

A proposal to tag ISR events at BES III

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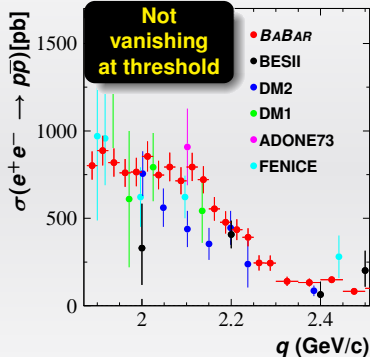
Motivations



$$\sigma(e^+e^- \rightarrow p\bar{p}) = \frac{4\pi\alpha^2\beta_p C}{3q^2} \left[|G_M|^2 + \frac{2M_p^2}{q^2} |G_E|^2 \right]$$

Coulomb factor

$$C_{\beta_p \rightarrow 0} \sim \frac{\pi\alpha}{\beta_p}$$



At the threshold

$$\sigma(e^+e^- \rightarrow p\bar{p})(4M_p^2) = \frac{\pi^2\alpha^3}{2M_p^2} \frac{\beta_p}{\beta_p} |G^p(4M_p^2)|^2$$

$$\sigma(e^+e^- \rightarrow p\bar{p})(4M_p^2) = 850 |G^p(4M_p^2)|^2 \text{ pb}$$

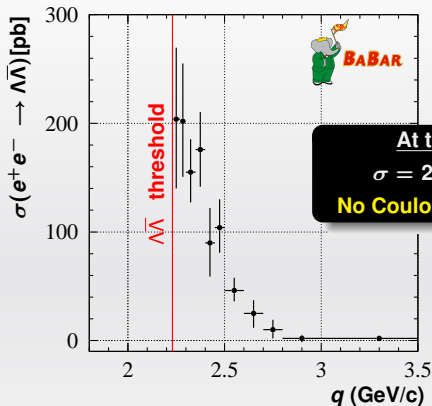


$|G^p(4M_p^2)| \equiv 1$
as pointlike fermion pairs!

Using the ISR technique with only few fb^{-1} of integrated luminosity *BESIII* can easily achieve the *BABAR* statistics

$$e^+e^- \rightarrow \Lambda\bar{\Lambda}$$

PRD76, 092006 (BABAR)

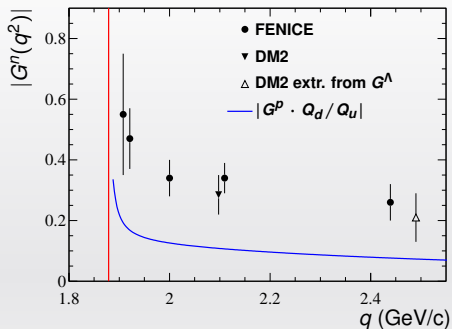


Only at the J/ψ mass BESIII can increase the BABAR statistics at least by a factor of two because of a better Λ reconstruction resolution (only one Λ reconstructed)

Λ polarization for free $\Rightarrow G_E^\Lambda - G_M^\Lambda$ relative phase



$$e^+ e^- \rightarrow n\bar{n}$$



- Measured only once by FENICE at ADONE
- $\int \mathcal{L} = 500 \text{ nb}^{-1}$ (15' at BESIII)
- ~ 100 candidates $n\bar{n}$ events!
- $\sigma(n\bar{n}) > \sigma(p\bar{p})$?
- **Not zero at threshold?**

BESIII has the unique possibility to measure this cross section

No other experiments at present and in near future will be able to perform such a measurement

Why Initial State Radiation



- Existing results, obtained by **BABAR** (ISR), show interesting and unexpected behaviors, mainly at thresholds, for

$$e^+e^- \rightarrow p\bar{p}$$

and

$$e^+e^- \rightarrow \Lambda\bar{\Lambda}$$

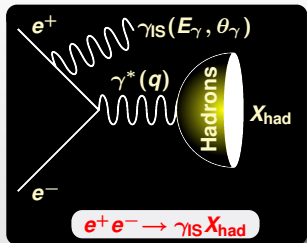
- Only one measurement (**FENICE** with energy scan) for

$$e^+e^- \rightarrow n\bar{n}$$

There are physical limits in reaching the threshold of many of these channels via energy scan (stable hadrons produced at rest can not be detected)

The Initial State Radiation technique provides a unique tool to access threshold regions working at higher resonances

Initial State Radiation



- $\frac{d^2\sigma}{dE_\gamma d\theta_\gamma} = W(E_\gamma, \theta_\gamma) \cdot \sigma_{e^+e^- \rightarrow X_{had}}(s)$

- $W(E_\gamma, \theta_\gamma) = \frac{\alpha}{\pi x} \left(\frac{2 - 2x + x^2}{\sin^2 \theta_\gamma} \right)$

- $s = q^2$, q X_{had} momentum
- E_γ, θ_γ .. CM γ_{IS} energy, scatt. ang.
- E_{CM} CM e^+e^- energy
- $x = E_\gamma/2E_{CM}$

Advantages

- All energies (q^2) at the same time
 \downarrow
 Better control on systematics
 (e.g. greatly reduced point to point)
- Detected ISR \Rightarrow full X_{had} angular coverage
- CM boost \Rightarrow $\left\{ \begin{array}{l} \text{at threshold } \epsilon \neq 0 \\ \text{energy resolution } \sim 1 \text{ MeV} \end{array} \right.$

Why ISR at zero degrees



Proposal for a zero-degree detector

- J/ψ , $\psi(2S)$, $\psi(3770)$ resonances decay with high BR's to final states with π^0 and γ_{FS} (final state)
- At BESIII these decay channels represent severe backgrounds for typical ISR final states with γ_{IS} detected at wide angle

- π^0 and final γ angular distributions are isotropic
- ISR angular distribution is peaked at small angles

