PHYSICAL REVIEW D **88,** 064019 (2013)

Solar System constraints to nonminimally coupled gravity

Orfeu Bertolami[*,†]

Departamento de Física e Astronomia, Faculdade de Ciências, Universidade do Porto, Rua do Campo Alegre 687, 4169-007 Porto, Portugal

Riccardo March[‡]

Istituto per le Applicazioni del Calcolo, CNR, Via dei Taurini 19, 00185 Roma, Italy, and INFN-Laboratori Nazionali di Frascati (LNF), Via E. Fermi 40 Frascati, 00044 Roma, Italy

Jorge Páramos[§]

Instituto de Plasmas e Fusão Nuclear, Instituto Superior Técnico, Universidade de Lisboa, Av. Rovisco Pais 1, 1049-001 Lisboa, Portugal

(Received 5 June 2013; published 9 September 2013)

We extend the analysis of Chiba *et al.* [Phys. Rev. D **75**, 124014 (2007)] of Solar System constraints on $f(R)$ gravity to a class of nonminimally coupled (NMC) theories of gravity. These generalize $f(R)$ theories by replacing the action functional of general relativity with a more general form involving two functions $f^1(R)$ and $f^2(R)$ of the Ricci scalar curvature R. While the function $f^1(R)$ is a nonlinear term in the action, analogous to $f(R)$ gravity, the function $f^2(R)$ yields a NMC between the matter Lagrangian density $\mathcal{L}_m$ and the scalar curvature. The developed method allows for obtaining constraints on the admissible classes of functions $f^1(R)$ and $f^2(R)$, by requiring that predictions of NMC gravity are compatible with Solar System tests of gravity. Then we consider a NMC model which accounts for the observed accelerated expansion of the Universe and we show that such a model cannot be constrained by the present method.

DOI: 10.1103/PhysRevD.88.064019 PACS numbers: 04.20.Fy, 04.80.Cc, 04.25.Nx

I. INTRODUCTION

One of the greatest challenges of contemporary physics is to make sense of the fact that, at Solar System level, there is no evidence that an extension of general relativity (GR) is required to account for all observed gravitational phenomena (see Ref. [1] for a recent account), even though, from the theoretical point of view, GR is not a fully satisfactory theory. Indeed, GR exhibits singularities and is incompatible with quantum mechanics; furthermore, in order to account for the cosmological data, new states such as dark matter and dark energy are required.

As a possible alternative to this standard scenario, it is equally plausible that GR is actually an effective version of a more general theory of gravity. More recently, a great deal of interest has been dedicated to the so-called $f(R)$ theories [2]; these can be further generalized by considering that matter and curvature are nonminimally coupled [3], an idea that gives rise to many interesting features and has spanned several studies: these include the impact on stellar observables [4], the so-called energy conditions [5], the equivalence with multi-scalar-tensor theories [6], the possibility to account for galactic [7] and cluster [8] dark matter, cosmological perturbations [9], a mechanism for mimicking a cosmological constant at astrophysical scales [10], postinflationary reheating [11] or the current accelerated expansion of the Universe [12], the dynamical impact of the choice of the Lagrangian density of matter [13,14], gravitational collapse [15], its Newtonian limit [16] and existence of closed timelike curves [17].

In this work, we study whether a nonminimally coupled theory of gravity can be assessed using Solar System observables. It follows an analogous analysis, performed by Chiba, Smith and Erickcek [18] for generic $f(R)$ theories. In Ref. [18] the authors find a set of conditions that, when satisfied by the function $f(R)$, lead to the prediction that the value of the parametrized post-Newtonian (PPN) parameter γ is given by $\gamma = 1/2$, which is not in agreement with Solar System tests of gravity. Hence, the analysis of Ref. [18] can be considered as a tool to rule out $f(R)$ theories that satisfy a suitable set of conditions. Particularly, it turns out that the $1/R^n$ $(n > 0)$ gravity theory, proposed by Carroll *et al.* [19] to account for the observed accelerated expansion of the Universe, is ruled out by this analysis.

In the present paper we consider a class of NMC theories of gravity where the action functional of GR is replaced with a more general form involving two functions $f^1(R)$ and $f^2(R)$ of the Ricci scalar curvature R. The function $f^1(R)$ has a role analogous to $f(R)$ gravity, and the function $f^2(R)$ yields a nonminimal coupling between the matter Lagrangian density $\mathcal{L}_m$ and the scalar curvature. When $f^2(R) = 0$, NMC gravity reduces to $f(R)$ gravity.

[*]Also at Instituto de Plasmas e Física Nuclear, Instituto Superior Técnico, Av. Rovisco Pais, 1, 1049-001, Lisboa, Portugal.
[†]orfeu.bertolami@fc.up.pt
[‡]r.march@iac.cnr.it
[§]paramos@ist.edu

We extend the analysis of Ref. [18] in order to develop a general framework for the study of Solar System constraints to NMC gravity. Then we apply the results of our analysis to a couple of case studies. Particularly, we consider the NMC model proposed by Bertolami *et al.* [12] to account for the observed accelerated expansion of the Universe. This model posits an inverse power-law NMC $f^2(R) \propto 1/R^n$ term in the action functional, and can be considered as a natural extension of $1/R^n$ $(n > 0)$ gravity to a nonminimally coupled case. We show that, differently from pure $1/R^n$ gravity, the NMC model of Ref. [12] cannot be constrained or excluded by the method developed in this work. Hence such a NMC model remains, in this respect, a viable theory of gravity.

The manuscript is organized as follows: in Secs. II and III we present our model and the assumptions adopted to ascertain the effect of the NMC in the Solar System. In Secs. IV and V, we carry out the suitable linearization of the relevant equations and derive the conditions required for applying the long range limit. Sections VI, VII, and VIII then address the solutions to the obtained set of equations. Section IX tackles the compatibility of the model under scrutiny with the various assumptions used to assess its impact at Solar System scales. Finally, we present our conclusions. Appendix A accounts for some technical aspects used to obtain the solution for linearized field equations, while Appendix B discusses a correction to the numerical values considered in Ref. [12].

II. NONMINIMALLY COUPLED GRAVITY

In the present work we consider gravitational theories with an action functional of the form [3],

$$S = \int \left[\frac{1}{2} f^1(R) + [1 + f^2(R)]\mathcal{L}_m \right] \sqrt{-g} d^4x, \tag{1}$$

where $f^i(R)$ $(i = 1, 2)$ are functions of the Ricci scalar curvature R, $\mathcal{L}_m$ is the Lagrangian density of matter and g is the metric determinant. The standard Einstein-Hilbert action is recovered by taking

$$f^1(R) = 2\kappa(R - 2\Lambda), \qquad f^2(R) = 0, \tag{2}$$

where $\kappa = c^4/16\pi G_N$ and Λ is the cosmological constant. Here, G_N is Newton's gravitational constant: as we will show, an effective gravitational constant G arises due to the composite effect of $f^1(R)$ and $f^2(R)$.

The variation of the action functional with respect to the metric $g_{\mu\nu}$ yields the field equations

$$\begin{aligned}(f^1_R + 2f^2_R \mathcal{L}_m)R_{\mu\nu} - \frac{1}{2} f^1 g_{\mu\nu} \\ = (1 + f^2)T_{\mu\nu} + (\nabla_\mu \nabla_\nu - g_{\mu\nu}\Box)(f^1_R + 2f^2_R \mathcal{L}_m),\end{aligned} \tag{3}$$

where $f^i_R \equiv df^i/dR$. In the following we assume that matter behaves as dust, i.e. a perfect fluid with negligible pressure and an energy-momentum tensor described by

$$T_{\mu\nu} = \rho u_\mu u_\nu, \qquad u_\mu u^\mu = -1, \tag{4}$$

where $\rho = \rho(r, t)$ is the matter density and u_μ is the four-velocity. The trace of the energy-momentum tensor is $T = -\rho$. We use $\mathcal{L}_m = -\rho$ for the Lagrangian density of matter (see Ref. [13] for a discussion).

III. ASSUMPTIONS ON THE METRIC AND ON FUNCTIONS $f^1(R)$ AND $f^2(R)$

We now seek the metric that describes the spacetime around a spherical body such as the Sun in the weak-field limit of NMC gravity. Such a metric will be regarded as a perturbation of a background spacetime around which we linearize the field equations. We take the background metric to be a flat Friedmann-Robertson-Walker (FRW) metric

$$ds^2 = -dt^2 + a^2(t)(dr^2 + r^2 d\Omega^2), \tag{5}$$

with scale factor $a(t)$ [we set $a(t) = 1$ at the present time]. Such a FRW metric solves the field equations (3) for a spatially uniform cosmological dust energy-momentum tensor, $T^{\cos}_{\mu\nu}$, the trace of which is $-\rho^{\cos}(t)$. We denote the Ricci scalar curvature of the background spacetime by $R_0 = R_0(t)$.

We assume that the spacetime around a spherical star is written (in spherical coordinates) by the following perturbation of the background metric:

$$\begin{aligned}ds^2 = -[1 + 2\Psi(r, t)]dt^2 \\ + a^2(t)([1 + 2\Phi(r, t)]dr^2 + r^2 d\Omega^2),\end{aligned} \tag{6}$$

where $|\Psi(r, t)| \ll 1$ and $|\Phi(r, t)| \ll 1$. The Ricci curvature of the perturbed spacetime is expressed as the sum

$$R(r, t) = R_0(t) + R_1(r, t). \tag{7}$$

As expected, we will show that the time scale of variations in Ψ, Φ and R_1 is much longer than the one of Solar System dynamics, such that

$$\Psi(r, t) \simeq \Psi(r), \quad \Phi(r, t) \simeq \Phi(r), \quad R_1(r, t) \simeq R_1(r). \tag{8}$$

Following Ref. [18], in the linearization of the field equations, both around and inside the star, we assume that

$$|R_1(r, t)| \ll R_0(t). \tag{9}$$

Such an assumption implies that the scalar curvature R of the perturbed spacetime remains close to the cosmological value R_0 inside the star. In $f(R)$ theories this condition is satisfied, for instance, by the model proposed in Ref. [19], where

$$f^1(R) = 2\kappa\left(R - \frac{\mu^4}{R}\right), \qquad f^2(R) = 0, \tag{10}$$

as shown in Refs. [18,20]. Such a behavior for the curvature differs from the usual scenario of GR, where the above condition breaks down inside the body, since the mass density of the star is larger than the cosmological mass density. This issue will play a central role in the application of the framework here developed to the NMC model proposed in Ref. [12]. Naturally, the validity of condition Eq. (9) will depend on the particular choice of $f^1(R)$ and $f^2(R)$, and thus can be used to constrain these functions.

