ON UPPER LIMITS FOR GRAVITATIONAL RADIATION

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Abstract

A procedure with a Bayesian approach for calculating upper limits to gravitational wave bursts from coincidence experiments with multiple detectors is described, where the detection efficiency for small signals is taken into consideration. The Bayesian approach to the upper limit estimation is confronted with the unified approach for the case when no events are observed in presence of a non-zero background.

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1 Introduction

After the initial experiments with room temperature resonant detectors, the new generation of cryogenic gravitational wave (GW) antennas entered long term data taking operation in 1990 (EXPLORER [1]), in 1991 (ALLEGRO [2]), in 1993 (NIOBE [3]), in 1994 (NAUTILUS [4]) and in 1997 (AURIGA [5]).

Searches for coincident events between detectors have been performed. Between EXPLORER and NAUTILUS and between EXPLORER and NIOBE in the years 1995 and 1996 [6]. Between ALLEGRO and EXPLORER with data recorded in 1991 [7]. In both cases no significative coincidence excesses were found and an upper limit to GW bursts was calculated [7].

A recent search [9] of coincidences between EXPLORER and NAUTILUS in the period June-December 1998 using event selection algorithms based on the energy of the events and on the directionality of the detectors has shown a small coincidence excess when the detectors are favorably oriented towards the Galactic Centre. Nevertheless the evidence is too small to allow a claim of detection of gravitational waves, and upper limit estimations retain their validity.

Finally, the first network of five widely spaced detectors (IGEC) has put a new upper limit on events in the Galaxy[8].

However, the upper limit determination has been done under the hypothesis that the signal-to-noise ratio (SNR) is very large. According to theoretical estimations the signals expected from cosmic GW sources are extremely feeble, so small that extremely sensitive detectors are needed. In fact, according to present knowledge, the detectors available today have not yet reached the sensitivity to detect even a few events per year.

Thus it is important to study the problem of the upper limit determination in the cases the SNRs of the observed *events* are not large. In order to do this we have to discuss our definition of *event*.

The raw data from a resonant GW detector are filtered with a filter matched to short bursts [10]. We describe now in more detail the procedure used for the GW detectors of the Rome group, EXPLORER and NAUTILUS.

After the filtering of the raw-data, *events* are extracted as follows. Be $x(t)$ the filtered output of the electromechanical transducer which converts the mechanical vibrations of the bar in electrical signals. This quantity is normalized, using the detector calibration, such that its square gives the energy innovation E_f of the oscillation for each sample, expressed in kelvin units. In absence of signals, for well behaved noise due only to the thermal motion of the bar and to the electronic noise of the amplifier, the distribution of $x(t)$ is normal with zero mean. The variance (average value of the square of $x(t)$) is called $effective\ temperature$ and is indicated with T_{eff}. The distribution of $x(t)$ is

$$f(x) = \frac{1}{\sqrt{2\pi T_{eff}}} e^{\frac{x^2}{2T_{eff}}} \tag{1}$$

For extracting *events* (in absence of signals the events are just due to noise) we set a threshold in terms of a critical ratio defined by

$$CR = \frac{|x| - < |x| >}{\sigma(|x|)} = \frac{\sqrt{SNR_f} - \sqrt{\frac{2}{\pi}}}{\sqrt{1 - \frac{2}{\pi}}} \tag{2}$$

where $\sigma(|x|)$ is the standard deviation of $|x|$ and we put

$$SNR_f = \frac{E_f}{T_{eff}} \tag{3}$$

The threshold is set at a value CR such to obtain, in presence of thermal and electronic noise alone, a number of events which can be easily exchanged among the other groups who participate to the data exchange. For about one hundred *events* per day the threshold corresponds to an energy $E_t = 19.5\ T_{eff}$.

We calculate now the theoretical probability to detect a signal with a given SNR, in presence of a well behaved Gaussian noise. We put $y = (s + x)^2$ where $s \equiv \sqrt{SNR}$ is the signal we look for and x is the gaussian noise. We obtain easily [11]

$$probability(SNR) = \int_{SNR_t}^{\infty} \frac{1}{\sqrt{2\pi y}} e^{-\frac{(SNR+y)}{2}} cosh(\sqrt{y \cdot SNR}) dy \tag{4}$$

We put $SNR_t = 19.5$ for the present EXPLORER and NAUTILUS detectors.