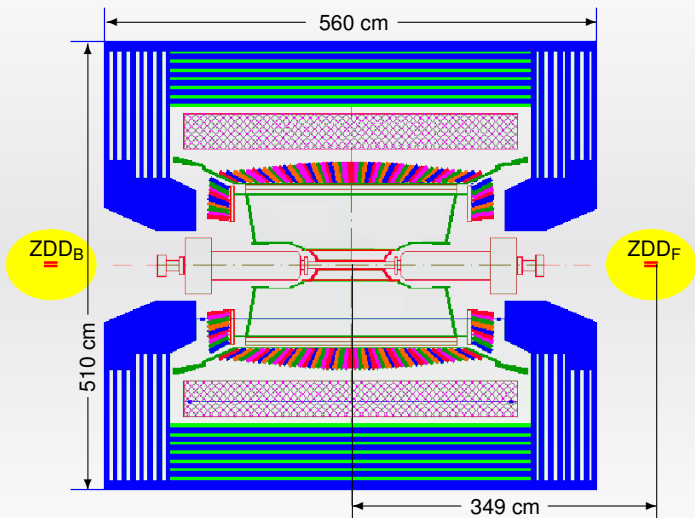


A zero-degree radiative photon tagger will suppress most of these backgrounds

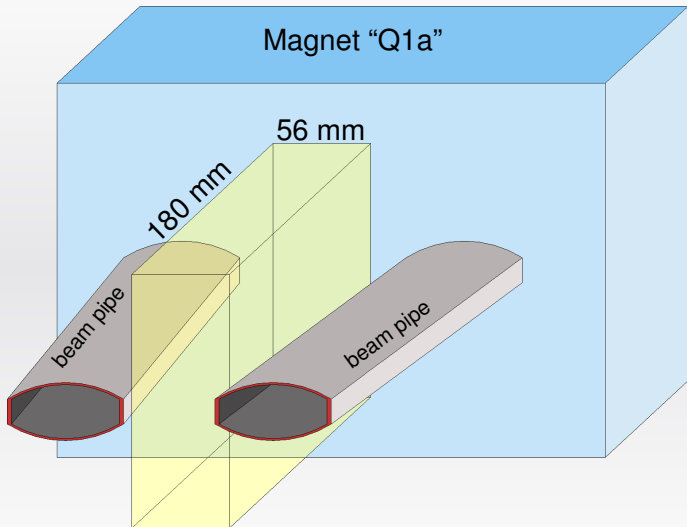
We propose to upgrade (July 2011?) the present luminosity monitor with a new zero-degree detector (ZDD), with a better energy resolution, to tag ISR photons as well as to measure the luminosity

Design and Installation
two options:
LYSO and Pb-Scint

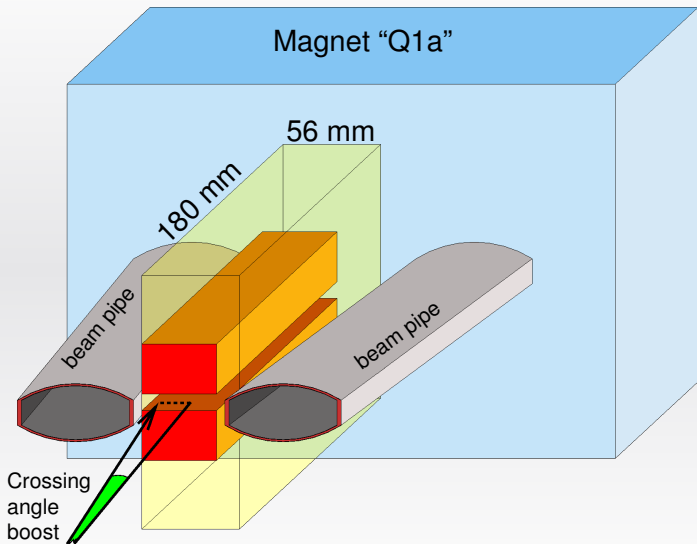
BESIII and ZDD



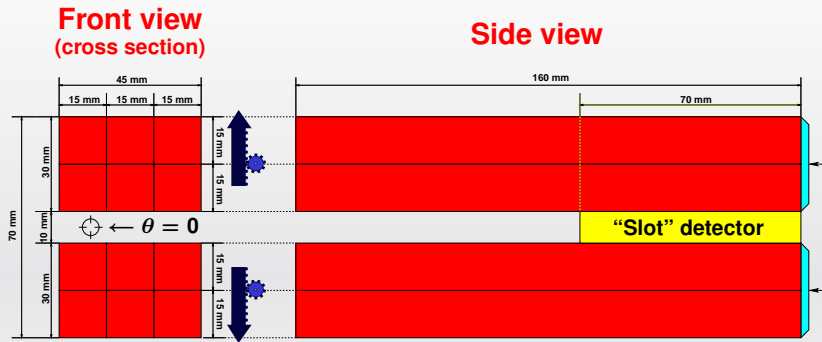
Available space



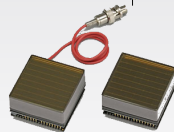
Available space



LYSO design

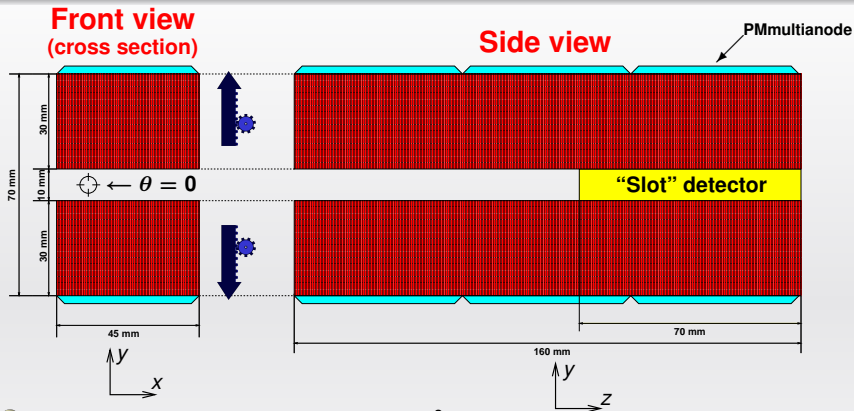


- Two 3×2 matrices of $1.5 \times 1.5 \times 16 \text{ cm}^3$ of LYSO bars
- Mechanical vertical motion to vary the opening
- Total volume 432 cm^3
- Readout with 2 PMmultianode
- Possible Luminosity-monitor "Slot" detector in the last 7 cm
Identical structure but reduced size w.r.t. the actual one