We consider that all derivatives of functions $f^1(R)$ and $f^2(R)$ exist at the present value of $R_0(t)$. Since we assume that $|R_1| \ll R_0$, we can Taylor expand $f^i(R)$ around $R = R_0$ to evaluate $f^i(R_0 + R_1)$ and $f^i_R(R_0 + R_1)$, for $i = 1, 2$. Neglecting terms nonlinear in R_1, we get

$$\begin{aligned} \left| f^i(R_0) + \frac{df^i}{dR}(R_0)R_1 \right| &\gg \left| \frac{1}{k!}\frac{d^k f^i}{dR^k}(R_0)R_1^k \right|, \\ \left| f^i_R(R_0) + \frac{df^i_R}{dR}(R_0)R_1 \right| &\gg \left| \frac{1}{k!}\frac{d^k f^i_R}{dR^k}(R_0)R_1^k \right|, \end{aligned} \tag{11}$$

for all $k > 1$ and $i = 1, 2$. Following Ref. [18], we introduce the useful notation (for $i = 1, 2$),

$$f^i_0 \equiv f^i(R_0), \quad f^i_{R0} \equiv \frac{df^i}{dR}(R_0), \quad f^i_{RR0} \equiv \frac{d^2 f^i}{dR^2}(R_0). \tag{12}$$

IV. LINEARIZATION OF THE TRACE OF THE FIELD EQUATIONS

The trace of the field equations (3) is given by

$$\begin{aligned} &(f^1_R + 2f^2_R \mathcal{L}_m)R - 2f^1 + 3\Box(f^1_R + 2f^2_R \mathcal{L}_m) \\ &\quad = (1 + f^2)T. \end{aligned} \tag{13}$$

The energy-momentum tensor is decomposed in the following way:

$$T_{\mu\nu} = T^{\rm cos}_{\mu\nu} + T^{\rm s}_{\mu\nu}, \qquad \rho = \rho^{\rm cos} + \rho^{\rm s}, \tag{14}$$

where $\rho^{\rm cos} = \rho^{\rm cos}(t)$ is the cosmological matter density and $\rho^{\rm s} = \rho^{\rm s}(r)$ is the stellar matter density. The traces of the energy-momentum tensor contributions are denoted by $T^{\rm cos}$ and $T^{\rm s}$, respectively. We denote by $\mathrm{R_S}$ the radius of the star and assume that both the function $\rho^{\rm s}(r)$ and its derivative are continuous across the surface of the star, such that

$$\rho^{\rm s}(\mathrm{R_S}) = \frac{d\rho^{\rm s}}{dr}(\mathrm{R_S}) = 0. \tag{15}$$

We also write $\mathcal{L}^{\rm cos}_m = -\rho^{\rm cos}$ and $\mathcal{L}^{\rm s}_m = -\rho^{\rm s}$, so that $\mathcal{L}_m = \mathcal{L}^{\rm cos}_m + \mathcal{L}^{\rm s}_m$. As a consequence of our definitions, we have that $\rho(r,t) = \rho^{\rm cos}(t) + \rho^{\rm s}(r)$ inside the star.

The background curvature R_0 solves the trace Eq. (13) with matter source given by $T^{\rm cos}$:

$$\begin{aligned} &(f^1_{R0} + 2f^2_{R0}\mathcal{L}^{\rm cos}_m)R_0 - 2f^1_0 + 3\Box(f^1_{R0} + 2f^2_{R0}\mathcal{L}^{\rm cos}_m) \\ &\quad = (1 + f^2_0)T^{\rm cos}. \end{aligned} \tag{16}$$

We now linearize Eq. (13) using the first-order Taylor expansions of the functions $f^i(R)$ and $f^i_R(R)$ around $R = R_0 \neq 0$. Since $R = R_0 + R_1$, using condition Eq. (9), we neglect $O(R_1^2)$ contributions, but keep the cross term $R_0 R_1$. Moreover, using the fact that R_0 solves Eq. (16), we eliminate in the linearized trace equation terms that are independent of R_1, with the exception of those containing the matter source $T^{\rm s} = \mathcal{L}^{\rm s}_m$. The application of the above procedure yields

$$\begin{aligned} &[-f^1_{R0} + f^2_{R0}\mathcal{L}_m + (f^1_{RR0} + 2f^2_{RR0}\mathcal{L}_m)R_0]R_1 \\ &\quad + 3\Box[(f^1_{RR0} + 2f^2_{RR0}\mathcal{L}_m)R_1] \\ &\quad = (1 + f^2_0)T^{\rm s} - 2f^2_{R0}\mathcal{L}^{\rm s}_m R_0 - 6\Box(f^2_{R0}\mathcal{L}^{\rm s}_m). \end{aligned} \tag{17}$$

In order to compute the term

$$\Box[(f^1_{RR0} + 2f^2_{RR0}\mathcal{L}_m)R_1], \tag{18}$$

we consider the approximation $R_1(r,t) \simeq R_1(r)$, that will be verified later, obtaining

$$\Box(f^1_{RR0}R_1) = f^1_{RR0}\Box R_1 + R_1\Box f^1_{RR0}, \tag{19}$$

and

$$\begin{aligned} \Box(f^2_{RR0}\mathcal{L}_m R_1) &= -f^2_{RR0}\rho^{\rm cos}\Box R_1 - R_1\Box(f^2_{RR0}\rho^{\rm cos}) \\ &\quad - f^2_{RR0}\Box(\rho^{\rm s}R_1) - \rho^{\rm s}R_1\Box f^2_{RR0}. \end{aligned} \tag{20}$$

By definition,

$$\begin{aligned} \Box R_1(r) &= g^{rr}\frac{d^2R_1}{dr^2} - g^{\mu\nu}\Gamma^r{}_{\mu\nu}\frac{dR_1}{dr}, \\ \Box(\rho^{\rm s}(r)R_1(r)) &= g^{rr}\frac{d^2(\rho^{\rm s}R_1)}{dr^2} - g^{\mu\nu}\Gamma^r{}_{\mu\nu}\frac{d(\rho^{\rm s}R_1)}{dr}, \end{aligned} \tag{21}$$

where $\Gamma^\lambda{}_{\mu\nu}$ are the Christoffel symbols of the metric Eq. (6). Neglecting terms in Eq. (17) that involve products of R_1 or its spatial derivatives with Ψ, Φ and their spatial derivatives [since such products turn out to be of order $o(1/c^2)$], we may approximate

$$\Box R_1 \simeq \nabla^2 R_1, \qquad \Box(\rho^{\rm s}R_1) \simeq \nabla^2(\rho^{\rm s}R_1), \tag{22}$$

where ∇^2 denotes the three-dimensional flat space Laplacian. Taking into account that $f^2_{RR0} = f^2_{RR0}(t)$, it follows that

$$\begin{aligned} \Box(f^2_{RR0}\mathcal{L}_m R_1) &\simeq -R_1[\rho^{\rm s}\Box f^2_{RR0} + \Box(f^2_{RR0}\rho^{\rm cos})] \\ &\quad + \nabla^2(f^2_{RR0}\mathcal{L}_m R_1). \end{aligned} \tag{23}$$

Collecting these results, we thus find

$$\begin{aligned} &\Box[(f^1_{RR0} + 2f^2_{RR0}\mathcal{L}_m)R_1] \\ &\quad \simeq [\Box(f^1_{RR0} - 2f^2_{RR0}\rho^{\rm cos}) - 2\rho^{\rm s}\Box f^2_{RR0}]R_1 \\ &\quad\quad + \nabla^2[(f^1_{RR0} + 2f^2_{RR0}\mathcal{L}_m)R_1]. \end{aligned} \tag{24}$$

The same steps are also applied to the term

$$\Box(f^2_{R0}\mathcal{L}^s_m) = -f^2_{R0}\Box\rho^s - \rho^s\Box f^2_{R0}, \tag{25}$$

found in Eq. (17); substituting the obtained expressions into Eq. (17), we obtain

$$\begin{aligned}&3\nabla^2[(f^1_{RR0} + 2f^2_{RR0}\mathcal{L}_m)R_1] + (-f^1_{R0} + f^2_{R0}\mathcal{L}_m)R_1 + (f^1_{RR0} + 2f^2_{RR0}\mathcal{L}_m)R_0R_1 + 3[\Box(f^1_{RR0} - 2f^2_{RR0}\rho^{\cos}) - 2\rho^s\Box f^2_{RR0}]R_1\\ &\quad = -(1+f^2_0)\rho^s + 2f^2_{R0}\rho^s R_0 + 6\rho^s\Box f^2_{R0} + 6f^2_{R0}\nabla^2\rho^s.\end{aligned} \tag{26}$$

We define the potential

$$U(r,t) = [f^1_{RR0}(t) + 2f^2_{RR0}(t)\mathcal{L}_m(r,t)]R_1(r), \tag{27}$$

and the mass parameter

$$m^2 = \frac{1}{3}\left[\frac{f^1_{R0} - f^2_{R0}\mathcal{L}_m}{f^1_{RR0} + 2f^2_{RR0}\mathcal{L}_m} - R_0 - \frac{3\Box(f^1_{RR0} - 2f^2_{RR0}\rho^{\cos}) - 6\rho^s\Box f^2_{RR0}}{f^1_{RR0} + 2f^2_{RR0}\mathcal{L}_m}\right], \tag{28}$$

assuming that $f^1_{RR0} + 2f^2_{RR0}\mathcal{L}_m \neq 0$. Note that $m = m(t)$ outside the spherical body, where $\rho^s = 0$ and $\mathcal{L}_m = -\rho^{\cos}(t)$. For $f^2(R) = 0$, the mass formula presented in Ref. [18] for $f(R)$ theories is recovered. A negative mass squared $m^2 < 0$ could generically produce a gravitational instability, as the solution of Eq. (32) would lead to radial oscillations of the potential U with wavelength and frequency $\sim|m|^{-1}$.

In the remainder of this study, we will assume that $|mr| \ll 1$ within the Solar System, so that the contribution of any mass parameter is negligible and any putative oscillations evolve with a wavelength and period much larger than the typical time scale of Solar System dynamics.

Using the expressions for U and m^2, the equation for R_1 can be written as

$$\begin{aligned}\nabla^2U - m^2U = &-\frac{1}{3}(1+f^2_0)\rho^s + \frac{2}{3}f^2_{R0}\rho^s R_0\\ &+ 2\rho^s\Box f^2_{R0} + 2f^2_{R0}\nabla^2\rho^s.\end{aligned} \tag{29}$$

The assumption $|mr| \ll 1$ at Solar System scales signals a long-range extra force due to the nontrivial functions $f^i(R)$. If the mass parameter is negative, this implies that the time scale of oscillations is much larger than the one ruling Solar System dynamics.