2 Upper limit determination

We consider M detectors and search for M-fold coincidences over a total period of time t_m during which all detectors are in operation. Be $\bar{n}$ the average number of accidental coincidences (due to chance) and n_c the number of coincidences which are found within a given time window.

For events which have a Poissonian distribution in time the expected average number of M-fold accidental coincidences is given [16] by

$$\bar{n} = M w^{M-1} \prod_{k}^{1,M} n_k \tag{5}$$

where n_k is the event density of the k^{th} detector.

The accidental coincidence distribution can be estimated experimentally by proper shifting [12] the event occurrence times of each detector. In the case of Poissonian distribution the average number of the M-fold accidental coincidences coincides with that given by eq. 5. The comparison between n_c and $\bar{n}$ allows to reach some conclusion about the detection of GW or to establish an upper limit to their existence.

In paper [7] and in the previous paper [13] the upper limit has been estimated as follows. It has been found that, for various energy levels of the observed events, the number n_c was smaller than or did not exceeded significantly $\bar{n}$. Such numbers n_c, one for each energy level, were used for calculating the upper limit. A Poissonian distribution of the number of the observed events was considered together with the hypothesis of an isotropic distribution in the sky of the GW sources. The value of h (adimensional perturbation of the metric tensor) was then derived $from$ the energy levels, using the detector cross-section for gravitational waves.

This procedure can be objected on two points:

a)The most important point is that, as shown in [14], for SNR small and up to values of a few dozens, the energy of an event is *not* the energy of the GW absorbed by the detector. This means that we cannot deduce the value of h directly from the energy levels of the observed events;

b)In addition, the efficiency of detection, again for SNR values up to one or two dozens, is rather smaller than unity, and this changes the upper limit, particularly at small SNR.

We introduce a new procedure for estimating the upper limit, which circumvents the difficulties indicated in the above two points.

The problem to determine the upper limit has been discussed in several papers. In particular in paper [17], as indicated by the PDG, [18], and, more recently, in paper [19]. According to [19] the upper limit can be calculated using the relative belief updating ratio[20]

$$\mathcal{R}(n_{GW}, n_c, \bar{n}) = e^{-n_{GW}}(1 + \frac{n_{GW}}{\bar{n}})^{n_c} \tag{6}$$

referring to a given period t_m of data taking. This function is proportional to the likelihood and it allows to infer the probability to have n_{GW} signals for given priors (using the Bayes's theorem). It has already been used in High Energy Physics [21, 22, 23].

In Appendix A some properties on the $\mathcal{R}$ function are reviewed.

We calculate the upper limit by solving the equation

$$R(n_{GW}, n_c, \bar{n}) = 0.05 \tag{7}$$

We remark that 5% does not represent a probability but it is an useful way to put a limit independently on the priors[1)]. In the case the prior is taken to be uniform then the probability is just 5%. If the prior, based on previous knowledge, is not uniform (see ref [20], Section 7 Table 1) then the probability can be larger or smaller than 5%. As shown in ref [20], after a few experiments asymptotically the prior distribution becomes irrelevant.

Eq. 7 has a very interesting solution. Putting $n_c = 0$ we find $n_{GW} = 2.99$, independent on the value of the background $\bar{n}$. If we use the calculations of ref. [17] we find that, for $n_c = 0$ and $\bar{n} = 0$, the upper limit is 3.09 (almost identical to the previous one) *but* it decreases for increasing $\bar{n}$. The reason for this different behavior is due to the non-Bayesan character of the calculations made in [17], as discussed in [24] and reported in Appendix B.

Suppose we have $n_c = 0$ and $\bar{n} \neq 0$. This certainly means that the number of accidentals, whose average value can be determined with any desired accuracy, has undergone

1) To avoid confusion we call this limit *standard sensitivity bound* [19]. We remark that this limit becomes the standard *upper limit* if an uniform prior is used.