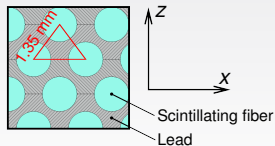


PMmultianode

Pb-Scintillating fibers design à la Kloe



- Kloe module prototype: $400 \times 60 \times 23 \text{ cm}^3$ available for cutting
- Two bricks of $4.5 \times 3 \times 16 \text{ cm}^3$ of Pb-Scint with fibers along the y axis
- Mechanical vertical motion to vary the opening
- Readout with 6 PMmultianode
- Possible Luminosity-monitor "Slot" detector



Physical properties of materials

Material	LYSO	Pb-Scint (Kloe prototype)
Density (g/cm^3)	7.4	5.3
Radiation Length (cm)	1.1	1.6
Molière Radius (cm)	1.9	2.9
Decay Constant (ns)	40-44	2.4
Peak Emission (nm)	428	460
Radiation Hardness (rad)	$\sim 10^8$	$\sim 10^6$

- $\sigma_{\text{Bre}}(\text{ZDD}/4) = 2.6 \text{ mb}$
- $T = 1.5 \times 10^7 \text{ s}$
- $\bar{\mathcal{L}} = 1.5 \times 10^{32} \text{ cm}^{-2}\text{s}^{-1}$
- $E_{c.m.} = 3.77 \text{ GeV}$

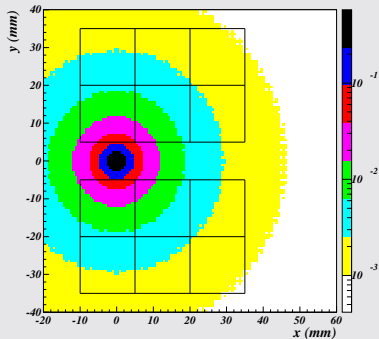


$$\frac{\text{dose absorbed}}{\text{year}} = \begin{cases} 0.4 \frac{\text{Mrad}}{\text{year}} & \text{LYSO} \\ 1 \frac{\text{Mrad}}{\text{year}} & \text{Scint} \end{cases}$$

Energy Resolution

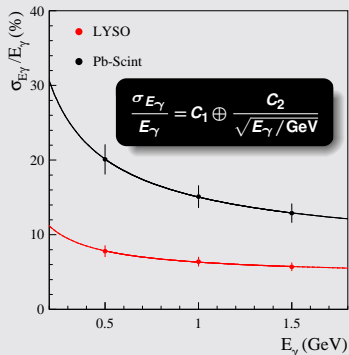
Energy resolution, the ISR case

ISR angular distribution on ZDD



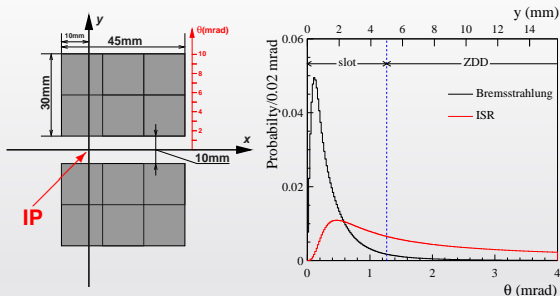
	LYSO	Pb-Scint
E_γ (GeV)	$\sigma_{E_\gamma}/E_\gamma$	$\sigma_{E_\gamma}/E_\gamma$
1.5	5.7%	12.9 %
1.0	6.4%	15.1 %
0.5	7.8%	20.1 %

Energy resolution for ISR



	LYSO	Pb-Scint
C_1	4.3%	6.9 %
C_2	4.6%	13.4 %

$e^+e^- \rightarrow e^+e^-\gamma$ Bremsstrahlung simulation



- $E_{\text{beam}} = 1.89 \text{ GeV}$
- $E_{\gamma}^{\text{min}} = 50 \text{ MeV}$
- $\sigma_{\text{Bre}}(4\pi) = 353 \text{ mb}$
- $\sigma_{\text{Bre}}(\text{ZDD}) = 10 \text{ mb}$
- $\mathcal{L} = 8 \times 10^{32} \text{ cm}^{-2} \text{ s}^{-1}$

- ISR in ZDD **13.7%** of total solid angle
- Bremsstrahlung in ZDD **2.8%** of total solid angle
- Bremsstrahlung rate in a single ZDD element (upper or lower):

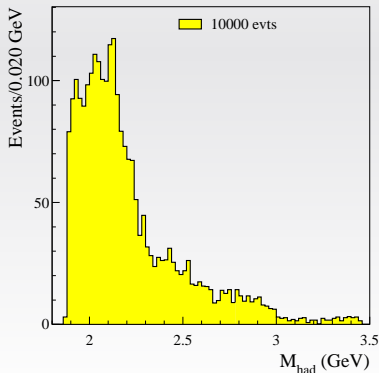
800 kHz at $\mathcal{L} = 3 \times 10^{32} \text{ cm}^{-2} \text{ s}^{-1}$

2.1 MHz at $\mathcal{L} = 8 \times 10^{32} \text{ cm}^{-2} \text{ s}^{-1}$

$n\bar{n}$ physics case

The $n\bar{n}\gamma_{\text{IS}}$ physics case

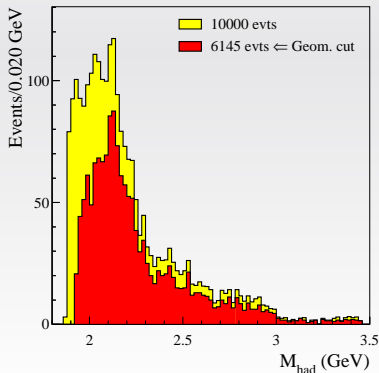
- $e^+e^- \rightarrow n\bar{n}\gamma_{\text{IS}}$ at a center of mass energy: $E_{c.m.} = 3.77 \text{ GeV}$
- IS photon energy range: $50 \text{ MeV} \leq E_{\gamma_{\text{IS}}} \leq (E_{c.m.}/2) \left(1 - 4M_n^2/E_{c.m.}^2\right)$
- Beam pipe suppresses **sinc. rad. bkg.** and γ_{IS} with $E_{\gamma_{\text{IS}}} < 50 \text{ MeV}$
- γ_{IS} in ZDD and only antineutron detected in BESIII
- \bar{n} annihilates in the scintillator with **probability** $\sim 100\%$
- \bar{n} annihilation star detected in TOF ($\Delta t_{\text{TOF}} = 150 \text{ ps}$)