V. SOLUTION FOR R_1

Outside the star, Eq. (29) reads $\rho^s = 0$ and we obtain

$$\nabla^2U = m^2(t)U, \tag{30}$$

so that U behaves as a Yukawa potential with a characteristic length $1/m(t)$ evolving on a cosmological time scale,

$$U \sim \frac{e^{-mr}}{r} \sim \frac{1}{r}, \tag{31}$$

or, if m^2 is negative, as an oscillating potential with strength $\sim 1/r$. The approximation $U \sim 1/r$ stems from the assumption that $|mr| \ll 1$ within the Solar System: we may thus drop the mass term m^2U in Eq. (29) outside the spherical body. Moreover, standard approximation properties of solutions of differential equations permit us to neglect this mass term also inside the spherical body, where the mass m^2 depends both on r and t, whenever $|mr| \ll 1$. Equation (29) then becomes

$$\nabla^2U = \eta(t)\rho^s(r) + 2f^2_{R0}\nabla^2\rho^s, \tag{32}$$

with the definition

$$\eta(t) = -\frac{1}{3}(1+f^2_0) + \frac{2}{3}f^2_{R0}R_0 + 2\Box f^2_{R0}. \tag{33}$$

Outside the spherical body, $\rho^s = 0$ and we may use the divergence theorem to obtain

$$U(r,t) = -\frac{\eta(t)}{4\pi}\frac{M_S}{r}, \tag{34}$$

where M_S is the total gravitational mass of the spherical body. Using Eq. (27), this implies that

$$R_1(r,t) = \frac{\eta(t)}{4\pi(2f^2_{RR0}\rho^{\cos} - f^1_{RR0})}\frac{M_S}{r}. \tag{35}$$

For $f^2(R) = 0$, this expression reduces to the solution for R_1 found in Ref. [18]. Notice that, although R_1 depends on time through $R_0(t)$ and $\rho^{\cos}(t)$, the time scale of its variation (comparable to the current Hubble time being much bigger than the one of Solar System dynamics) ensures the approximation $R_1(r,t) \simeq R_1(r)$.

Inside the spherical body, Eq. (32) implies that

$$\frac{d}{dr}(U - 2f^2_{R0}\rho^s) = \frac{\eta(t)}{4\pi}\frac{M(r)}{r^2}, \tag{36}$$

where $M(r)$ is the gravitational mass inside a sphere of radius r, defined as

$$M(r) \equiv 4\pi \int_0^r \rho^s(\xi)\xi^2 d\xi, \qquad M_S = M(R_S). \tag{37}$$

Since the potential U must be continuous, it is profitable to rewrite this equation in terms of the dimensionless variable $x \equiv r/R_S$ and dimensionless function

$$y \equiv \frac{U(x)}{U(x=1)} = -\frac{4\pi R_S U(x)}{\eta(t) M_S}, \tag{38}$$

so that Eq. (36) becomes

$$\frac{d}{dx}\left(y + \frac{8\pi f^2_{R0}}{\eta(t)}\frac{R_S}{M_S}\rho^s\right) = -\frac{M(x)}{M_S x^2}. \tag{39}$$

In order to derive $y(x)$ from the above, we require prior knowledge of the density profile inside the spherical body, ρ^s; to do so, we assume that the latter may be expanded as a Taylor series,

$$\rho^s = \rho^s_0 \sum_{i=0} a_i x^i, \tag{40}$$

where $\rho^s_0 \sim 10^5\ \mathrm{kg/m^3}$ is the central density and $a_0 = 1$. We thus get

$$M(r) = 4\pi\rho^s_0 R^3_S \sum_{i=0} \frac{a_i}{i+3} x^{i+3}, \tag{41}$$

so that

$$M_S = 4\pi\rho^s_0 R^3_S \sum_{i=0} \frac{a_i}{i+3}, \tag{42}$$

and Eq. (39) may be integrated between x and $x = 1$ to obtain

$$y = \frac{\sum_{i=0}\frac{a_i}{i+2}}{\sum_{i=0}\frac{a_i}{i+3}} - \frac{\sum_{i=0} a_i x^i\left[\frac{2f^2_{R0}}{\eta(t)R^2_S} + \frac{x^2}{(i+2)(i+3)}\right]}{\sum_{i=0}\frac{a_i}{i+3}}. \tag{43}$$

Using Eqs. (43) and (27), we thus obtain

$$\frac{R_1}{R_0} = \frac{\eta}{4\pi[2f^2_{RR0}(\rho^{\cos} + \rho^s) - f^1_{RR0}]}\frac{M_S}{R_0 R_S} y. \tag{44}$$

Equation (44) must be used to check if the perturbative approach $|R_1| \ll R_0$ is valid within the spherical body. Outside it, it suffices to compare Eq. (35) with the expression for R_0 found from a cosmological solution of NMC gravity.

The condition $|R_1| \ll R_0$ implies that the Ricci curvature $R = R_0 + R_1$ of the perturbed spacetime is close to the cosmological value R_0 at Solar System scales, and also inside the spherical body, even though the metric Eq. (6) of the perturbed spacetime is fairly close to the Minkowski metric.

In theories where $f^2(R) = 0$, such a condition is satisfied for $|mr| \ll 1$, with r varying from Solar System scales to the star interior, and $f^1_{R0}/f^1_{RR0} \sim R_0$ [2,18]. However, such theories yield the value $\gamma = 1/2$ which does not satisfy Solar System tests of gravity. Theories which do not satisfy the condition $|R_1| \ll R_0$ inside the spherical body are characterized by a large mass m, such that $|mr| \gg 1$ at Solar System scales [2]. For $f^2(R) = 0$, this could render viable, due to decoupling, a minimally coupled model of gravity; for GR, the condition $|R_1| \ll R_0$ is not satisfied in the star interior. In this study, we consider this issue for $f^2(R) \neq 0$.

VI. LINEARIZATION OF THE FIELD EQUATIONS

In this section we linearize the field equations (3). We denote by $[R_0]^\mu_\nu$ the components of the Ricci tensor in the considered background metric. The tensor $[R_0]^\mu_\nu$ solves the field equations (3) with matter source given by $T^{\cos\mu}_\nu$:

$$\begin{aligned}([R_0]^\mu_\nu - \nabla^\mu\nabla_\nu + \delta^\mu_\nu\Box)(f^1_{R0} + 2f^2_{R0}\mathcal{L}^{\cos}_m) \\ -\frac{1}{2}f^1_0\delta^\mu_\nu = (1 + f^2_0)T^{\cos\mu}_\nu.\end{aligned} \tag{45}$$

We now linearize Eqs. (3) using the first-order Taylor expansions of the functions $f^i(R)$ and $f^i_R(R)$ around $R = R_0$, for $i = 1, 2$. Using Eq. (45) and neglecting time derivatives of the background metric, we obtain the following system of equations in R^μ_ν:

$$\begin{aligned}&(f^1_{R0} + 2f^2_{R0}\mathcal{L}^{\cos}_m)(R^\mu_\nu - [R_0]^\mu_\nu) + 2f^2_{R0}\mathcal{L}^s_m R^\mu_\nu \\ &\quad + (f^1_{RR0} + 2f^2_{RR0}\mathcal{L}_m)R_1 R^\mu_\nu - f^2_{R0}R_1 T^\mu_\nu \\ &\quad - \frac{1}{2}f^1_{R0}R_1\delta^\mu_\nu - f^1_{RR0}(\nabla^\mu\nabla_\nu - \delta^\mu_\nu\Box)R_1 \\ &\quad - 2f^2_{RR0}(\nabla^\mu\nabla_\nu - \delta^\mu_\nu\Box)(\mathcal{L}_m R_1) \\ &= (1 + f^2_0)T^{s\mu}_\nu + 2f^2_{R0}(\nabla^\mu\nabla_\nu - \delta^\mu_\nu\Box)\mathcal{L}^s_m.\end{aligned} \tag{46}$$

The R^0_0 component is thus given by

$$\begin{aligned}R^0_0 &= -\frac{1}{1+2\Psi}\left[\frac{1}{a^2}\nabla^2\Psi - 3\left(H^2 + \frac{dH}{dt}\right)\right] \\ &\simeq -\nabla^2\Psi + 3\left(H^2 + \frac{dH}{dt}\right),\end{aligned} \tag{47}$$

while R_{rr} reads

$$\begin{aligned}R_{rr} &= a^2\frac{1+2\Phi}{1+2\Psi}\left(3H^2 + \frac{dH}{dt}\right) \\ &\quad - \frac{1}{1+2\Psi}\frac{d^2\Psi}{dr^2} + \frac{1}{(1+2\Psi)^2}\left(\frac{d\Psi}{dr}\right)^2 \\ &\quad + \frac{2}{r}\frac{1}{1+2\Phi}\frac{d\Phi}{dr} + \frac{1}{(1+2\Phi)(1+2\Psi)}\frac{d\Phi}{dr}\frac{d\Psi}{dr} \\ &\simeq -\frac{d^2\Psi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr} + 3H^2 + \frac{dH}{dt}.\end{aligned} \tag{48}$$

By neglecting the terms involving functions Ψ and Φ in the previous expressions we get the corresponding components of the tensor $[R_0]^\mu_\nu$.

We can simplify Eqs. (46) by neglecting terms involving the product of R_1, Ψ, Φ and their derivatives with H and dH/dt. Moreover, following Ref. [18], we neglect terms that are nonlinear functions of the metric perturbations Ψ and Φ, and we neglect terms involving products of R_1 by Ψ and Φ. Such approximations permit us to replace the d'Alembert operator $\Box$ with the flat space Laplace operator ∇^2. The 00 and rr components of Eqs. (46) then become, respectively,

$$(f^1_{R0}+2f^2_{R0}\mathcal{L}_m)\left(\nabla^2\Psi+\frac{1}{2}R_1\right) - \nabla^2[(f^1_{RR0}+2f^2_{RR0}\mathcal{L}_m)R_1] = (1+f^2_0)\rho^s - 2f^2_{R0}\nabla^2\rho^s, \tag{49}$$

and

$$(f^1_{R0}+2f^2_{R0}\mathcal{L}_m)\left(-\frac{d^2\Psi}{dr^2}+\frac{2}{r}\frac{d\Phi}{dr}\right)-\frac{1}{2}f^1_{R0}R_1 + \frac{2}{r}f^1_{RR0}\frac{dR_1}{dr}+\frac{4}{r}f^2_{RR0}\frac{\partial(\mathcal{L}_mR_1)}{\partial r} = \frac{4}{r}f^2_{R0}\frac{d\rho^s}{dr}. \tag{50}$$

In the next sections we shall compute the solutions Ψ and Φ of these equations.