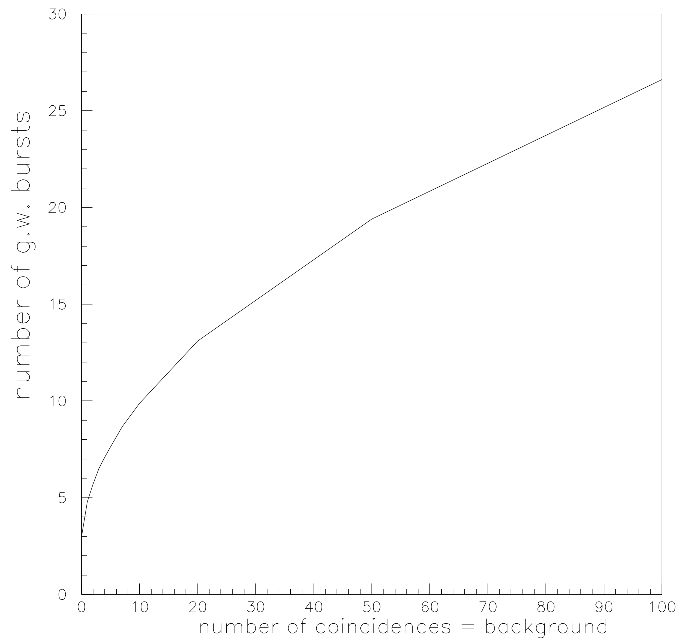

Figure 1: Number of GW signals expected for the sensitivity limit of 5% versus the number of coincidences equal to the average number of accidentals.

a fluctuation. For larger $\bar{n}$ values, smaller is the (*a priori*) probability that such fluctuation occur. Thus one could reason that it is less likely that a number n_{GW} be associated to a large value of $\bar{n}$, since the observation gave $n_c = 0$.

According to the Bayesan approach instead, one cannot ignore the fact that the observation $n_c = 0$ had already being made at the time the estimation of the upper limit is considered. The Bayesan approach requires that, given $n_c = 0$ and $\bar{n} \neq 0$, one evaluate the *chance* that a number n_{GW} of signals exist. This *chance* of a possible signal is referred to the observation already made and, rather obviously, it cannot depend on the previous fluctuation of the background, since the presence of a signal cannot be related to the background due to the detector.

Mathematically,(as shown in [24] and in the Appendix B), it is easy to demonstrate that that due to the Poissonian character of the number of accidentals this *relative chance* (for $n_c = 0$) is indeed independent on $\bar{n}$.

It can be seen, comparing the results of [17] with those of [20], that the Bayesan upper limits are for all values of n_c and $\bar{n}$ (except $n_c = \bar{n} = 0$), greater than those obtained with the non-Bayesan procedure. In our opinion the Bayesan approach has to be preferred, and so we do in this paper.

If we have $n_c \neq 0$ then we apply eq. 6. It is interesting to show the result for the case $n_c = \bar{n} \neq 0$ for the standard sensitivity bound of 5%. The result is given in fig.1 We note that for $n_c = \bar{n}$ and $n_{GW} << \bar{n}$ eq. 6 can be approximated with

$$n_{GW} \approx \sqrt{6\,\bar{n}} \tag{8}$$

From the result shown in fig.1 it appears evident that the lowest upper limit is obtained for $n_c \sim \bar{n} \sim 0$. In order to obtain $\bar{n} \sim 0$ one can raise the threshold used for determining the events. However in doing this one diminish the efficiency of detection, as shown in eq.4. Whether the procedure to raise the threshold is convenient or not, it depends on the numerical effects of the two competing operations. Certainly for large GW signals, when the detection efficiency is always unity, it is much better to have a threshold that gives $\bar{n} = 0$. For smaller signals one has to consider specific cases. However it can be seen that in the most interesting cases it is better to raise the threshold until we get $\bar{n} \sim 0$. This will be shown in the section where we reconsider the upper limit obtained with ALLEGRO and EXPLORER in 1991 [7].

In the estimation of the upper limit we consider the efficiency of detection, which we indicate with $\epsilon_k(SNR)$ where k refers to the k^{th} detector. For EXPLORER and NAUTILUS the theoretical efficiency is obtained from eq. 4.

