8.B.6 Nuclear Physics B27 (1971) 381-397. North-Holland Publishing Company

HARD-PHOTON EMISSION IN e^+e^- REACTIONS

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Received 4 January 1971

Abstract: The radiative corrections to lowest order for the reactions e^+e^- giving any final state f, different from e^+e^-, are calculated, taking into account the contributions coming from the photons emitted by the electrons. Then an application of these results to the process $e^+e^- \to \pi^+\pi^-$ is done in the experimental conditions of ACO. The difference with previous results arises from: (i) a rigorous calculation of Feynman diagrams without using the "quasi-réel" process approximation, (ii) the error in the Coulomb terms, and (iii) a precise evaluation of the photon phase space.

1. INTRODUCTION

Since the time experiments have been performed with storage rings, people have calculated radiative corrections to e^+e^- reactions. Most of them use a soft-photon approximation. Some take into account the hard-photon contribution. For example in ref. [1], Mosco calculates the radiative corrections to lowest order to the total cross section of e^+e^- giving $\mu^+\mu^-$, taking into account the contribution of the high-energy photons emitted by the electrons. In ref. [2], Tavernier calculates the radiative corrections for e^+e^- giving $\pi^+\pi^-$ using an approximation for the photon phase space and the "quasi-réels" processes approximation (ref. [3]) for the photon emitted by the electrons.

In this paper, considering only the photon emitted by the initial electrons, we establish a general formula for the radiative corrections in α^3 to the process e^+e^- giving any final state f (except e^+e^-), the cross section of this process being calculated in the one-photon-exchange approximation (fig. 1). Then we apply the general formula to a particular case: e^+e^- giving $\pi^+\pi^-$ in the experimental conditions of ACO. The resolution of the photon phase space and the calculations have been done with a good accuracy, using a computer. In section 2, we establish, as a rigorous result of quantum electrodynamics, a formula for the differential cross section of $e^+e^- \to \gamma f$, coming from the two graphs of fig. 2 (subsect. 2.2.1).

We show, in particular ultra relativistic conditions, the validity of the

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"quasi-rêels" processes approximation (ref. [3]). We show that in that case there is a factorization of the differential cross section, factorization in $e \to e\gamma$ and $e^+e^- \to f$ (subsect. 2.2.2). But these ultrarelativistic conditions are not realized in the storage rings where for a ρ-experiment the energy of one beam is around 400 MeV. So we study the error made, if we use that factorized formula instead of the rigorous one, for the differential cross section. We find that the error is less than 14%. It can reach 14% in particular cases (subsect. 2.2.1). However as a result, we find that for the total cross section (when the integration on the final lines f has been performed in the whole space), the factorization of the "quasi-rêels" approximation is really valid (subsect. 2.3).

In subsect. 2.2.3 we study the angular distribution of the emitted photon. We find that even in the "quasi-rêels" processes approximation, it is wrong to say that most of the photons are emitted with an angle θ less than $1/\gamma = m/E$ (where θ is the angle between the direction of flight of the electron and the direction of flight of the photon, m the mass of the electron and E its energy). For $E = 400$ MeV, $1/\gamma = 10^{-3}$, and we find that 25% of the photons are emitted with $\theta > \frac{1}{10}\pi$ and 60% with $\theta > \frac{1}{100}\pi$.

After, we establish a general formula $d\sigma = d\sigma_0(1+\delta)$ for the radiative corrections desired (subsect. 2.4.3). For this we have to take into account the renormalization diagrams of the fourth order in e for the vertex $e^+e^-\gamma$. (subsect. 2.4.1). The contribution of these diagrams has been calculated in two different ways: by Feynman rules, and by dispersion relations and analytic continuation (ref. [4]). The results which are the same in the two cases, are in disagreement with the results given in refs. [5,6]. In the ultrarelativistic limit, the difference is between the terms in π^2: $(2\alpha/\pi)(-\frac{1}{6}\pi^2)$ for the ref. [5], $(2\alpha/\pi)(\frac{1}{12}\pi^2)$ for the ref. [6] and $(2\alpha/\pi)(\frac{1}{3}\pi^2)$ for us.