VII. SOLUTION FOR Ψ

Using Eqs. (32), (33), and (49) becomes

$$(f^1_{R0}+2f^2_{R0}\mathcal{L}_m)\left(\nabla^2\Psi+\frac{1}{2}R_1\right) = \frac{2}{3}(1+f^2_0+f^2_{R0}R_0+3\Box f^2_{R0})\rho^s. \tag{51}$$

We assume that $f^1_{R0}+2f^2_{R0}\mathcal{L}_m\neq 0$ and, following Ref. [18], decompose Ψ as the sum of two functions, $\Psi=\Psi_0+\Psi_1$, such that

$$\nabla^2\Psi_0 = \frac{2}{3}\frac{1+f^2_0+f^2_{R0}R_0+3\Box f^2_{R0}}{f^1_{R0}+2f^2_{R0}\mathcal{L}_m}\rho^s, \qquad \nabla^2\Psi_1 = -\frac{1}{2}R_1. \tag{52}$$

Using Eq. (15), integration through the divergence theorem yields for the function Ψ_0 outside of the star,

$$\Psi_0(r,t) = -\frac{1}{6\pi}(1+f^2_0+f^2_{R0}R_0+3\Box f^2_{R0})\frac{M^*}{r}+C_0, \qquad M^* = 4\pi\int_0^{\mathrm{R_S}}\frac{\rho^s(x)}{f^1_{R0}+2f^2_{R0}\mathcal{L}_m(x)}r^2dr, \tag{53}$$

with C_0 an integration constant. The function Ψ_1 is computed in Appendix A, where it is shown that, under the additional condition

$$\left|\frac{f^1_{R0}+2f^2_{R0}\mathcal{L}_m}{f^1_{RR0}+2f^2_{RR0}\mathcal{L}_m}\right| r^2 \ll 1, \tag{54}$$

assumed to be valid both inside and outside the star, we have

$$\Psi_1(r,t) = \Psi^*_1(r,t)+C_1, \qquad |\Psi^*_1(r,t)| \ll |\Psi_0(r,t)-C_0|, \tag{55}$$

where C_1 is another integration constant. Condition Eq. (54) is satisfied for instance by functions of the type $f^1(R)\sim R^m$, $f^2(R)\sim R^n$, at least for a suitable range of values of the exponents, and its meaning will be discussed at the end of this section. By requiring that $\Psi(r,t)$ vanishes as $r\to+\infty$, we obtain that $C_0+C_1=0$. The validity of the Newtonian limit requires that $\Psi(r)$ is proportional to M_S/r, leading to the following constraint on the functions $f^1(R)$ and $f^2(R)$:

$$|2f^2_{R0}|\rho^s(r) \ll |f^1_{R0}-2f^2_{R0}\rho^{\cos}(t)|, \qquad r\leq \mathrm{R_S}. \tag{56}$$

We now get the solution for Ψ outside of the star,

$$\Psi(r,t) = -\frac{1+f^2_0+f^2_{R0}R_0+3\Box f^2_{R0}}{6\pi(f^1_{R0}-2f^2_{R0}\rho^{\cos})}\frac{M_S}{r}, \qquad r\geq \mathrm{R_S}. \tag{57}$$

For $f^2(R)=0$, this expression reduces to the solution for Ψ found in Ref. [18]. The expression for Ψ yields a gravitational coupling slowly varying in time,

$$G = \frac{\omega(t)}{6\pi(f^1_{R0}-2f^2_{R0}\rho^{\cos})}, \qquad \omega(t) = 1+f^2_0+f^2_{R0}R_0+3\Box f^2_{R0}. \tag{58}$$

As expected, the time scale $\dot{G}/G$ is much longer than the one of Solar System dynamics. Hence we have approximately $G\simeq$ const and $\Psi(r,t)\simeq\Psi(r)$.

By comparing with available bounds on $\dot{G}/G$ (see Ref. [21] for an updated review), Eq. (58) can in principle be used to constrain $f^1(R)$ and $f^2(R)$.

We may now check the assumption $|R_1|\ll R_0$ outside the spherical body. Using the solution Eq. (35) for R_1 and the expression Eq. (58) of the effective gravitational constant G, we have, for $r\geq \mathrm{R_S}$,

$$\left|\frac{R_1}{R_0}\right| \leq \frac{3}{2R_0}\frac{GM_S}{\mathrm{R_S}}\left|\frac{\eta(t)}{\omega(t)}\right|\cdot\left|\frac{f^1_{R0}-2f^2_{R0}\rho^{\cos}}{f^1_{RR0}-2f^2_{RR0}\rho^{\cos}}\right|. \tag{59}$$

Then, the assumption $|R_1|\ll R_0$ used in the linearization of the field equations places the following additional constraint on functions $f^1(R)$ and $f^2(R)$:

$$\left|\frac{\eta(t)}{\omega(t)}\right|\cdot\left|\frac{f^1_{R0}-2f^2_{R0}\rho^{\cos}}{f^1_{RR0}-2f^2_{RR0}\rho^{\cos}}\right| \ll R_0\left(\frac{\mathrm{R_S}}{GM_S}\right). \tag{60}$$

If the following condition is satisfied,

$$\left|\frac{\eta(t)}{\omega(t)}\right| \lesssim 1, \tag{61}$$

as shall be checked later, then the above amounts to

$$\left|\frac{f^1_{R0}-2f^2_{R0}\rho^{\cos}}{f^1_{RR0}-2f^2_{RR0}\rho^{\cos}}\right| \ll R_0\left(\frac{\mathrm{R_S}}{GM_S}\right) \approx 4.7\times10^5 R_0. \tag{62}$$

Since $R_0 r^2 \sim H^2 r^2 \ll GM_S/R_S$ for the current Hubble parameter H and r of the order of Solar System scales, we find that condition Eq. (62) is much stronger than Eq. (54).

If condition Eq. (62) is satisfied and the effective gravitational constant G is identified with Newton's gravitational constant, using Eq. (60) we have $|R_1/R_0| \ll 1$. For $f^2(R) = 0$, conditions Eqs. (54) and (62) reduce to $|f^1_{R0}/f^1_{RR0}|r^2 \ll 1$ and $|f^1_{R0}/f^1_{RR0}| \ll R_0 R_S/(GM_s)$ found in Ref. [18]. This condition is satisfied for instance by the theory of $1/R^n$ gravity, proposed in Ref. [19], where

$$f^1(R) = 2\kappa\left(R - \frac{\mu^{2+2n}}{R^n}\right), \quad n > 0, \qquad f^2(R) = 0. \tag{63}$$

This theory satisfies also the condition $|mr| \ll 1$ at Solar System scales [18].

VIII. SOLUTION FOR Φ

We now compute the solution Φ under condition Eq. (54). For $r \geq R_S$, Eq. (50) becomes

$$\begin{aligned}(f^1_{R0} + 2f^2_{R0}\mathcal{L}^{\cos}_m)\left(-\frac{d^2\Psi}{dr^2} + \frac{2}{r}\frac{d\Phi}{dr}\right) - \frac{1}{2}f^1_{R0}R_1 \\ + \frac{2}{r}(f^1_{RR0} + 2f^2_{RR0}\mathcal{L}^{\cos}_m)\frac{dR_1}{dr} = 0.\end{aligned} \tag{64}$$

Using the solution Eq. (35) for R_1, we have

$$\frac{R_1}{dR_1 dr} = -r. \tag{65}$$

Since $\rho^{\cos}(t) \ll \rho^s(r)$ for $r < R_S$ and $|r - R_S|$ large enough, using Eq. (56) we have also

$$|2f^2_{R0}|\rho^{\cos}(t) \ll |f^1_{R0}|. \tag{66}$$

Using these results and Eq. (54), we have

$$\begin{aligned}&\left|\frac{f^1_{R0}R_1/2}{(2/r)(f^1_{RR0} + 2f^2_{RR0}\mathcal{L}^{\cos}_m)(dR_1/dr)}\right| \\ &\quad \simeq \frac{1}{4}\left|\frac{f^1_{R0} + 2f^2_{R0}\mathcal{L}^{\cos}_m}{f^1_{RR0} + 2f^2_{RR0}\mathcal{L}^{\cos}_m}\right| r^2 \ll 1.\end{aligned} \tag{67}$$

It follows that the term $f^1_{R0}R_1/2$ can be neglected in Eq. (64), which now becomes

$$\frac{d\Phi}{dr} = \frac{r}{2}\frac{d^2\Psi}{dr^2} - \left[\frac{f^1_{RR0} - 2f^2_{RR0}\rho^{\cos}}{f^1_{R0} - 2f^2_{R0}\rho^{\cos}}\right]\frac{dR_1}{dr}. \tag{68}$$

Substituting in this equation the derivatives of functions R_1 and Ψ, computed from Eqs. (35) and (57), respectively, we obtain

$$\Phi(r, t) = \frac{1 + f^2_0 + 4f^2_{R0}R_0 + 12\Box f^2_{R0}}{12\pi(f^1_{R0} - 2f^2_{R0}\rho^{\cos})}\frac{M_S}{r}, \tag{69}$$

for $r \geq R_S$. As expected, setting $f^2(R) = 0$ reduces this expression to the solution for Φ found in Ref. [18]. Again, we have $\Phi(r, t) \simeq \Phi(r)$.

Using the expressions of Ψ and Φ, we get the PPN parameter γ:

$$\gamma = \frac{1}{2}\left[\frac{1 + f^2_0 + 4f^2_{R0}R_0 + 12\Box f^2_{R0}}{1 + f^2_0 + f^2_{R0}R_0 + 3\Box f^2_{R0}}\right]. \tag{70}$$

Thus, the parameter γ is completely defined by the background metric and its value can be obtained by computing the cosmological solution of NMC gravity. Inserting $f^2(R) = 0$ yields the value $\gamma = 1/2$ as it has been found in Ref. [18]. In particular, the $1/R^n$ gravity model given by Eq. (63) also predicts $\gamma = 1/2$. However, notice that formula (70) cannot be applied when the functions $f^i(R)$ reduce to their GR expressions, since in this case the mass parameter m, defined in Eq. (28), is ill defined (and divergent), so that the assumptions of our computations are not satisfied.

The obtained results show that, in order for a cosmologically viable nonminimally coupled model to be compatible with Solar System tests, one of the following conditions has to be satisfied:

(i) Either the condition $|mr| \ll 1$ at Solar System scales is not satisfied, or nonlinear terms in R_1 are not negligible in the Taylor expansions Eqs. (11) (which happens if the perturbative condition $|R_1| \ll R_0$ is not satisfied), so that the present analysis does not apply;

(ii) If both conditions of point (i) are satisfied, then the condition Eq. (56) of validity of the Newtonian limit has to be satisfied, and the value of γ given by Eq. (70) has to satisfy the constraint from the Cassini measurement $\gamma = 1 + (2.1 \pm 2.3) \times 10^{-5}$ (cf. Ref. [1]).

The mass m^2, which is a function $m^2 = m^2(r, t)$ given by Eq. (28), has to be computed by using the cosmological solution $R_0(t)$, $\rho^{\cos}(t)$. In the following section, we implement the obtained criteria for the cosmological scenario posited in Ref. [12].