We must relate the h values of the GW to the energy E absorbed by the detectors. We have to consider that the absorbed energy depends on the direction of the impinging GW and on its polarization. For taking care of the various polarization we use the average value dividing the cross section by a factor of two. We then have [15]

$$h = 1.13\ 10^{-17}\sqrt{E} \tag{9}$$

with the energy E expressed in kelvin unit. This formula is valid only if the GW arrives perpendicularly to the detector axis ($\theta = 90^o$). For a given direction we calculate the absorbed energy using the $sin(\theta)^4$ dependency. We also consider that for an isotropic distribution of sources the number of possible GW impinging directions is proportional to $sin(\theta)^2$.

The procedure for calculating the upper limit is accomplished thru the following points:

a) consider various values of h;

b) assume an isotropic distribution of the GW sources;

c) for each direction θ and for each h calculate the absorbed energy $E(\theta)$ by means of eq. 9 and the $sin^4(\theta)$ dependency;

d) for each detector calculate the SNR for the adsorbed energy by taking into consideration the noise $T_{eff,k}$:

$$SNR_k(\theta) = \frac{E(\theta)}{T_{eff,k}}, \qquad k = 1, .., M \tag{10}$$

e) using the individual efficiencies $\epsilon_k(SNR_k(\theta))$ consider the total efficiency $\epsilon_t(\theta) = \prod_k^{1,M} \epsilon_k(SNR_k(\theta))$;

f) integrate $\epsilon_t(\theta)$ over θ with the weight $sin^2(\theta)$, because of the assumed isotropic distribution of the sources;

Table 1: Procedure for calculating the upper limit with two detectors. We assume that one detector has noise $T_{eff} = 1\ mK$, the other one has noise $T_{eff} = 2\ mK$. For each value of h we give: maximum energy adsorbed by the detector (for $sin^4(\theta) = 1$), SNR and efficiency of detection for each detector, total weighted efficiency (having considered an isotropic distribution of the GW sources. Due to the angular weighting $\epsilon_{total} < \epsilon_A \epsilon_B$). The upper limit is given by $\frac{2.99}{\epsilon_{total}}$.

h 10^{18}	E_{abs} $[mK]$	Detector A SNR_A	ϵ_A %	Detector B SNR_B	ϵ_B %	ϵ_{tot} %	upper limit
2	31	31	0.88	15.5	0.32	0.12	16
3	70	70	1	35	0.93	0.56	4.3
4	126	126	1	63	1	0.76	3.6
10	783	783	1	392	1	0.95	3.1

g) from eq.6, given n_c and $\bar{n}$, we obtain n_{GW}. We then divide n_{GW} by the result of point f) and obtain for each value of h the upper limit during the measuring time t_m.

We remark that in this case we have not used the energy of the observed events, as done instead previously [13, 7].

The total efficiency is calculated with the following eq. 11.

$$\epsilon_{tot}(h) = \frac{\int_0^{\frac{\pi}{2}} \prod_k^{1,M} \epsilon_k(SNR_k(\theta)) sin^2(\theta) d\theta}{\frac{\pi}{4}} \tag{11}$$

For more clarity we show in table 1 some of the steps needed for our calculation, using two parallel detectors and $n_c = 0$. We use the efficiency given by eq. 4, valid for a well behaved noise[2)].

3 Ricalculation of the upper limit with the data of ALLEGRO and EXPLORER in 1991

In a previous paper [7] the upper limit for GW bursts was calculated, using the data recorded by ALLEGRO and EXPLORER in 1991. We wish now to recalculate the upper limit according the considerations discussed in this paper.

In 1991 the EXPLORER data filtering was done differently from that described in this Introduction. For both ALLEGRO and EXPLORER the output of the electromechanical transducer was sent to lock-ins referred to the frequencies of the resonant modes. Then the outputs of the lock-ins (in phase and in quadrature) were filtered searching for delta-like signals and combined for obtaining the energy innovation, which we still indicate with E_f. In this case the probability to have an event (above threshold SNR_t) due to a signal with given SNR is obtained (see ref. [2, 14]) with the following equation:

$$probability(SNR) = \int_{SNR_t}^{\infty} e^{-(SNR+y)} I_o(2\sqrt{y \cdot SNR}) dy \tag{12}$$

[2)] The real data often show a non gaussian behaviour. In this case the efficiency differs from the theoretical one given by eq.4, but one can easily make use of the efficiency experimentally measured.