In section 3, we apply the results of the first part to the process $e^+e^- \to \pi^+\pi^-$, using for π form factor the model of Gounaris-Sakuraï. This experiment has been done at ACO [10].

We shall study the consequence of the angular cuts done over the angle of the projection of the tracks on the transverse plane and "diffusion plane". (See ref. [10].) One finds that the percentage of events with tracks making on angle of more than 10% in the transverse view is very small (less than 10%: see table 2) and one compares this result with those obtained in the general treatment of the first part. Then we have compared our results with those given in refs. [2, 10]. The discrepancy comes from three reasons: an error about the renormalization graphs (terms in π^2) already mentioned; an error coming from the use of a spherical phase space for the photon [2]; a theoretical difference due to the use by Tavernier of the "quasi-rêels" Bremsstrahlung process approximation. (We see in part one that this is really an approximation when the integration over the final-state particles is not done in the whole space.)

Then we study the radiative correction as a function of the angular cuts, in order to see if their choice is a crucial one. A nearly symmetrical variation is found when the cuts are moved within the experimental error of $\pm 3^o$ (figs. 9 and 10). At last we draw the curves giving the consequences on the radiative correction of some variation of the ρ-meson parameters (figs. 11 and 12).

2. DERIVATION OF THE GENERAL FORMULA

2.1. *Generalities. Notations*

2.1.1. p_+, p_-, q are the energy-momentum vectors respectively for e^+, e^- and the photon, $q = (\boldsymbol{q}, q_0)$, m is the electron mass, p_f is the total energy momentum vector of the final state f, and the Lorentz metric is (+++-) so $p_+^2 = p_-^2 = -m^2$. The differential cross sections for the processes $e^+e^- \to f$ and $e^+e^- \to \gamma f$, calculated with the graphs of figs. 1 and 2 can be written:

$$d\sigma_0(e^+e^- \to f) = \frac{1}{[(p_+p_-)^2 - m^4]^{1/2}} \frac{e^4}{4s_0^2} A^0_{\mu\nu} \{f\}^{\mu\nu},$$

$$\{f\}^{\mu\nu} = \sum_{\text{pol f}} (2\pi)^4 \delta_4(p_f - p_+ - p_-) \, d\rho \; \mathcal{F}^\mu \mathcal{F}^{\nu *} \tag{1}$$

with

$$\mathcal{F}_\mu = \langle f|_{\text{out}} J^{\text{em}}_\mu(o)|o\rangle, \qquad s_0 = -(p_+ + p_-)^2,$$

$$A^0_{\mu\nu} = \{p_{+\mu} p_{-\nu} + p_{+\nu} p_{-\mu} + \tfrac{1}{2} s_0 g_{\mu\nu}\}, \qquad \text{see ref. [7]},$$

$$d\sigma_B = \frac{1}{[(p_+p_-)^2 - m^4]^{1/2}} \frac{e^6}{4s^2} \frac{q_0 \, dq_0 \, d\Omega_{\boldsymbol{q}}}{2(2\pi)^3} A_{\mu\nu} \{f\}^{\mu\nu}.$$

For $A_{\mu\nu}$ see subsect. 2.2.1, $\qquad s = -(p_+ + p_- - q)^2$. (2)

In the following we consider the case where $\mathcal{F}_\mu$ can be written as

$$\mathcal{F}_\mu = k_\mu \cdot \mathcal{F}',$$

where k is a real vector with positive k^2 (as a consequence of the conservation of the electromagnetic current $k \cdot p_f = 0$), and $\mathcal{F}'$ is a complex scalar. Let us define $F_\mu = k_\mu |\mathcal{F}'|$, then $\mathcal{F}_\mu \mathcal{F}^*_\nu = F_\mu F_\nu$ and there is a simplification of notations.