IX. APPLICATION

Following Ref. [12], let us consider the case study

$$f^1(R) = 2\kappa R, \qquad f^2(R) = \left(\frac{R}{R_n}\right)^{-n}, \qquad n > 0, \tag{71}$$

where R_n is a constant; the linear choice of $f^1(R)$ serves to highlight the impact of the NMC between matter and curvature on the dynamics. Notice that the correct GR limit of a power-law coupling between matter and curvature is not attained by setting $n = 0$ [as this simply doubles the minimal coupling, $f^2(R) = 0$], but by imposing $R_n \to 0$ (for positive n, i.e. an inverse power law).

The above choice yields a cosmological scenario where the contribution of the NMC dominates the dynamics and a constant (negative) deceleration parameter is obtained, $q < 0$; this, however, is attained due to the large value of $f^2_{R0}\rho^{\cos}$ and its temporal derivatives, not the NMC itself, which remains subdominant, $f^2_0 \ll 1$.

This mechanism implies a direct relation between the exponent n and the latter [12],

$$q = -1 + \frac{3}{2(1+n)}; \tag{72}$$

that is, a de Sitter solution with exponential scale factor is ruled out. Thus, the scale factor $a(t)$ of the background metric and the cosmological matter density $\rho^{\cos}(t)$ follow the temporal evolution

$$a(t) = a_0\left(\frac{t}{t_0}\right)^{2(1+n)/3}, \tag{73}$$

and

$$\begin{aligned} \rho^{\cos}(t) &= \rho_0^{\cos}\left(\frac{t_0}{t}\right)^{2(1+n)}, \\ \rho_0^{\cos} &= (1+n)\frac{8}{3}\frac{\kappa}{t_0^2}\left[\frac{4(1+n)(1+4n)}{3}\left(\frac{t_n}{t_0}\right)^2\right]^n, \end{aligned} \tag{74}$$

where t_0 is the current age of the Universe and $t_n \equiv 1/\sqrt{R_n}$; the latter expression stems from the covariant conservation of the energy-momentum tensor, which remains valid since the Lagrangian density is given by $\mathcal{L}_m = -\rho^{\cos}$ (see Ref. [13] for a discussion).

Equation (73) yields

$$\begin{aligned} H &= \frac{\dot a}{a} = \frac{2(1+n)}{3t}, \\ R_0 &= 6(\dot H + 2H^2) = \frac{4(1+n)(1+4n)}{3t^2}, \end{aligned} \tag{75}$$

where $H \equiv \dot a/a$ is the Hubble parameter. Since the current value of the former is $H \sim 70$ (km/s)/Mpc [22] and the deceleration parameter is of order $q_0 \sim -1$, we get that $R_0 \sim (10^{14}\ \mathrm{AU})^{-2}$, to be compared with the relevant range for the Solar System, $r \lesssim \mathrm{R}_{SS} \sim 100$ AU.

Inserting the expression for the scalar curvature R_0 into Eq. (71), we get

$$f_0^2 = \left[\frac{3}{4(1+n)(1+4n)}\left(\frac{t_0}{t_n}\right)^2\right]^n. \tag{76}$$

We recall that the choice for the Lagrangian density $\mathcal{L}_m = -\rho^{\cos}$ implies that the energy-momentum tensor of matter is conserved, $\nabla_\mu T^{\mu\nu} = 0 \rightarrow \dot\rho^{\cos} = -3H\rho^{\cos}$. From Eq. (75), we get

$$\ddot R_0 = \frac{3}{2}\frac{(\dot R_0)^2}{R_0} = \frac{6}{t^2}R_0 = \frac{9R_0^2}{2(1+n)(1+4n)}, \tag{77}$$

and, together with the expressions below, valid for a power-law NMC,

$$f^2_{R0} = -n\frac{f_0^2}{R_0}, \qquad f^2_{RR0} = n(n+1)\frac{f_0^2}{R_0^2}, \tag{78}$$

we get

$$\frac{\Box(f^2_{R0})}{f_0^2} \approx \frac{n}{f_0^2}\left[\frac{d^2}{dt^2}\left(\frac{f_0^2}{R_0}\right) + 3H\frac{d}{dt}\left(\frac{f_0^2}{R_0}\right)\right] = \frac{3}{2}\frac{3+4n}{1+4n}n, \tag{79}$$

as well as

$$\begin{aligned} \frac{\Box(f^2_{RR0}\rho^{\cos})}{f^2_{RR0}\rho^{\cos}} &\approx -\frac{R_0^2}{f_0^2\rho^{\cos}}\left[\frac{d^2}{dt^2}\left(\frac{f_0^2}{R_0^2}\rho^{\cos}\right) + 3H\frac{d}{dt}\left(\frac{f_0^2}{R_0^2}\rho^{\cos}\right)\right] \\ &= -\frac{3}{2}\frac{2n+3}{4n^2+5n+1}R_0, \end{aligned} \tag{80}$$

and

$$\begin{aligned} \frac{\Box f^2_{RR0}}{f^2_{RR0}} &\approx -\frac{R_0^2}{f_0^2}\left[\frac{d^2}{dt^2}\left(\frac{f_0^2}{R_0^2}\right) + 3H\frac{d}{dt}\left(\frac{f_0^2}{R_0^2}\right)\right] \\ &= -\frac{3}{2}\frac{4n^2+13n+10}{4n^2+5n+1}R_0. \end{aligned} \tag{81}$$

As expected, the d'Alembertian terms cannot be neglected, as they are comparable to $R_0 \sim H^2$.

From Eq. (79) and the definitions Eqs. (33) and (58), we obtain

$$\eta(t) = -\frac{1}{3} + \frac{28n^2+21n-1}{3(1+4n)}f_0^2 \tag{82}$$

and

$$\omega(t) = 1 + \frac{28n^2+33n+2}{2(1+4n)}f_0^2. \tag{83}$$

In the discussion following Eq. (60), the assumption Eq. (61) was put forward. We are now in a position to check it directly, obtaining

$$\begin{aligned} \left|\frac{\eta(t)}{\omega(t)}\right| &= \frac{1}{3}, \qquad f_0^2 \ll 1, \\ \left|\frac{\eta(t)}{\omega(t)}\right| &= \frac{2}{3}\left|\frac{28n^2+21n-1}{28n^2+33n+2}\right| \le \frac{2}{3}, \qquad f_0^2 \gg 1. \end{aligned} \tag{84}$$

This indicates that for either a large or negligible value of the NMC, the condition Eq. (61) holds. In Eq. (B3) of Appendix B, it is shown that the intermediate regime $f_0^2 \sim 1$ (for a power-law NMC) also yields the ratio $|\eta(t)/\omega(t)| \sim 0.5$, so that Eq. (61) is always valid, as assumed before.

We now show that condition Eq. (62) [and, as a result, Eq. (54)] is valid for the chosen NMC, Eq. (71); from Eqs. (78) and (B1) from Appendix B, we get

$$\left|\frac{f^1_{R0}+2f^2_{R0}\mathcal{L}_m}{f^1_{RR0}+2f^2_{RR0}\mathcal{L}_m}\right| = \left|\frac{\kappa R_0 + n f_0^2\rho^{\cos}(1+\frac{\rho^s}{\rho^{\cos}})}{n(n+1)f_0^2\rho^{\cos}(1+\frac{\rho^s}{\rho^{\cos}})}\right| R_0 = \left|\frac{6n+1+2n\frac{\rho^s}{\rho^{\cos}}}{2n(n+1)(1+\frac{\rho^s}{\rho^{\cos}})}\right| R_0. \quad (85)$$

Away from the spherical body, $\rho^s = 0$ and the above reads

$$\left|\frac{f^1_{R0}+2f^2_{R0}\mathcal{L}_m}{f^1_{RR0}+2f^2_{RR0}\mathcal{L}_m}\right| = \frac{6n+1}{2n(n+1)} \ll \frac{\mathrm{R_S}}{GM_S} \rightarrow n \gg 10^{-6}. \quad (86)$$

If the contribution of ρ^s inside the spherical body dominates the above, this becomes

$$\left|\frac{f^1_{R0}+2f^2_{R0}\mathcal{L}_m}{f^1_{RR0}+2f^2_{RR0}\mathcal{L}_m}\right| \approx \frac{R_0}{1+n}, \quad (87)$$

which is smaller than $\mathrm{R_S}/(GM_S)$ for any value of the exponent n. If the latter is so small that the density contributions do not dominate Eq. (85), then one falls back into Eq. (87). Thus, one concludes that Eqs. (54) and (62) are valid for $n \gg 10^{-6}$, which includes the values $n = 4$, 10 considered in Ref. [12].

Several values for the exponent n have been evaluated in previous studies, ranging from studies of hydrostatic equilibrium [4] or spherical collapse [15] to galactic [7] and cluster [8] dark matter, dark energy [12] and postinflationary preheating [11]. All scenarios assumed a linear $f^1(R) = 2\kappa R$, except for the latter—where $f^1(R) = 2\kappa(R + R^2/6M^2)$ (the so-called Starobinsky inflation).

In all of these studies, it has been argued that any particular power-law form for the NMC represents the dominant behavior of a more evolved function $f^2(R)$ in each regime (i.e. typical scalar curvature associated with the context under scrutiny, from astrophysics to cosmology). As an example, a particular set (n, R_n) that accounts for e.g. galactic dark matter was shown to be irrelevant to implement a generalized preheating after inflation (and vice versa). This argument is also used concerning the plethora of forms used for the curvature term in $f(R)$ theories.

The same reasoning should apply here: for completeness, the full set of power-law contributions considered in the mentioned studies should be used, that is, $f^2(R) = \sum_i (\frac{R}{R_i})^{-i}$. However, since this quantity (and its derivatives) must be evaluated at its cosmological value $R = R_0(t)$, it suffices to retain the cosmologically dominant term, as studied in Ref. [12]. Thus, the results here obtained cannot be used to constrain the power-law NMC functions used to account for astrophysical scenarios (including galactic and cluster dark matter).

With the above in mind, we recall the two examples presented numerically in Ref. [12], where

$$\begin{aligned} n = 4&:\ t_4 = \frac{t_0}{4} \rightarrow f_0^2 = \left(\frac{12}{85}\right)^4 \approx 4\times 10^{-4}, \\ n = 10&:\ t_{10} = \frac{t_0}{2} \rightarrow f_0^2 = \left(\frac{3}{451}\right)^{10} \approx 10^{-22}, \end{aligned} \quad (88)$$

confirming that the NMC is indeed perturbative, as indicated above. However, notice that due to a miscalculation in Ref. [12] (described in Appendix B), we should instead consider the values

$$\begin{aligned} n = 4&:\ t_4 = 0.10 t_0 \rightarrow f_0^2 = 0.63, \\ n = 10&:\ t_{10} = 0.043 t_0 \rightarrow f_0^2 = 0.29. \end{aligned} \quad (89)$$

Although the NMC is considerably higher, this does not have an impact on any of the results described below, since we never resort to its actual value, but instead to Eqs. (74) and (75) and ensuing analytical results.