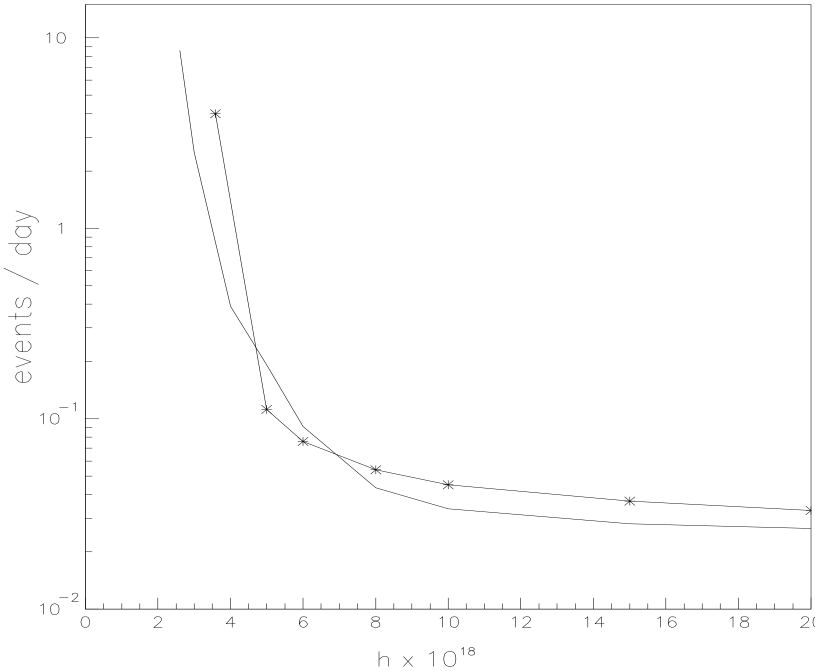

Figure 2: The asterisks indicate the upper limit calculated in [7]. The other line indicates the standard sensitivity bound evaluated with the Bayesan approach.

Here $y = \frac{E_f}{T_{eff}}$, I_o is the modified Bessel function of order zero, and the noise temperature T_{eff} is the average value of the energy innovation E_f.

We recall that in a time period of 123 days 70 coincidences were found with a background of 59.3. For extracting the events the ALLEGRO threshold was $SNR_t = 11.5$ with a noise temperature $T_{eff} \sim 8\ mK$. For EXPLORER the threshold was $SNR_t = 10$ also with $T_{eff} \sim 8\ mK$. Applying eq.6 we find an upper limit of $n_{GW} = 37$ over the 123 days.

According to the previous considerations we can raise the event threshold, say for EXPLORER, in order to reduce the number of accidentals. For instance, for a threshold $SNR_t = 24$ we get $n_c = 1$ and $\bar{n} = 0.74$, obtaining, from eq. 6, the value $n_{GW} = 4.8$.

Thus the procedure for calculating the upper limit with the Bayesan approach when we have data at various thresholds, including cases with n_c and $\bar{n}$ different from zero, is the following.

Start with n_c and $\bar{n}$ for various thresholds and use eq.6 for obtaining n_{GW} at each threshold. Calculate the upper limit for various values of h as shown in the previous section. For each h take as upper limit the smallest value among those obtained by varying the threshold. Clearly at large h values, when we get $n_c = 0$, the standard sensitivity bound is, for the entire period of time, $n_{GW} = 2.99$.

The result is shown in fig.2 together with that obtained previously in [7]. It turns out that the two upper limits are similar, if an uniform prior is assumed.

The reason for this is due to the fact that in applying the previous algorithm [7] we started from an energy level higher than the largest energy of the detected (accidental) coincidences, thus obtaining, at this level ($n_c = \bar{n} = 0$) an upper limit of 3.09 very close to the value 2.99 obtained with the Bayesan approach. The similarity of the results at lower h values is accidental. In the previous algorithm the increase at lower h is due only to the increase of the number $\bar{n}$ of accidentals. In the present algorithm the increase is due to the smaller efficiency of detection and to the increase in n_{GW} which roughly goes with $\sqrt{\bar{n}}$ (eq.8).

In spite of the similar numerical results, we believe that the procedure proposed here which does not extract the value of h from the energy levels of the accidental coincidence is more correct. Furthermore the use of the Bayesan approach allows to obtain a more general result expressed by the $\mathcal{R}$ function, and this can be used for obtaining probabilities using priors based on previous knowledge.