For instance we can write:

$$A^0_{\mu\nu}\{f\}^{\mu\nu} = \sum_{\text{pol f}} (2\pi)^4 \delta_4(p_f - p_+ - p_-) \, d\rho_f [2(p_+F)(p_-F) + \tfrac{1}{2} s_0 F^2]. \tag{3}$$

Notice that it is really the case when the final state f is composed of two spinless particles. For the general case note that $A_{\mu\nu}\{f\}^{\mu\nu}$ is a symmetric

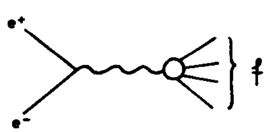

Fig. 1. $e^+e^- \to f$ in the one-photon-exchange approximation.

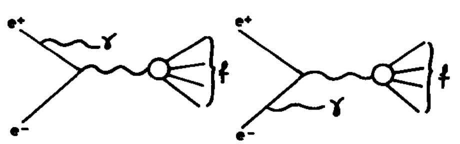

Fig. 2. Graphs contributing to $e^+e^- \to \gamma f$.

bilinear form $h(\mathcal{F}, \mathcal{F}^*)$. So if we write the vector $\mathcal{F}$ as $\mathcal{F} = F_R + F_I$ where F_R and F_I are real vectors then:

$$h(\mathcal{F}, \mathcal{F}^*) = h(F_R, F_R) + h(F_I, F_I).$$

So the formula for the general case is obtained by summing the two formulas where F equals respectively to F_R and F_I.

2.1.2. In storage rings the laboratory system is the e^+e^- c.m. system. So we shall apply all our formulas in that system.

In that frame

$$p_\pm = (E, \pm \boldsymbol{p}), \qquad F = (F_0, \boldsymbol{F}), \qquad q = (q_0, \boldsymbol{q}),$$

$$|\boldsymbol{p}| = p, \qquad |\boldsymbol{F}| = \hat{F}, \qquad |\boldsymbol{q}| = q_0.$$

We define the angles as follows (fig. 3):

$$\theta = (\boldsymbol{p}, \boldsymbol{q}), \qquad \psi = (\boldsymbol{p}, \boldsymbol{F}), \qquad \varphi = (\boldsymbol{q}, \boldsymbol{F}),$$

$\Phi_{\boldsymbol{q}}$ and $\Phi_{\boldsymbol{F}}$ are the azimuthal angles. We have the angular relation:

$$\cos\varphi = \cos\psi \cos\theta + \sin\psi \sin\theta \cos(\Phi_{\boldsymbol{q}} - \Phi_{\boldsymbol{F}}).$$

In storage rings, we are also in an ultrarelativistic limit. For $E = 300$ MeV, $\frac{m}{E} = 1.7 \times 10^{-3}$, so we can neglect, with a very good approximation, m^2/E^2 compared with unity. In that case $d\sigma_0$ can be written as

$$d\sigma_0 = \frac{e^4}{64E^4} \sum_{\text{pol f}} F^2 \sin^2\psi \, (2\pi)^4 \, \delta_4(p_f - p_+ - p_-) \, d\rho_f. \tag{4}$$

2.2. *Differential cross section for* $e^+e^- \to \gamma f$

2.2.1. The calculation of the two graphs of fig. 2, by Feynman techniques and the use of the conservation of the electromagnetic current ($p_f \cdot \mathcal{F} = (p_+ + p_- - q) \cdot \mathcal{F} = 0$), give for $A_{\mu\nu}$ of formula (2):

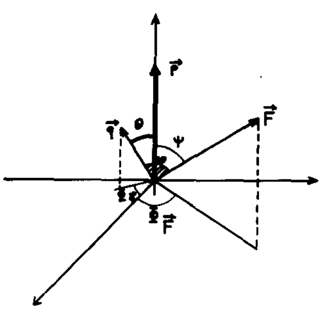

Fig. 3. Polar system of reference.