A. Long range regime, $|mr| \ll 1$

Using Eqs. (71), (80), and (81), we are now able to compute the mass parameter given by Eq. (28), obtaining

$$\begin{aligned} m^2 &= \frac{\mu\rho^{\cos} + \nu\rho^s}{\rho^{\cos}+\rho^s} R_0, \\ \mu &\equiv -\frac{8n^3 + 4n^2 - 18n + 1}{6n(n+1)(4n+1)}, \\ \nu &\equiv \frac{28n^2 + 111n + 89}{6(n+1)(4n+1)}. \end{aligned} \quad (90)$$

Notice that the roots of the denominator of both μ and ν are nonpositive, while the NMC used in a cosmological setting assumes a positive exponent n [12].

Figure 1 shows the variation of $\mu(n)$ and $\nu(n)$: for $n > 0$, we see that both functions are $O(10)$ or below: since $\rho^s \gg \rho^{\cos}$ inside the spherical body—except for a vanishingly thin surface layer—the mass parameter is given inside it by $m^2 \sim \nu R_0$ (for all values of n, since ν has no roots); in the outside, we have $m^2 = \mu R_0$.

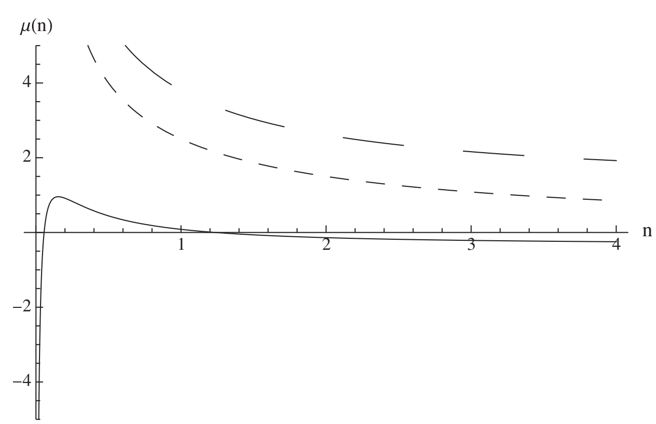

FIG. 1. Quantities $\mu(n)$ (full), $\nu(n)$ (long dash) and $\epsilon(n)$ (short dash), defined in Eqs. (90) and (95) as a function of the exponent n.

For $n \sim 0$, the function μ grows to large (negative) values; if $\mu\rho^{\cos} \gg \nu\rho^{s}$, the mass parameter inside the spherical body is given by

$$m^2 \approx \frac{\rho^{\cos}}{\rho^{s}}\mu R_0. \tag{91}$$

Since $\mu \sim -1/6n$ for $n \sim 0$, the validity of the long-range regime yields

$$|mr| \leq |m\mathrm{R_S}| \ll 1 \rightarrow n \gg \frac{\rho^{\cos}}{\rho^{s}}\frac{\mathrm{R_S^2}R_0}{6} \sim 10^{-66}, \tag{92}$$

using $\rho^{\cos} \sim 10^{-27}$ kg/m^3, $\rho^{s} \lesssim \rho_0^{s} \sim 10^5$ kg/m^3 (the central density of the Sun), $R_0 \sim (10^{14}\ \mathrm{AU})^{-2}$ and $\mathrm{R_S} = 1.4 \times 10^9\ \mathrm{m} \sim 5 \times 10^{-3}$ AU.

By the same token, away from the spherical body we get

$$|mr| \gtrsim |m\mathrm{R}_S| \ll 1 \rightarrow n \gg \frac{\mathrm{R}_S^2 R_0}{6} \sim 10^{-25}, \tag{93}$$

a stronger constraint than the one above, but extremely mild nonetheless.

B. Newtonian regime

The previously discussed Eq. (56) provides the condition for the validity of the Newtonian regime adopted in this study. Using the previous expressions Eqs. (71) and (74), we find that $f_{R0}^2\rho^{\cos}(t)/\kappa = \mathrm{const}$, and this condition can be recast as

$$\left|\frac{\kappa}{f_{R0}^2\rho^{\cos}(t)} - 1\right|\rho^{\cos} = \left(3 + \frac{1}{2n}\right)\rho^{\cos}(t) \gg \rho^{s}(r) \rightarrow n \ll \frac{\rho^{\cos}}{2\rho^{s}} \sim 10^{-33}, \tag{94}$$

which is incompatible with the constraint $n \gg 10^{-25}$ required for the long-range condition $|mr| \ll 1$ to be valid outside the spherical body. Nevertheless, we cannot yet conclude that the Newtonian limit is not valid for $n \gg 10^{-25}$, since we have still to check the validity of Eq. (11), i.e. our assumptions that terms nonlinear in R_1 are negligible in the Taylor expansions of $f^i(R)$ and $f_R^i(R)$. This will be the subject of Sec. IX D.

C. PPN parameter γ

If nonlinear terms in R_1 were negligible in the Taylor expansions Eqs. (11), then the result of the preceding section implies that the Newtonian approximation would not be valid in the Solar System, whenever $|mr| \ll 1$, i.e. $n \gg 10^{-25}$. Thus we cannot rely on the result presented here for its impact at Solar System scales, i.e. the expression for the PPN γ parameter, Eq. (70).

D. Perturbative regime, $|R_1| \ll R_0$

We now check our assumption that $|R_1| \ll R_0$. At the end of Sec. VII, in order to check such an assumption outside the spherical body, we have used the inequality $GM_\mathrm{S}\mathrm{R_S} \ll 1$, where G is the effective gravitational constant defined in Eq. (58). However, the result of Sec. IX B shows that in the long-range regime $|mr| \ll 1$ we cannot rely on the validity of the Newtonian limit, so that we are prevented from using the effective gravitational constant G in this way. Hence, in order to estimate the ratio R_1/R_0, in the sequel we resort to Newton's gravitational constant G_N, which we recall is defined by $\kappa = c^4/16\pi G_N$.

1. Outer solution

We first assess the validity of the perturbative condition $|R_1| \ll R_0$ outside the spherical body.

Using Eqs. (35), (74), and (75), we get

$$\begin{aligned}\frac{R_1}{R_0} &= \eta(t)\frac{R_0}{8\pi n(n+1)f_0^2\rho^{\cos}}\frac{M_\mathrm{S}}{r}\\ &= \eta(t)\frac{1+4n}{6\pi n f_0^2 t_0^2\rho^{\cos}}\frac{M_\mathrm{S}}{r} = \epsilon\eta(t)\frac{G_N M_\mathrm{S}}{r},\\ \epsilon &\equiv \frac{1+4n}{n(1+n)}.\end{aligned} \tag{95}$$

We see that the function ϵ, plotted in Fig. 1, has no positive roots, but diverges at $n = 0$. Thus, the perturbative condition $|R_1| \ll |R_0|$ requires that

$$n \gg |\eta(t)|\frac{G_N M_\mathrm{S}}{\mathrm{R_S}} \approx 2.1 \times 10^{-6}|\eta(t)|, \tag{96}$$

and, since Eq. (B3) of Appendix B shows that $\eta(t) = 6.24$ for $n = 4$ and $\eta(t) = 6.68$ for $n = 10$, we conclude that $n \gg 10^{-5}$, a much stronger constraint that those obtained in Eqs. (92) and (93) of the preceding section.

2. Inner solution

We now assess the validity of the perturbative condition $|R_1| \ll R_0$ inside the spherical body.

We address Eq. (44): using Eqs. (43) and (78), the former reads

$$\frac{R_1}{R_0} = \frac{\rho^{s}}{\rho^{\cos} + \rho^{s}}\frac{1 - z(t)w(x)}{1+n}, \tag{97}$$

defining the dimensionless form function

$$w(x) \equiv \frac{\rho_0^{s}}{\rho^{s}}\sum_{i=0}\frac{a_i}{i+2}\left(1 - \frac{x^{i+2}}{i+3}\right), \tag{98}$$

and coupling

$$\begin{aligned}z(t) &\equiv \frac{\eta(t)\mathrm{R_S^2}}{2f_{R0}^2}\\ &= -\frac{\eta(t)}{2n}\left(\frac{4(1+4n)(1+n)}{3}\right)^{n+1}\left(\frac{\mathrm{R_S}}{t}\right)^2\left(\frac{t_n}{t}\right)^{2n}\\ &= -\eta(t)\frac{1+4n}{n}4\pi G_N\mathrm{R_S^2}\rho^{\cos}(t),\end{aligned} \tag{99}$$

again using Eq. (75). The above may be recast as

$$z(t) = -\eta(t)\frac{1+4n}{n}4\pi\frac{G_N M_\mathrm{S}}{\mathrm{R_S}}\frac{\rho^{\cos}\mathrm{R_S^3}}{M_\mathrm{S}}, \tag{100}$$

clearly showing that, since $G_N M_S/R_S \sim 2 \times 10^{-6}$ and $\rho^{\cos} R_S^3/M_S \sim 10^{-31}$, $z(t)$ is vanishingly small unless $n \ll 10^{-37}$ [since $|\eta(t)| \leq 7.0$, as discussed in the Appendix B).

At the surface of the spherical body, $x = 1$, we have $\rho^s = \rho_0^s \sum_{i=0} a_i = 0$, so that

$$\frac{R_1}{R_0} = -\frac{z(t)\rho_0^s}{(n+1)\rho^{\cos}} \sum_{i=0} \left(\frac{a_i}{i+3}\right) = -\frac{1}{4\pi} \frac{z(t)}{(n+1)\rho^{\cos}} \frac{M_S}{R_S^3}, \tag{101}$$

and, using Eqs. (99) and (95) is matched at the surface, as expected.

To assess the behavior inside the spherical body, we consider the following model of the density profile of the Sun [23]:

$$\rho^s(r) = \rho_0^s(1 - 5.74x + 11.9x^2 - 10.5x^3 + 3.34x^4), \tag{102}$$

depicted in Fig. 2. As discussed below, the overall result is not qualitatively affected by the specific density model. Notice that this fourth-order expression obeys the constraint $\rho^s(R_S) = 0$ and $(d\rho^s/dr)(R_S) \simeq 0$.

The density $\rho^s(r)$ rises beyond $\rho^{\cos}$ immediately after the surface of the spherical body: for the chosen density profile Eq. (102), we find numerically that $\rho^s \gg \rho^{\cos} \rightarrow x < 1\text{–}10^{-31}$. Thus, this thin surface layer may be safely disregarded, and Eq. (97) is approximated by

$$\frac{R_1}{R_0} \approx \frac{1 - z(t)w(x)}{1+n}. \tag{103}$$

Figure 3 plots the form function $w(x)$ for the density profile above. For comparison, two unrealistic cases are also depicted: $w_1(x)$, obtained from a linear density $\rho^s = \rho_0^s(1-x)$, and $w_0(x)$, derived from a constant one, $\rho^s = \rho_0^s$.