- 10000 events with $\gamma_{\text{IS}} \rightarrow$ ZDD
- $M_{\text{had}} = E_{c.m.} \sqrt{1 - 2E_{\gamma_{\text{IS}}}/E_{c.m.}}$
- Geometrical cut:
 - $\bar{n} \rightarrow$ BESIII
 - No constraint in n

The $n\bar{n}\gamma_{IS}$ physics case

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$n\bar{n}\pi^0$ and $n\bar{n}\gamma_{FS}$ backgrounds

- $e^+e^- \rightarrow n\bar{n}\pi^0$ and $e^+e^- \rightarrow n\bar{n}\gamma_{FS}$ (with a final state photon) are important backgrounds for $e^+e^- \rightarrow n\bar{n}\gamma_{IS}$

- Assuming $\sigma(e^+e^- \rightarrow n\bar{n}\pi^0) \simeq \sigma(e^+e^- \rightarrow p\bar{p}\pi^0)$:

$$\frac{\text{Ev}(n\bar{n}\pi^0)}{\text{Ev}(n\bar{n}\gamma_{IS})} [M_{\Upsilon(4S)}] \simeq R_{BABAR} = \frac{\text{Ev}(p\bar{p}\pi^0)}{\text{Ev}(p\bar{p}\gamma_{IS})} [M_{\Upsilon(4S)}] = 0.06$$

- In BESIII, directly at the $\psi(3770)$ mass:

$$R_{\text{BESIII}} = 0.06 \times \underbrace{\left(\frac{0.012}{3 \times 10^{-6}} \right)}_{p\bar{p}\pi^0 \text{ cross section ratio}} \times \underbrace{\left(\frac{1}{10.7} \right)}_{\text{Lum. ratio}} = 22.4$$

$\gamma_{IS} \rightarrow \text{ZDD}$

ZDD solid angle
BESIII solid angle

$$\frac{2 \cdot (2.4 \cdot 5.3 / 349^2)}{4\pi \cos \theta_{\min}} = 3.8 \cdot 10^{-5}$$



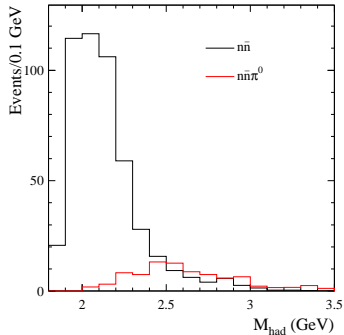
$$\frac{\text{Ev}(n\bar{n}\pi^0, \pi^0 \rightarrow 0^0)}{\text{Ev}(n\bar{n}\gamma_{IS}, \gamma_{IS} \rightarrow 0^0)} = 0.0008$$

$$\frac{\text{Ev}(n\bar{n}\gamma_{FS}, \gamma_{FS} \rightarrow 0^0)}{\text{Ev}(n\bar{n}\gamma_{IS}, \gamma_{IS} \rightarrow 0^0)} \sim 0.0001$$

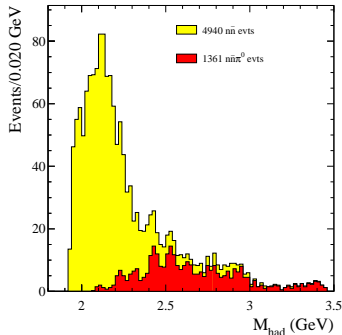
The $n\bar{n}\pi^0\gamma_{\text{IS}}$ background

$e^+e^- \rightarrow n\bar{n}\pi^0\gamma_{\text{IS}}$
is a severe background
having the IS photon

Yields from *BABAR*

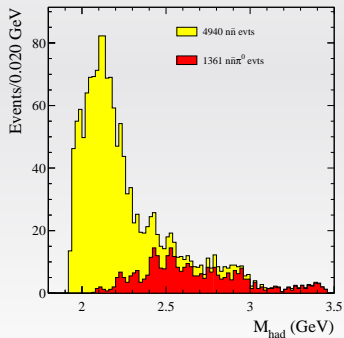


- After geometrical cut
- High contamination at high M_{had}



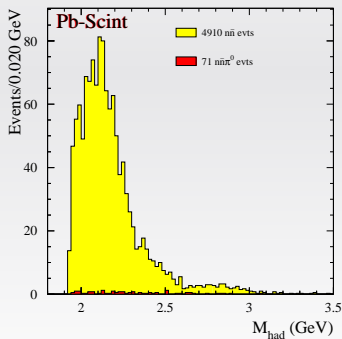
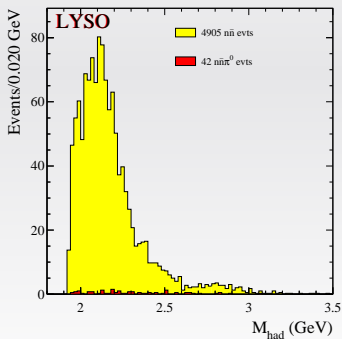
The $n\bar{n}\pi^0\gamma_{\text{IS}}$ background reduction

- **Geometrical cut on \bar{n}**
- **π^0 detection in BESIII:** at least one of the π^0 photons with $E_\gamma > 50$ MeV in BESIII not in a 200 mrad cone around \bar{n} direction
- **Kinematic fit:** $\chi^2 \leq 10$