$$A^{\mu\nu} = g^{\mu\nu}\left(\frac{qp_+}{qp_-}+\frac{qp_-}{qp_+}\right) + 2m^2\left(\frac{p_+^\mu}{qp_-}-\frac{p_-^\mu}{qp_+}\right)\left(\frac{p_+^\nu}{qp_-}-\frac{p_-^\nu}{qp_+}\right) - (p_+ + p_- - q)^2$$
$$\times\left\{\tfrac{1}{2}g^{\mu\nu}\left(\frac{p_+}{qp_+}-\frac{p_-}{qp_-}\right)^2 - \frac{p_+^\mu p_+^\nu + p_-^\mu p_-^\nu}{(qp_+)(qp_-)}\right\}. \tag{5}$$

In the e^+e^- c.m. system and in the ultrarelativistic case of storage rings ($m^2/E^2 \ll 1$) we have

$$A^{\mu\nu}F_\mu F_\nu = \frac{\sin^2\theta}{(1-(p^2/E^2)\cos^2\theta)^2} f(0)$$
$$+ \frac{m^2}{E^2}\frac{g(0)}{(1-(p^2/E^2)\cos^2\theta)^2} + \frac{\sin^2\theta}{(1-(p^2/E^2)\cos^2\theta)^2}[f(\theta)-f(0)], \tag{6}$$

with

$$f(\theta) = F^2\left[2(1+\cos^2\theta) - \frac{8E(E-q_0)}{q_0^2}\left\{\frac{(2E-q_0)^2\cos^2\psi + q_0^2\cos^2\varphi}{(2E-q_0)^2 - q_0^2\cos^2\varphi} - 1\right\}\right],$$

$$f(0) = \frac{8E^2F^2_{\theta=0}}{q_0^2}\frac{(1-(q_0/2E))^2(1-(q_0/E)+(q_0^2/2E^2))}{(1-(q_0/E)+(q^2/4E^2)\sin^2\psi)}\sin^2\psi,$$

$$g(0) = F^2_{\theta=0}\frac{4(1-(q_0/2E)^2}{(1-(q_0/E)+(q_0^2/4E^2)\sin^2\psi)}\sin^2\psi.$$

This formula is valid for θ between 0 and $\frac{1}{2}\pi$. For θ between $\frac{1}{2}\pi$ and π we must change $f(0)$ and $g(0)$ by $f(\pi)$ and $g(\pi)$, that is to say $F^2_{\theta=0}$ by $F^2_{\theta=\pi}$. Notice that in the general case we have not $F^2_{\theta=\pi} = F^2_{\theta=0}$, nevertheless in the case of a two spinless particles final state F^2 depends only on $s = -(p_+ + p_- - q)^2 = 4E(E-q_0)$ so $F^2_{\theta=\pi} = F^2_{\theta=0}$.

Formula (6) is written in such a way that the first term is the preponderant one. This is due to the fact that $\sin^2\theta/(1-(p^2/E^2)\cos^2\theta)^2$ has two peaks very near $\theta = 0$ and $\theta = \pi$. To understand the preponderance one can compare the different integrals given in appendix 1.

Let us study now the preponderance of the first term in front of the others. When E/m is great enough that we neglect 1 in front of $\log E/m$ we must keep only the first term in formula (6). But let us remark that to have $\log E/m \geqslant 10^3$ we must have $E/m \geqslant e^{10^3}$ which is gigantic. At the energies of storage rings let us look at the error made if we keep only the first term. For example at $E = 300$ MeV, for a two spinless particles final state, for $\psi = \frac{1}{2}\pi$ and after integration in $d(\cos\theta)$ we find that the error can reach 14% if the emitted photon is very hard (q_0 close to E).

2.2.2. Factorization. We show that if we keep only the first term in formula (6) there is a factorization of the differential cross section $d\sigma_B$ ((2)). If we do this approximation, in $f(0)$ we consider that the photon has been emitted at $\theta = 0$ by the electron e^+, because e^+ is "quasi-réel" after emission:

$$(p_+ - q)^2 = -m^2 - 2(\boldsymbol{p}\boldsymbol{q} - Eq_0) = -m^2 - 2(p - E)q_0 ,$$

which is of order m^2.

If it has been emitted at $\theta = 0$ by the electron e^-, e^- would not have been "quasi-rêel" because $(p_- - q)^2 = 4Eq_0$.