Clearly, the peak around $x \sim 0.5$ for the form function $w_4(x)$ derived from Eq. (102) appears because the density has a minimum at $x \approx 0.52$ (which is an unphysical artifact of the fourth-order approximation). We also see that $w_1(x)$ is an approximate envelope of $w_4(x)$, i.e. presents an approximate behavior without the aforementioned peak.

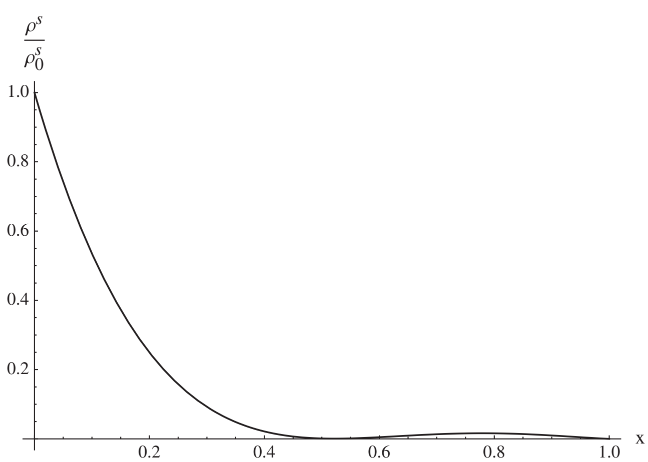

FIG. 2. Fourth-order approximation of the density profile inside the spherical body [Eq. (102)].

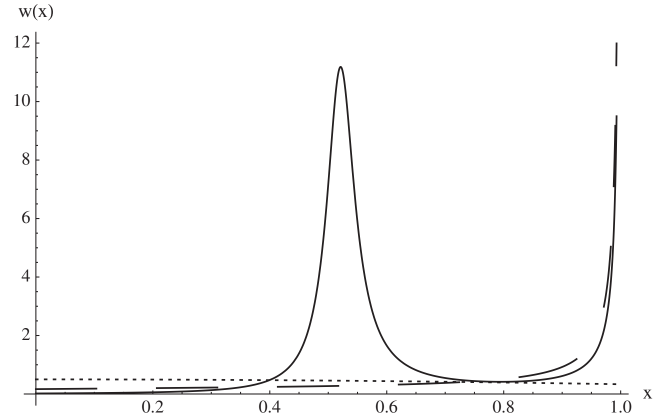

FIG. 3. Form functions $w(x)$ for the density profile Eq. (102) [$w_4(x)$, full], linear [$w_1(x)$, dashed] and constant profile [$w_0(x)$, dotted].

If $n \ll 10^{-37}$, a large coupling $z(t) \gg 1$ arises and we get $R_1 \approx -z(t)w(x)R_0$. This result breaks the perturbative condition underlying this work; moreover, in this case the long-range condition $|mr| \ll 1$ is not satisfied.

The converse case $n > 10^{-37}$ (which comprises $n = 4$ or $n = 10$, the two scenarios studied in Ref. [12]) leads to

$$\frac{R_1}{R_0} \approx \frac{1}{1+n}, \tag{104}$$

which is valid for the full interior of the spherical body, with the exception of a very thin surface layer signaling the transition to the outer solution. Notice that this result is not dependent on the adopted density model, as the vanishingly small value of $z(t)$ absorbs any peaks that may arise in the form factor $w(x)$.

Equation (104) implies that the condition $|R_1| \ll R_0$ is not satisfied when $n \sim 1$ or $n < 1$. For $n = 4$, the value of the curvature perturbation R_1 is one fifth of the cosmological background curvature R_0, while $n = 10$ yields a smaller $1/11$ factor. At first sight, this result allows us to validate the perturbative condition $|R_1| \ll R_0$, or at least it leads to the conclusion that $n \gg 1$, in order to get a larger separation between $|R_1|$ and R_0. However, this is not the case: indeed, if we expand the power-law NMC Eq. (71) up to third order and insert Eq. (104),

$$\begin{aligned} f^2(R) &\simeq f_0^2\left[1 - n\frac{R_1}{R_0} + \frac{n(n+1)}{2}\left(\frac{R_1}{R_0}\right)^2 \right. \\ &\qquad \left. - \frac{1}{6}n(n+1)(n+2)\left(\frac{R_1}{R_0}\right)^3\right] \\ &= f_0^2\left[1 - \frac{n}{n+1} + \frac{n}{2(n+1)} - \frac{n(n+2)}{6(n+1)^2}\right], \end{aligned} \tag{105}$$

we conclude that, for any exponent $n > 1$, the nonlinear terms in the Taylor expansion of $f^2(R)$ cannot be disregarded. An analogous result holds for the Taylor expansion of the function $f_R^2(R)$. It follows that conditions Eq. (11) are not respected.

A third possibility remains: that the coupling $z(t)$ is such that it enables a small numerator in Eq. (103). This implies that $z(t)w(x) \sim 1$, which requires an approximately constant form function $w(x) \approx$ const. However, since $z(t)$ is determined by the choice of the cosmologically relevant NMC, this would lead to an unphysical fine-tuning of the form function $w(x)$, and is thus deemed unfeasible.

Given the above discussion, we conclude that the perturbative regime is not compatible with the scenario posited in Ref. [12], and thus the method here developed cannot be applied to constrain the latter using Solar System observables.

E. Postinflationary reheating model

Following Ref. [11], we now consider the model

$$f^1(R) = 2\kappa\left(R + \frac{R^2}{6M^2}\right), \qquad f^2(R) = 2\xi\frac{R}{M^2}, \qquad (106)$$

which adds a nonminimal coupling to the standard preheating scenario in the context of Starobinsky inflation [24]. In Eq. (106), M has dimensions of mass and ξ is a dimensionless parameter. The mass parameter m^2, defined in Eq. (28), is proportional to M^2. Since M^2 is large in Starobinsky gravity, the condition $|mr| \ll 1$ is not satisfied inside the Solar System and we cannot use the present analysis to constraint the NMC model given by Eq. (106).

X. CONCLUSIONS AND OUTLOOK

We have analyzed the constraints that the NMC Eq. (71) should fulfill in order to be consistent with the regimes considered in this work. This is summarized as follows:

(i) Long-range regime $|m|r \ll 1$ within the Solar System, leading to $n \gg 10^{-25}$;

(ii) Newtonian approximation, leading to $n \ll 10^{-33}$;

(iii) Perturbative regime $|R_1| \ll R_0$, only viable if $z(t)w(x) \sim 1$ [see Eq. (103)], thus leading to an unphysical fine-tuning of the density profile inside the spherical body.

The lack of validity of the perturbative regime leads us to conclude that the mechanism proposed in Ref. [12] cannot be constrained or excluded by the method developed in the present paper.

This result, however, is not specific to the Sun or similar objects, but is characteristic of any relevant spherical body of astrophysical scale for which the weak field approximation can be used.

Nevertheless, this study provides a relevant set of tools with which to assess the local impact of proposals for a perturbative power-law NMC driving the accelerated expansion of the Universe. Notice that the procedure can also be applied for a NMC that does not follow a power-law form, as long as its temporal variation (and of its derivatives) is of the order of H^2.

Of course, in what concerns the cosmological context, a new set of issues associated with the treatment of cosmological perturbations must be considered in order to address the impact of the NMC (see Ref. [9]).

ACKNOWLEDGMENTS

The work of O. B. and J. P. is partially supported by FCT (Fundação para a Ciência e a Tecnologia, Portugal) under the project PTDC/FIS/111362/2009. The work of R. M. is partially supported by INFN (Istituto Nazionale di Fisica Nucleare, Italy), as part of the MoonLIGHT-2 experiment in the framework of the research activities of the Commissione Scientifica Nazionale n. 2 (CSN2). The authors wish to thank the referee for her/his valuable comments and suggestions.

APPENDIX A

We compute here the solution of Eqs. (52). We set

$$R_1(r,t) = A(t)\frac{M_S}{r} \qquad r \geq R_S,$$
$$A(t) = \frac{1 + f_0^2 - 2f_{R0}^2 R_0 - 6\Box f_{R0}^2}{12\pi(f_{RR0}^1 - 2f_{RR0}^2\rho^{\cos})}. \qquad (A1)$$

Using the divergence theorem, for $r \geq R_S$ we have

$$2r^2\frac{d\Psi_1}{dr} = \frac{1}{2}A(t)M_S(R_S^2 - r^2) - \int_0^{R_S} R_1(r,t)r^2 dr. \qquad (A2)$$

From the definition of function U and the generalized mean value theorem for integrals we have

$$\int_0^{R_S} R_1(r,t)r^2 dr = \frac{1}{f_{RR0}^1 - 2f_{RR0}^2(\rho^{\cos} + \rho^s(\xi))}\int_0^{R_S} U(r,t)r^2 dr, \qquad (A3)$$

for $\xi \in (0, R_S)$.

Using Eq. (32) and the divergence theorem, for $r \leq R_S$ we have

$$\frac{dU}{dr} = \frac{\eta(t)}{4\pi}\frac{M(r)}{r^2} + 2f_{R0}^2\frac{d\rho^s}{dr}, \qquad M(r) = \int_{B_r}\rho^s(x)d^3x, \qquad (A4)$$

where B_r is the ball of radius r centered at the center of the star. Imposing the condition $\lim_{r\to 0} U(t,r)r^3 = 0$, repeated integration by parts yields

$$\int_0^{R_S} U(r,t)r^2 dr = \frac{1}{3}U(t,R_S)R_S^3 - \frac{\eta(t)}{12\pi}\int_0^{R_S} M(r)r dr + \frac{1}{2\pi}f_{R0}^2 M_S. \qquad (A5)$$