4 Discussion

The best upper limit which can be obtained with an array of M identical parallel detectors in M^{pl} coincidence cannot go below the value 2.99, because this is the upper limit [20] when one finds *zero* coincidences independently on the background.

The basic advantage in using many detectors comes from the fact that with many detectors it is easier to obtain $\bar{n} \sim 0$, and thus (in absence of GW) $n_c = 0$. Because of the Poisson distributions, the average number of accidental coincidences for M detectors in a time window $\pm w$ is given by eq.5. On the time scale of 1 second (w=1 s) it turns out that $n_k << 1$. By increasing the number of detectors one obtains smaller values of $\bar{n}$, thus approaching the requirement to have $n_c = 0$ and then the lowest possible upper limit.

This is certainly true at large h values, where the detection efficiency for all detectors is unity. The result, as shown in fig.2, is a plateau. Instead it might be convenient at low h values to use the two most sensitive detectors, in order to have the largest possible efficiency of detection. The overall upper limit is then obtained by taking the smallest ones among the values of the various upper limit determinations.

The above procedure can be easily adjusted to the more general case of any distribution of the GW sources, and of non-parallel detectors.

5 APPENDIX A: The relative belief updating ratio

In this Appendix we report on relative belief updating ratio in the case of Poisson processes. The full derivation and discussion on this subject is in [20].

From the Bayes theorem, we have:

$$f(r\,|\,n_c, r_b) \propto f(n_c\,|\,r, r_b) \cdot f_\circ(r)\,, \tag{13}$$

where $r = n_{GW}/T$, with T the observation time, $r_b = \bar{n}/T$ and $f_\circ(r)$ is the prior probability density function. From (13) it follows, considering two possible values of r (r_1 and r_2), that

$$\frac{f(r_1\,|\,n_c, r_b)}{f(r_2\,|\,n_c, r_b)} = \underbrace{\frac{f(n_c\,|\,r_1, r_b)}{f(n_c\,|\,r_2, r_b)}}_{\textit{Bayes factor}} \cdot \frac{f_\circ(r_1)}{f_\circ(r_2)}\,. \tag{14}$$

The ratio of likelihoods is known as the Bayes factor and it quantifies the ratio of evidence provided by the data in favour of either hypothesis. The Bayes factor is considered to be practically objective because likelihoods (i.e. probabilistic description of the detector response) are usually much less critical than priors (see the extended discussion in [20]) The Bayes factor can be extended to a continuous set of hypotheses r, considering a function which gives the Bayes factor of each value of r with respect to a reference value r_{REF}. The reference value could be arbitrary, but for our problem we choose $r_{REF} = 0$, obtaining

$$\mathcal{R}(r; n_c, r_b) = \frac{f(n_c \,|\, r, r_b)}{f(n_c \,|\, r = 0, r_b)}\,, \tag{15}$$

This choice is convenient for comparing and combining the experimental results. The function $\mathcal{R}$ has nice intuitive interpretations which can be highlighted by reordering the terms of (14) in the form

$$\frac{f(r \,|\, n_c, r_b)}{f_\circ(r)} \Bigg/ \frac{f(r = 0 \,|\, n_c, r_b)}{f_\circ(r = 0)} = \frac{f(n_c \,|\, r, r_b)}{f(n_c \,|\, r = 0, r_b)} = \mathcal{R}(r; n_c, r_b) \tag{16}$$

(valid for all possible a priori r values). $\mathcal{R}$ has the probabilistic interpretation of relative belief updating ratio, or the geometrical interpretation of shape distortion function of the probability density function (p.d.f.). $\mathcal{R}$ goes to 1 for $r \to 0$, i.e. in the asymptotic region in which the experimental sensitivity is lost. As long as $\mathcal{R} = 1$, the shape of the p.d.f. (and therefore the relative probabilities in that region) remains unchanged. Instead, in the limit $\mathcal{R} \to 0$ (for large r) the final p.d.f. vanishes, i.e. the beliefs go to zero, no matter how strong they were before.