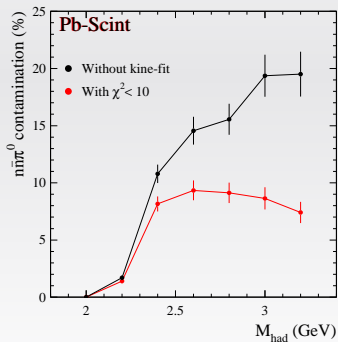
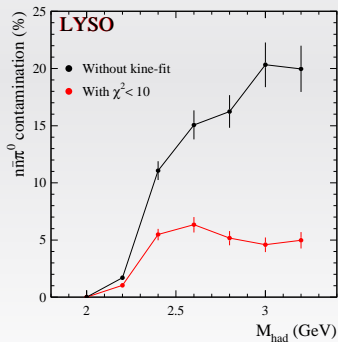
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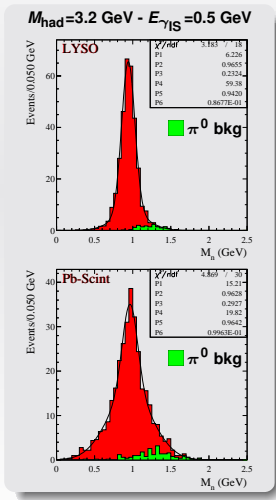
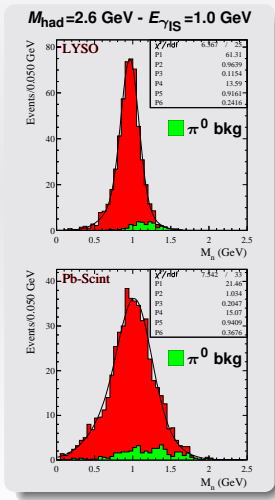
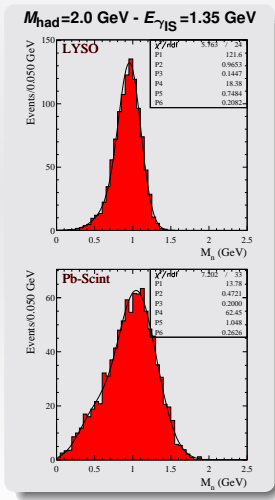
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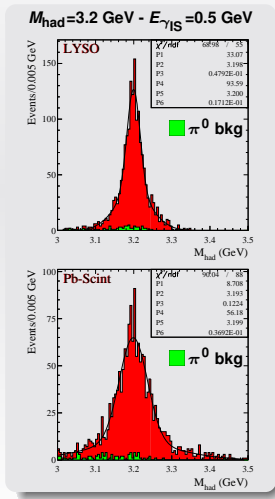
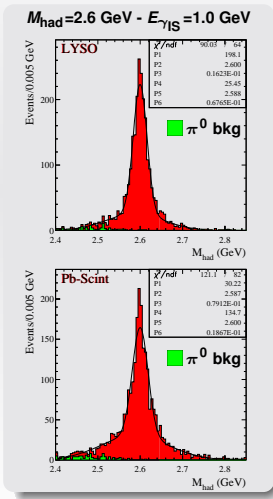
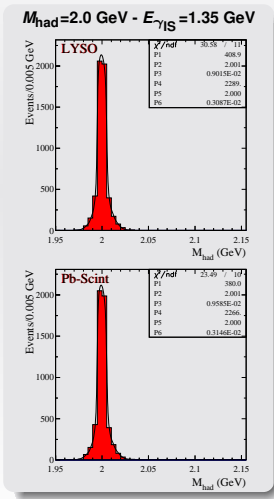
Energy resolution in $\bar{n}_{\gamma_{IS}}$ missing mass

- Events are generated with fixed value of $M_{had} = E_{c.m.} \sqrt{1 - 2E_{\gamma_{IS}}/E_{c.m.}}$
- The $\bar{n}_{\gamma_{IS}}$ missing mass is obtained only from experimental data



Energy resolution in M_{had} slices

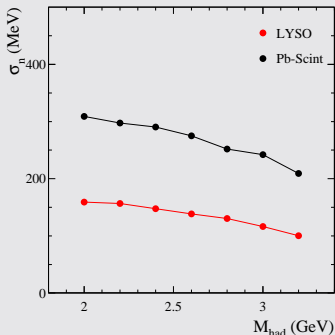
- Events are generated with fixed value of $M_{\text{had}} = E_{c.m.} \sqrt{1 - 2E_{\gamma_{\text{IS}}}/E_{c.m.}}$
- M_{had} is reconstructed using the kinematic fit procedure



Energy resolutions in summary

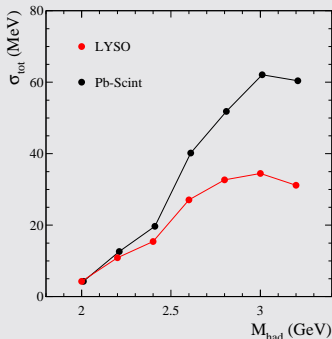
- Two-gaussian fit: $\sigma =$ half width of the area, symmetric w.r.t. the center of mass of the distributions, which contains the 68% of events

Width of the $\bar{n}\gamma_{1S}$ miss. mass



σ_n is dominated by $\delta p_{\bar{n}}$
small $M_{had} \Rightarrow$ large $E_{\gamma_{1S}} \Rightarrow$ small $\delta E_{\gamma_{1S}}$

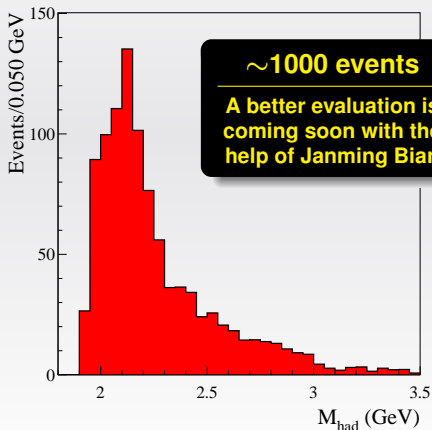
Energy resolution in M_{had} bins



σ_{tot} is dominated by $\delta E_{\gamma_{1S}}$
large $M_{had} \Rightarrow$ small $E_{\gamma_{1S}} \Rightarrow$ large $\delta E_{\gamma_{1S}}$