Substituting the previous results into Eq. (A2) yields, for $r \geq R_S$:

$$\Psi_1(r,t) = -\frac{1}{r}\frac{1}{f^1_{RR0} - 2f^2_{RR0}(\rho^{\cos} + \rho^s(\xi))} \times\left(\frac{\eta(t)}{24\pi}M_S R_S^2 + \frac{\eta(t)}{24\pi}\int_0^{R_S} M(r)rdr - \frac{1}{4\pi}f^2_{R0}M_S\right) - \frac{1}{4}A(t)M_S\left(\frac{R_S^2}{r}+r\right) + C_1, \tag{A6}$$

where C_1 is an integration constant. Now we estimate the various contributions to $\Psi_1(r,t)$. We have

$$I = \frac{|\eta(t)|}{24\pi r}\frac{1}{|f^1_{RR0} - 2f^2_{RR0}(\rho^{\cos} + \rho^s(\xi))|}\int_0^{R_S} M(r)rdr \leq \frac{|\eta(t)|}{48\pi}\frac{1}{|f^1_{RR0} - 2f^2_{RR0}(\rho^{\cos} + \rho^s(\xi))|}\frac{M_S R_S^2}{r}, \tag{A7}$$

from which, using $r \geq R_S$, follows

$$I \leq \frac{r^2}{8}\left|\frac{f^1_{R0} - 2f^2_{R0}\rho^{\cos}}{f^1_{RR0} - 2f^2_{RR0}(\rho^{\cos} + \rho^s(\xi))}\right|\left|\frac{\eta(t)}{\omega(t)}\right||\Psi_0 - C_0|, \tag{A8}$$

where $\omega(t)$ has been defined in Eq. (58). Using Eq. (66), we have

$$|2f^2_{R0}|(\rho^{\cos}(t) + \rho^s(r)) \ll |f^1_{R0}|, \quad r \leq R_S. \tag{A9}$$

Thus, the following approximation can be used:

$$|f^1_{R0} - 2f^2_{R0}\rho^{\cos}(t)| \simeq |f^1_{R0} - 2f^2_{R0}(\rho^{\cos}(t) + \rho^s(\xi))|, \tag{A10}$$

from which, using conditions Eqs. (54) and (61), it follows that

$$I \lesssim \frac{r^2}{8}\left|\frac{f^1_{R0} + 2f^2_{R0}\mathcal{L}_m(\xi,t)}{f^1_{RR0} + 2f^2_{RR0}\mathcal{L}_m(\xi,t)}\right|\left|\frac{\eta(t)}{\omega(t)}\right||\Psi_0 - C_0| \ll |\Psi_0(r,t) - C_0|, \tag{A11}$$

where we have used $|\eta(t)\omega(t)| \lesssim 1$ (as shown in Appendix B). Analogously, we have

$$II = \frac{1}{24\pi}\frac{|\eta(t)|}{|f^1_{RR0} - 2f^2_{RR0}(\rho^{\cos} + \rho^s(\xi))|}\frac{M_S R_S^2}{r} \ll |\Psi_0(r,t) - C_0|. \tag{A12}$$

By the same token, we get

$$III = |A(t)|\frac{M_S R_S^2}{4r} \simeq \frac{1}{16\pi}\frac{|\eta(t)|}{|f^1_{RR0} - 2f^2_{RR0}\rho^{\cos}|}\frac{M_S R_S^2}{r} \ll |\Psi_0(r,t) - C_0|. \tag{A13}$$

We can then estimate

$$IV = \frac{1}{4}|A(t)|M_S r \simeq \frac{3}{8}r^2\left|\frac{f^1_{R0} - 2f^2_{R0}\rho^{\cos}}{f^1_{RR0} - 2f^2_{RR0}\rho^{\cos}}\right|\left|\frac{\eta(t)}{\omega(t)}\right||\Psi_0(r,t) - C_0|, \tag{A14}$$

so that the estimate $IV \ll |\Psi_0(r,t) - C_0|$ follows in the same way as for the term I.

It remains to consider the term

$$V = \frac{1}{4\pi}\left|\frac{f^2_{R0}}{f^1_{RR0} - 2f^2_{RR0}(\rho^{\cos} + \rho^s(\xi))}\right|\frac{M_S}{r}. \tag{A15}$$

Using Eq. (56) and integrating over the volume of the star, we have

$$|f^2_{R0}|M_S \ll \frac{4\pi}{3}|f^1_{R0} - 2f^2_{R0}\rho^{\cos}(t)|R_S^3, \tag{A16}$$

from which, using Eq. (A9) and condition (62), we have, for $r \geq R_S$:

$$V \ll \frac{1}{3}\left|\frac{f^1_{R0} - 2f^2_{R0}(\rho^{\cos} + \rho^s(\xi))}{f^1_{RR0} - 2f^2_{RR0}(\rho^{\cos} + \rho^s(\xi))}\right|r^2 = \frac{1}{3}\left|\frac{f^1_{R0} + 2f^2_{R0}\mathcal{L}_m(\xi,t)}{f^1_{RR0} + 2f^2_{RR0}\mathcal{L}_m(\xi,t)}\right|r^2 \lesssim \frac{1}{3}R_0 r^2\frac{R_S}{GM_S} \sim H^2 r^2\frac{R_S}{GM_S} \sim 10^{-19}. \tag{A17}$$

Since the quantity $\Psi_0(r,t) - C_0$ turns out to be the Newtonian potential [see Eqs. (57) and (58)], we have $|V| \ll |\Psi_0(r,t) - C_0|$ for r of order of Solar System scales. Eventually we have

$$|\Psi_1(r,t) - C_1| \leq I + II + III + IV + V, \tag{A18}$$

and, collecting the above estimates, we find that

$$|\Psi_1(r,t) - C_1| \ll |\Psi_0(r,t) - C_0|. \tag{A19}$$

APPENDIX B

In this appendix, some aspects of the results presented in Ref. [12] are discussed, namely the relation between the cosmological matter density $\rho^{\cos}(t)$, the NMC $f^2(R)$ and the characteristic time scale t_n. Indeed, from Eqs. (74)–(78), we first derive

$$f^2_0\rho^{\cos} = \frac{6\kappa H^2}{1+n} = \frac{2\kappa R_0}{1+4n}, \tag{B1}$$

so that, using the definition of critical density $\rho_c = 6\kappa H^2$ and the relative matter density $\Omega_m \equiv \rho^{\cos}/\rho_c = 31.7\%$ [25], we obtain

$$f^2_0 = \frac{1}{(1+n)\Omega_m} < \frac{1}{\Omega_m} \approx 3.1, \tag{B2}$$

for a positive exponent n. Clearly, Eq. (88) does not obey the above result or, equivalently, is incompatible with the present value of Ω_m [conversely, Eq. (89) is obtained directly from the above, with $n = 4, 10$].

Given the above, Eqs. (82) and (83) show that

$$n=4:\ \eta=6.24, \quad \omega=11.80 \rightarrow \left|\frac{\eta}{\omega}\right| = 0.53,$$
$$n=10:\ \eta=6.68, \quad \omega=11.95 \rightarrow \left|\frac{\eta}{\omega}\right| = 0.56, \tag{B3}$$

instead of the values $\eta(t) = -1/3$, $\omega(t) = 1$ and $|\eta(t)/\omega(t)| = 1/3$ one would obtain if the contribution from the NMC was negligible [which, however, should not be directly interpreted as signaling the return to GR, since Eq. (35) is ill defined if $f^1(R) = R$ and $f^2(R) = 0$].

Notice that, as n grows large,

$$\eta(t) \sim -\frac{1}{3} + \frac{7}{3\Omega_m} \approx 7.03,$$
$$\omega(t) \sim 1 + \frac{7}{2\Omega_m} \approx 12.04 \rightarrow \left|\frac{\eta(t)}{\omega(t)}\right| \sim \frac{2}{3}\frac{7-\Omega_m}{7+2\Omega_m} = 0.583, \tag{B4}$$

so that these two quantities (which grow with the exponent n) vary within the limited ranges $-1/3 < \eta(t) < 7.03$ and $1 < \omega(t) < 12.04$, and thus do not lead to any pathological divergences in the results of this work. Furthermore, the condition Eq. (61) is upheld, as assumed in this work. These results do not change qualitatively for other values of the exponent n.

A relevant consequence of the above discussion is that the characteristic time scale t_n cannot be freely specified, but must be constrained by Eq. (B2). Thus, from Eq. (76), one is led to

$$t_n = t_0\sqrt{\frac{3}{4}\frac{[(1+n)\Omega_m]^{1/n}}{(1+n)(1+4n)}}, \tag{B5}$$

which yields the values $t_4 = 0.10t_0$ and $t_{10} = 0.043t_0$ used in Eq. (89). Notwithstanding this correction to the numerical values considered in Ref. [12], the results presented here remain valid, as they make full use of the analytical relations invoked throughout this work, instead of any specific numerical value.

[1] O. Bertolami and J. Páramos, *The Experimental Status of Special and General Relativity*, Handbook of Spacetime (Springer, Berlin, 2013); arXiv:1212.2177.
[2] A. De Felice and S. Tsujikawa, Living Rev. Relativity **13**, 3 (2010).
[3] O. Bertolami, C. G. Böhmer, T. Harko, and F. S. N. Lobo, Phys. Rev. D **75**, 104016 (2007).
[4] O. Bertolami and J. Páramos, Phys. Rev. D **77**, 084018 (2008).
[5] O. Bertolami and M. C. Sequeira, Phys. Rev. D **79**, 104010 (2009).
[6] O. Bertolami and J. Páramos, Classical Quantum Gravity **25**, 245017 (2008).
[7] O. Bertolami and J. Páramos, J. Cosmol. Astropart. Phys. 03 (2010) 009.
[8] O. Bertolami, P. Frazão, and J. Páramos, Phys. Rev. D **86**, 044034 (2012).
[9] O. Bertolami, P. Frazão, and J. Páramos, J. Cosmol. Astropart. Phys. 05 (2013) 029.
[10] O. Bertolami and J. Páramos, Phys. Rev. D **84**, 064022 (2011).
[11] O. Bertolami, P. Frazão, and J. Páramos, Phys. Rev. D **83**, 044010 (2011).
[12] O. Bertolami, P. Frazão, and J. Páramos, Phys. Rev. D **81**, 104046 (2010).
[13] O. Bertolami, F. S. N. Lobo, and J. Páramos, Phys. Rev. D **78**, 064036 (2008).
[14] O. Bertolami and J. Páramos, arXiv:1306.1177.
[15] J. Páramos and C. Bastos, Phys. Rev. D **86**, 103007 (2012).
[16] O. Bertolami and A. Martins, Phys. Rev. D **85**, 024012 (2012).
[17] O. Bertolami and R. Z. Ferreira, Phys. Rev. D **85**, 104050 (2012).
[18] T. Chiba, T. L. Smith, and A. L. Erickcek, Phys. Rev. D **75**, 124014 (2007).
[19] S. M. Carroll, V. Duvvuri, M. Trodden, and M. S. Turner, Phys. Rev. D **70**, 043528 (2004).
[20] K. Henttunen, T. Multamäki, and I. Vilja, Phys. Rev. D **77**, 024040 (2008).
[21] T. Chiba, Prog. Theor. Phys. **126**, 993 (2011).
[22] C. L. Bennett *et al.*, arXiv:1212.5225 [Astrophys. J. Suppl. Ser. (to be published)].
[23] http://spacemath.gsfc.nasa.gov.
[24] A. A. Starobinsky, Phys. Lett. B **91**, 99 (1980).
[25] P. A. R. Ade *et al.* (Planck Collaboration), arXiv:1303.5076.