Moreover there are some technical advantages in reporting the $\mathcal{R}$ function as a result of a search experiment:

- One deals with numerical values which can differ from unity only by a few orders of magnitude in the region of interest, while the values of the likelihood can be extremely low. For this reason, the comparison between different results given by the $\mathcal{R}$ function can be perceived better than in terms of likelihood.
- Since $\mathcal{R}$ differs from the likelihood only by a factor, it can be used directly in Bayes' theorem, which does not depend on constants, whenever probabilistic considerations are needed.
- The combination of different independent results on the same quantity r can be done straightforwardly by multiplying individual $\mathcal{R}$ functions.
- Finally, one does not need to decide a priori if he wants to apply a 'discovery' or an 'upper limit' analysis, as conventional statistics teaches. The $\mathcal{R}$ function represents the most unbiased way of presenting the results and everyone can draw their own conclusions.

6 APPENDIX B: Upper limits in the case that zero events are observed

In this Appendix we report on the intuitive solution to the "background dependence puzzle", in case that zero events are observed (as discussed in [24]). According to the (FC) "unified approach"[17] the upper limit is calculated using a revised version of the classical Neyman construction for confidence intervals. This approach is usually referred to as the "unified approach to the classical statistical analysis", and it aims to unify the treatment of upper limits and confidence intervals. On the Bayes side, according to [19], the upper limit may be calculated using the function $\mathcal{R}$ that is proportional to the likelihood.

Comparison between the two approaches is difficult for the general case. But we have noticed a special case which is easier to discuss. In this case the greater efficacy of one approach compared to the other one seems clear. This case is when the experiment gave no events, even in the presence of a background greater than zero.

When there are zero counts, the predictions obtained with the two methods are different. Our intuition would be satisfied by an upper limit that increases with the background level, and this is, in general, the case when the observation gives a number of events of the order of the background. However, when zero events are observed, the "unified approach" upper limit decreases if the background increases (a noisier experiment puts a better upper limit than a less noisy one, which seems absurd) while the Bayesian approach leads to the predictions that a constant upper limit will be found (the upper limit does not depend on the noise of the experiment). Various papers[25, 26, 27] have been devoted to the problem. In particular, in [26], the proposed method "gives limits that do not depend on background in the case of no observed events" (that is the Bayesian result).

In what follows we will give an explanation for the two results.

We remind the reader that the physical quantity for which a limit must be found is the events rate (i.e. a gravitational wave burst rate) r. Here we will assume stationary working conditions. For a given hypothesis r, the number of events which can be observed in the observation time T is described by a Poisson process which has an intensity equal to the sum of that due to background and that due to signal.

In general, the main ingredients in our problem are:

- we are practically sure about the expected rate of background events $r_b = n_b/T$ but not about the number of events that will actually be observed (which will depend on the Poissonian statistics);
- we have observed a number n_c of events but, obviously, we do not know how many of these events have to be attributed to background and how many (if any) to true signals.

Under the stated assumptions, the likelihood is

$$f(n_c \,|\, r, r_b) = \frac{e^{-(r+r_b)\,T}((r+r_b)\,T)^{n_c}}{n_c!}\,, \tag{17}$$

We will now concentrate on the solution given by the Bayesian approach.

As shown in Appendix A, the "relative belief updating ratio" $\mathcal{R}$ is defined as

$$\mathcal{R}(r; n_c, r_b, T) = \frac{f(n_c \,|\, r, r_b)}{f(n_c \,|\, r=0, r_b)}\,, \tag{18}$$

This function is proportional to the likelihood and it allows to infer the probability that rT signals will be observed for given priors (using the Bayes's theorem).

Under the hypothesis $r_b > 0$ if $n_c > 0$, $\mathcal{R}$ becomes

$$\mathcal{R}(r; n_c, r_b, T) = e^{-rT}\left(1 + \frac{r}{r_b}\right)^{n_c} . \tag{19}$$

(that is Eq. 6 in the text).

The upper limit, or -more properly- "standard sensitivity bound" [19], can then be calculated using the $\mathcal{R}$ function. The value r_{ssb} is obtained when

$$\mathcal{R}(r_{ssb}; n_c; r_b; T) = 0.05 \tag{20}$$

We remark that 5% does not represent a probability, but is a useful way to put a limit independently of the priors.

Eq. 19 when no events are observed, that is, when n_c=0, becomes:

$$\mathcal{R}(r; n_c = 0, r_b, T) = e^{-rT} \tag{21}$$

In this case we find $r_{ssb} = 2.99$, independently of the value of the background n_b.