Expected events

- One year of data taking: $T = 1.5 \times 10^7 \text{ s}$
- Average luminosity: $\overline{\mathcal{L}} = 1.5 \times 10^{32} \text{ cm}^{-2}\text{s}^{-1}$
- Detection efficiency: $\epsilon \sim 0.5$
- Center of mass energy: $E_{c.m.} = 3.77 \text{ GeV}$



Other possible physics items

R_{had} in the 1-3 GeV region

- Accessible had-CoM energy: $M_{\text{had}} = \sqrt{E_{\text{coll}}^2 - 2E_{\text{coll}}E_{\gamma\text{IS}}}$

- PDG: $\gamma\gamma 2$ and BESII (2-3 GeV) only

- ISR: small systematic error versus M_{had}

- ISR on ZDD: negligible π^0 background



- $|\Delta M_{\text{had}}| = |\Delta E_{\gamma\text{IS}}| E_{\text{coll}} / M_{\text{had}}$:
feasible only if $E_{\text{coll}} / M_{\text{had}} \sim 1$ (not for B -factories)

- BESIII: $E_{\text{coll}} \sim 3.5 \text{ GeV} \Rightarrow M_{\text{had}} \simeq 1 - 3 \text{ GeV}$



- LYSO: $|\Delta M_{\text{had}}| \simeq 150 \text{ MeV}$

- Pb-Scint: $|\Delta M_{\text{had}}| \simeq 300 \text{ MeV}$

- $|\Delta M_{\text{had}}|$ reduced by deconvolution techniques



To sum up

We have presented the two-options proposal to INFN committee for HEP, chaired by F. Ferroni:

- approved proposal with priority in the low cost Kloe option
- cut and test the Kloe-prototype funded
- in July we can ask for readout electronics budget

- We propose to install it in **summer 2011 shutdown**
- In the following years the crystals (LYSO?) option can be put ahead and eventually be installed during a following shutdown

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感谢您的关注！

Additional slides

Radiation hardness

- Radiation damages mostly due to Bremsstrahlung: $\sigma_{\text{Bre}}(\text{ZDD}/4) = 2.6 \text{ mb}$
- One year of data taking: $T = 1.5 \times 10^7 \text{ s}$
- Average luminosity: $\bar{\mathcal{L}} = 1.5 \times 10^{32} \text{ cm}^{-2}\text{s}^{-1}$
- Center of mass energy: $E_{c.m.} = 3.77 \text{ GeV}$

$$\frac{\text{Dose absorbed}}{\text{year}} = \frac{\text{energy deposited}}{\text{year} \cdot \text{mass}} = \left\{ \begin{array}{l} \frac{3 \times 10^{21} \text{ eV}}{0.12 \text{ kg}} = 4 \times 10^5 \frac{\text{rad}}{\text{year}} \quad \text{LYSO} \\ \frac{3 \times 10^{21} \cdot \frac{2}{13} \text{ eV}}{1.8 \times 10^{-2} \text{ kg}} = 10^6 \frac{\text{rad}}{\text{year}} \quad \text{Scint} \end{array} \right.$$

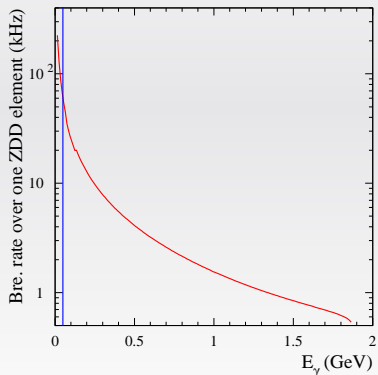
Declared hardness

- LYSO $\sim 10^8 \text{ rad}$
- Scint. $\sim 10^6 \text{ rad}$

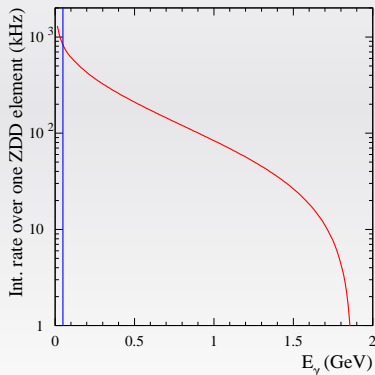
Bremsstrahlung rate

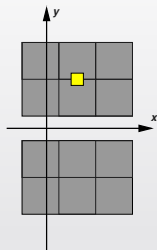
$$\mathcal{L} = 3 \times 10^{32} \text{ cm}^{-2} \text{ s}^{-1}$$

Bremsstrahlung rate
in 10 MeV E_γ intervals

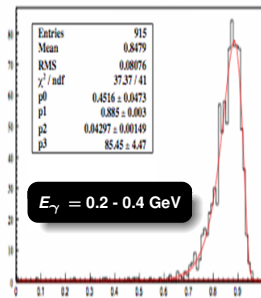
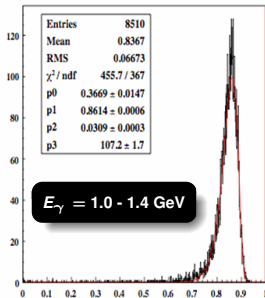


