
Imp. Clover, Nf=2 (+1)

- Asqtad \rightarrow Aubin and Blum
- DW \rightarrow Edinburg
- Imp. Clover \rightarrow QCDSF

⇒ need analysis with our improved observables
Running of QED coupling  (Preliminary)

$$\alpha(Q^2) = \frac{\alpha}{1-\Delta\alpha(Q^2)}$$

$$\Delta\alpha_{\text{had}}(Q^2) = 4\pi\alpha\Pi_R(Q^2)$$

apply same idea:

$$\Delta\bar{\alpha}_{\text{had}}(Q^2) = 4\pi\alpha\pi_R \left( \frac{Q^2}{H^2_{\text{phys}}} \cdot H^2 \right)$$

$$\Delta\alpha(M_0^2) = 5.72 (12) \cdot 10^{-3} \quad \text{LQCD}$$

$$\Delta\alpha(M_0^2) = 5.60 (06) \cdot 10^{-3} \quad \text{PHENO}$$

$$M_0 = 2.5 \text{GeV}$$
Adler function (preliminary)

\[
D(Q^2) = 12\pi^2 Q^2 \frac{d\Pi_R}{dQ^2} \rightarrow \overline{D}(Q^2) = D \left( \frac{Q^2}{H_{\text{phys}}^2} \right) \cdot H^2
\]
divergence cancelled by derivative

\[
\alpha_s^{(2)}(2 \text{ GeV}^2) = 0.263 (16)
\]
\[
\Lambda^{(2)} = 222 (27) \text{ MeV}
\]
The accuracy question

We need a precision $< 1\%$

- include up, down, strange and charm quarks
- include explicit isospin breaking
- include electromagnetism
- need computation of light-by-light contribution
- reach small quark mass $\rightarrow$ physical point
Simulation setup for $N_f = 2 + 1 + 1$
Configurations available through ILDG

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$a[fm]$</th>
<th>$L^3T/a^4$</th>
<th>$m_\pi[MeV]$</th>
<th>status</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.9</td>
<td>$\approx 0.085$</td>
<td>$24^348$</td>
<td>300 – 500</td>
<td>ready</td>
</tr>
<tr>
<td>1.95</td>
<td>$\approx 0.075$</td>
<td>$32^364$</td>
<td>300 – 500</td>
<td>ready</td>
</tr>
<tr>
<td>2.0</td>
<td>$\approx 0.065$</td>
<td>$32^364$</td>
<td>300</td>
<td>ready</td>
</tr>
<tr>
<td>2.1</td>
<td>$\approx 0.055$</td>
<td>$48^396$</td>
<td>300 – 500</td>
<td>running/ready</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$64^3128$</td>
<td>200</td>
<td>thermalizing</td>
</tr>
</tbody>
</table>

- trajectory length always one
- 1000 trajectories for thermalization
- $\geq 5000$ trajectories for measurements
Next, $\alpha_s^3$, contribution

\[
\gamma(k) \\ k_\rho \\
\text{had} \\
q_1\mu \\
q_2\nu \\
q_3\lambda \\
\mu(p) \\
\mu(p') \\
+ 5 \text{ permutations of the } q_i
\]

**light-by-light scattering**

involves 4-point function

\[
\Pi_{\mu\nu\alpha\beta}(q_1, q_2, q_3) = \int_{xyz} e^{iq_1\cdot x + iq_2\cdot y + iq_3\cdot z} \langle j_\mu(0) j_\nu(x) j_\alpha(y) j_\beta(z) \rangle
\]

\[j_\mu \text{ electromagnetic quark current}
\]

\[
j_\mu = \frac{2}{3} \bar{u}\gamma_\mu u - \frac{1}{3} \bar{d}\gamma_\mu d - \frac{1}{3} \bar{s}\gamma_\mu s + \frac{2}{3} \bar{c}\gamma_\mu c
\]
Momentum sources
(Alexandrou, Constantinou, Korzec, Panagopoulos, Stylianou)

following Göckeler et.al.

for renormalization: need Green function in momentum space

\[ G(p) = \frac{a^{12}}{V} \sum_{x,y,z,z'} e^{-ip(x-y)} \langle u(x) \bar{u}(z) J(z,z') d(z') \bar{d}(y) \rangle \]

e.g. \( J(z,z') = \delta_{z,z'} \gamma_\mu \) corresponds to local vector current sources:

\[ b^a_\alpha(x) = e^{ipx} \delta_\alpha_\beta \delta_{ab} \]

solve for

\[ D_{\text{latt}} G(p) = b \]

**Advantage:** very high, sub-percent precision data (only moderate statistics)

**Disadvantage:** need inversion for each momentum separately
Illustration of precision

use the momentum source method to attack the 4-point function as needed for light-by-light scattering (P. Rakow et.al., lattice'08)
Alternative approach

(Aubin, Blum, Chowdhury, Hayakawa, Izubuchi, Yamada, Yamazaki)

include both, QCD and QED ⇒ easier calculation

But: need cancelation of large terms

Subtraction of lowest order piece:

Subtraction term is product of separate averages of the loop and line

Gauge configurations identical in both, so two are highly correlated

In PT, correlation function and subtraction have same contributions except the light-by-light term which is absent in the subtraction

(taken from talk of T. Blum at Seattle workshop on light-by-light scattering)
Summary

- lattice is catching up for hadronic vacuum polarization
- a number of collaborations working on problem
  - Feng, Petschlies, Renner, K.J. et.al. \(\leftrightarrow\) (this talk)
  - Boyle, Del Debbio, Kerrane, Zanotti (Edinburgh)
  - Della Morte, Jäger, Jüttner, Wittig (Mainz)
  - Aubin and Blum (Riken)
- new lattice method for \(a_{\mu}^{\text{had}}\)
  \(\rightarrow\) prospect to match experimental precision, i.e. \(<0.5\%\)
- can be applied to further quantities
  - \(\Delta_{\alpha_{\text{QED}}}^{\text{had}}\), Adlerfunction, \(\Lambda_{\text{QCD}}\)
- On the way:
  - four flavour calculation
  - inclusion of isospin splitting and electromagnetism
- Challenge: light-by-light scattering

But it will take a while ...