We will not describe the well known (FC) procedure here, but we just observe that, according to this procedure, for $n_c = 0$ and $n_b = 0$, the upper limit is 3.09 (numerically almost identical to the Bayes' one) *but* it decreases as n_b increases (e.g. for $n_c = 0$ and $n_b = 15$ the upper (FC) limit at 95% CL is 1.47).

In an attempt to understand such different behaviour we will now discuss some particular cases. Suppose we have $n_c = 0$ and $n_b \neq 0$. This certainly means that the number of accidentals, whose average value can be determined with any desired accuracy, has undergone a fluctuation. The larger the n_b values, the smaller is the *a priori* probability that such fluctuations will occur. Thus one could reason that, since the observation gave $n_c = 0$, it is less likely that a number n_{gw} of real signals could have been associated with a large value of n_b, as predicted by the (FC) approach.

According to the Bayesian approach, instead, one cannot ignore the fact that the observation $n_c = 0$ has already being made at the time the estimation of the upper limit comes to be calculated. The Bayesian approach requires that, given $n_c = 0$ and $n_b \neq 0$, one evaluates the *chance* that a number n_{gw} of signals exists. This *chance* of a possible signal is applied to the observation that has already been made.

Suppose that we have estimated the average background, for example n_b=10, with a high degree of accuracy. In absence of signals, the a priori probability of observing zero events, due just to a background fluctuation, is given by

$$f_n = f(n_c = 0|n_b = 10) = e^{-n_b} = 4.5 \cdot 10^{-5} \tag{22}$$

Now, suppose that we have measured zero events, that is n_c=0. In general $n_c = (n_b + n_{gw})$. It is now nonsense to ask what the probability that n_c=0 is, since the experiment has already been made and the probability is 1.

We may ask how the a priori probability would be changed if n_{gw} signals were added to the background. We get

$$f_{sn} = f(n_c = 0|n_b = 10, n_{gw}) = e^{-(n_b+n_{gw})} \tag{23}$$

It is obvious that $f_{sn} \leq f_n$.
The right question to ask now, since we have already measured $n_c = 0$, is: what is that signal n_{gw} which would have reduced the probability f_n by a constant factor, for example 0.05 ?

$$f_{sn} = f_n \cdot 0.05 = e^{-n_b} \cdot e^{-n_{gw}} \tag{24}$$

Using Eqs. 22, 23 and 24 the solution is:

$$e^{-n_{gw}} = 0.05 \tag{25}$$

that is:

$$n_{gw} = 2.99 \tag{26}$$

independent on the background. Now suppose another situation, n_b=20, thus $f_n = 2.1 \cdot 10^{-9}$. Repeating the previous reasoning we still get the limit 2.99.

The meaning of the Bayesian result is now clear: we do not care about the absolute value of the a priori probability of getting $n_c = 0$ in the presence of noise alone. The observation of $n_c = 0$ means that the background gave zero counts by chance. Even if the a priori probability is very small, its value has no meaning once it has happened. The fact that the single background measurement turned out to be zero, either due to a zero average background or due to the observation of a low (a priori) probability event, must not change our prediction concerning possible signals.

For $n_c = 0$ we are certain that the number of events due to the background is zero. Clearly this particular situation gives more information about the possible signals. In the case $n_c \neq 0$, instead, it is not possible to distinguish between background and signal. The mathematical aspect of this is that the Poisson formula when $n_c = 0$ reduces to the exponential term only, and thus it is possible to separate the two contributions, of the signal (unknown) and of the noise (known).

We note that the different behaviour of the limit in the unified approach is due to the non-Bayesian character of the reasoning. In such an approach an event that has already occurred is considered "improbable": given the observation of $n_c = 0$ they still consider that the probability

$$f_{sn} = f(n_c = 0|n_b, n_{gw}) = e^{-(n_b+n_{gw})} \tag{27}$$

decreases as n_b increases. As a consequence they deduce that to a larger n_b corresponds a smaller upper limit n_{gw}.

Given the previous considerations, we must now admit that our intuition to expect an upper limit that increases with increasing background, even when $n_c = 0$, was wrong. We should have expected to predict a constant signal rate, as a consequence of the observation of zero events, independently of the background level.

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