Integrated
Bremsstrahlung rate





Deposited energy/ E_γ

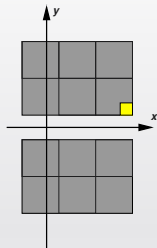


Log-normal distribution

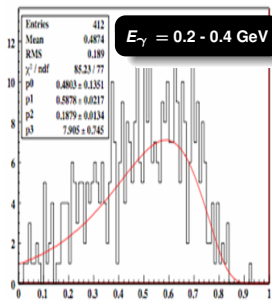
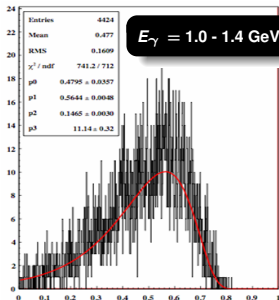
$$\frac{df}{dE} = \frac{\eta}{\sqrt{2\pi}\sigma_E\sigma_0} e^{-\frac{1}{2} \left[\frac{\ln^2 \left(1 - \frac{\eta(E-E_0)}{\sigma_E} \right)}{\sigma_0^2} + \sigma_0^2 \right]}$$

$$\sigma_0 = \frac{2}{2.35} \ln \left[\eta \frac{2.35}{2} + \sqrt{1 + \left(\eta \frac{2.35}{2} \right)^2} \right], \quad \sigma_E = \frac{\text{FWHM}}{2.35}$$

E_γ (GeV)	$\sigma_{E_\gamma} / E_\gamma$ Central (yellow square)
1.0 - 1.4	3.6%
0.2 - 0.4	4.9%



Deposited energy/ E_γ



E_γ (GeV)	$\sigma_{E_\gamma} / E_\gamma$ Central (yellow square)
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1.0 - 1.4	26.0%
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0.2 - 0.4	32.0%
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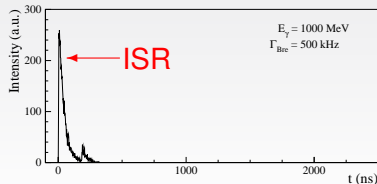
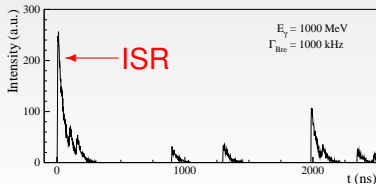
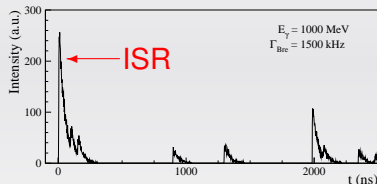
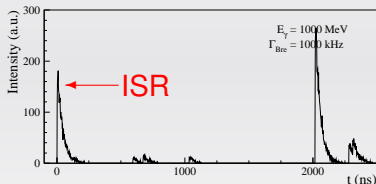
Pileup effect: signal generation

Maximum Bremsstrahlung rate expected 2.1 MHz (ZDD/4)

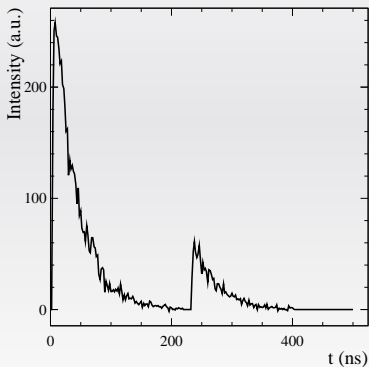
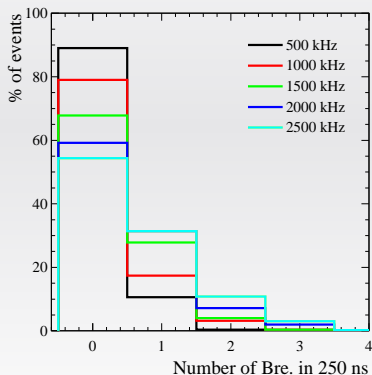
- Flash ADC: 500 MS/s, 8-bit resolution
- LYSO signal:

$$\text{Intensity} = e^{-t/\tau_r}(1 - e^{-t/\tau_d})$$

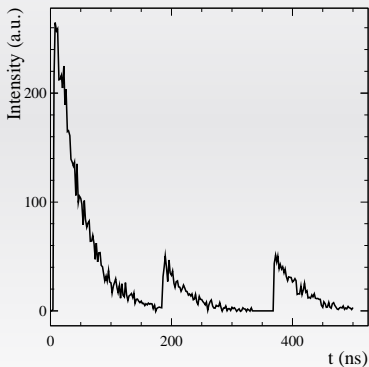
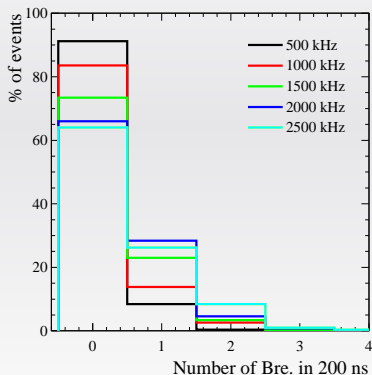
rising time $\tau_r = 2$ ns, decay time $\tau_d = 40$ ns



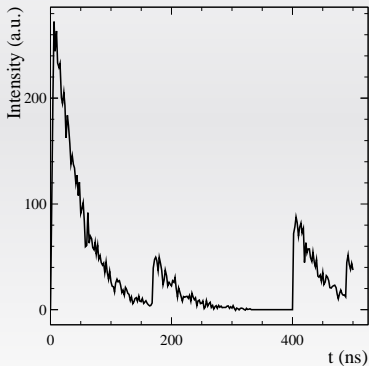
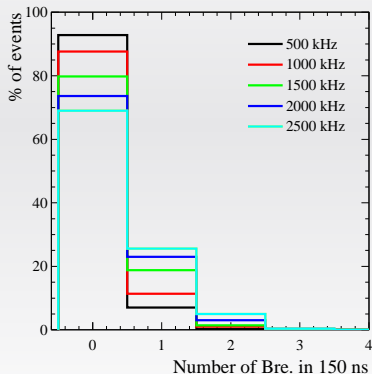
Probability of pileup as a function of the Bremsstrahlung rate



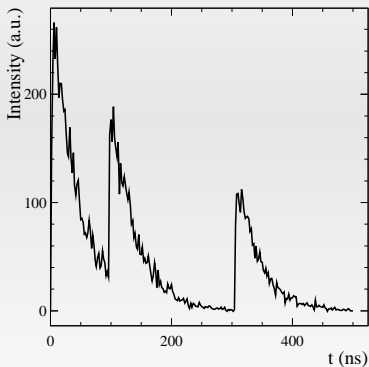
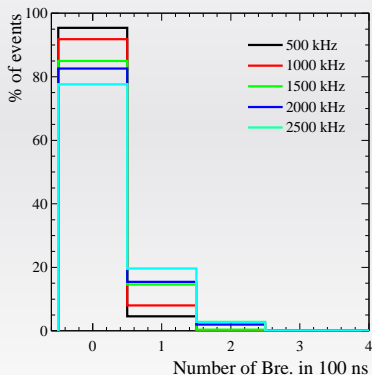
Probability of pileup as a function of the Bremsstrahlung rate



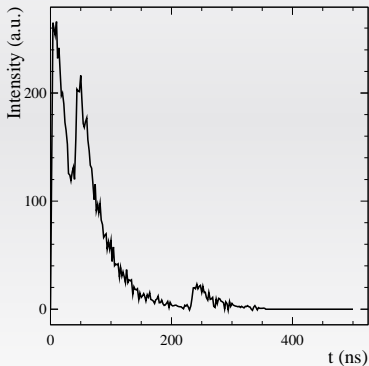
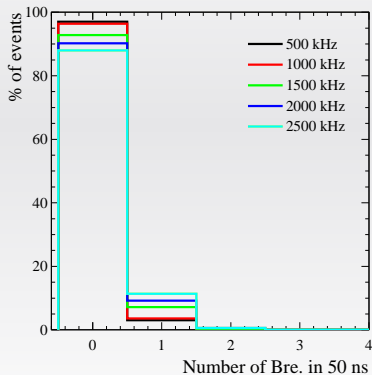
Probability of pileup as a function of the Bremsstrahlung rate



Probability of pileup as a function of the Bremsstrahlung rate



Probability of pileup as a function of the Bremsstrahlung rate

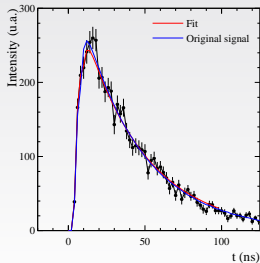
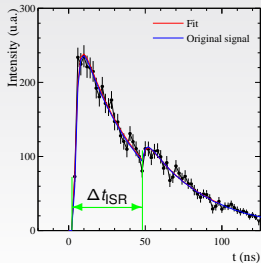
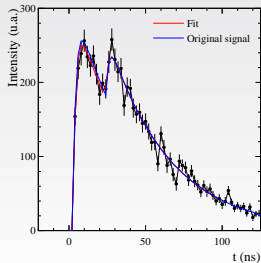


Pileup effect: evaluation

- Signals have been generated at various rates with

$$\text{Intensity} = E_{\gamma\text{IS}} \cdot e^{-t/\tau_{\text{decay}}} (1 - e^{-t/\tau_{\text{raise}}})$$

- E.g. at 2500 kHz:
31.6% has $\Delta t_{\text{ISR}} < 160 \text{ ns} \sim 4 \text{ decay times}$
- We fit these signals to verify our capability to distinguish ISR and Bremsstrahlung contributions



Pileup effect₂ in $T = 160$ ns

- The fit goodness is expressed as $(\sigma_E/E)_{\text{fit}} = (E_{\text{gen}} - E_{\text{fit}})/E_{\text{gen}}$, where E_{gen} is the generated ISR amplitude and E_{fit} is its fitted value
- We consider as a reference accuracies: $\left\{ \begin{array}{l} 7\% \sim \frac{\sigma_E}{E} \text{ LYSO} \\ 15\% \sim \frac{\sigma_E}{E} \text{ Pb-Scint} \end{array} \right.$
- $E_{\gamma_{\text{IS}}} \in [0.5 \text{ GeV}, 1.5 \text{ GeV}]$, mild dependence on $E_{\gamma_{\text{IS}}}$

rate (kHz)	Pileup in 160 ns (%)	$(\sigma_E/E)_{\text{fit}} > 7\%$ (%)	$(\sigma_E/E)_{\text{fit}} > 15\%$ (%)
2500	30	9.4	4.8
2100	26	8.1	4.2
1000	14	4.3	2.2
800	10	3.2	1.6

The $n\bar{n}_{\gamma_{IS}}$ physics case: kinematic fit

Inputs (6)

- \bar{n} 3-momentum (TOF)
- γ_{IS} 3-momentum (ZDD)

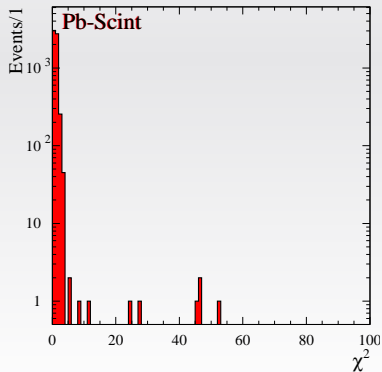
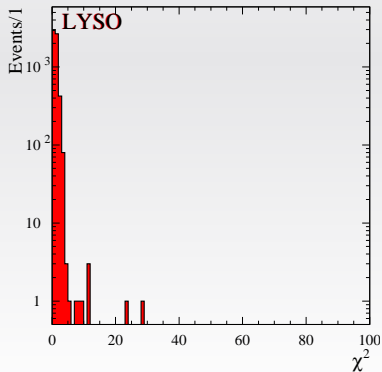
Constraints (4)

- 4-momentum cons.

Unknowns (3)

- n 4-momentum

$$\chi^2 = \sum_{\text{tracks}} \sum_i \frac{(p_i^{\text{exp}} - p_i^{\text{fit}})^2}{\sigma_{p_i}^2}$$



The $n\bar{n}\gamma_{\text{IS}}$ physics case: kinematic fit

Inputs (6)

- \bar{n} 3-momentum (TOF)
- γ_{IS} 3-momentum (ZDD)

Constraints (4)

- 4-momentum cons.

Unknowns (3)

- n 4-momentum

$$\chi^2 = \sum_{\text{tracks}} \sum_i \frac{(p_i^{\text{exp}} - p_i^{\text{fit}})^2}{\sigma_{p_i}^2}$$