So let us define by $d\sigma_0'$ the differential cross section $d\sigma_0(e^+e^- \to f)$ of formula (1) where we have changed p_+ by $p_+ - q$.

Considering $d\sigma_B$ of formula (2) in which we have taken for $A_{\mu\nu}F^{\mu}F^{\nu}$ the first term of formula (6), we find the following factorization:

$$d\sigma_B = \frac{2e^2}{(2\pi)^3} \frac{\sin^2\theta}{(1 - (p^2/E^2)\cos^2\theta)^2} \frac{dq_0}{q_0} d\Omega_{\boldsymbol{q}} \left(1 - \frac{q_0}{E} + \frac{q_0^2}{2E^2}\right) d\sigma_0' . \tag{7}$$

This is true for θ between 0 and $\frac{1}{2}\pi$, we find the same result for θ between $\frac{1}{2}\pi$ and π.

2.2.3. Angular distribution of the emitted photon. If we keep only the first term in formula (6) the angular distribution is given by $\sin^2\theta/(1 - (p^2/E^2)\cos^2\theta)^2$. To study it, let us consider

$$h(\theta_M) = \int_0^{\theta_M} \frac{\sin^2\theta}{(1 - (p^2/E^2)\cos^2\theta)^2} d(\cos\theta)$$

$$= -\tfrac{1}{2} + \log\frac{2E}{m} + \frac{m^2}{2E^2} \frac{\cos\theta_M}{(1 - (p^2/E^2)\cos^2\theta_M)} - \tfrac{1}{2}\log\left(\frac{E + p\cos\theta_M}{E - p\cos\theta_M}\right) . \tag{8}$$

At $E = 400$ MeV, let us look at $h(\theta_M)/h(\frac{1}{2}\pi)$ for some values of θ_M.

θ_M	$\frac{1}{2}\pi$	$\frac{1}{4}\pi$	$\frac{1}{10}\pi$	$\frac{1}{100}\pi$
$\frac{h(\theta_M)}{h(\frac{1}{2}\pi)}$	1	0.87	0.74	0.40

A study of the angular distribution with the all formula (6) gives approximately the same results. So it is wrong to say that the greatest part of the photons are emitted in a cone of angle $\theta_M < 1/\gamma = m/E = 10^{-3}$. Moreover:

$$h(\tfrac{1}{2}\pi) = -\tfrac{1}{2} + \log 2\gamma \sim \log\gamma \quad \text{when} \quad \gamma \to \infty, \qquad h\left(\frac{1}{\gamma}\right) \sim \tfrac{1}{2}\log 2 - \tfrac{1}{4} ,$$

so

$$\frac{h\left(\frac{1}{\gamma}\right)}{h(\frac{1}{2}\pi)} \sim 0 \qquad \text{when} \qquad \gamma \to \infty ,$$

and the previous assumption is definitely wrong when the energy E goes to infinity.

2.3. *Total cross section for* $e^+e^- \to \gamma f$ (*integrated over the final lines* f)

This total cross section is given by formula (2) where we replace $\{f\}^{\mu\nu}$ by $\{f\}_I^{\mu\nu} = \int_f \{f\}^{\mu\nu}$; $\{f\}_I^{\mu\nu}$ depends only on $p_f = p_+ + p_- - q$, and the conservation of the electromagnetic current implies that $\{f\}_I^{\mu\nu}$ is of the following form:

$$\{f\}_I^{\mu\nu} = \left(g^{\mu\nu} - \frac{p_f^\mu p_f^\nu}{p_f^2}\right) f_0(-p_f^2).$$

Using $A^0_{\mu\nu}$ of formula (1) and $A_{\mu\nu}$ of formula (5) we find respectively the total cross section $\sigma_0(s_0)$ (for $e^+e^- \to f$) and $d\sigma_B^T$ (for $e^+e^- \to \gamma f$). In the e^+e^- c.m. system and in the ultra-relativistic limit $m^2/E^2 \ll 1$ and $s/m^2 = 4E(E-q_0)/m^2 \gg 1$, after an integration over $d\Omega_q$ (angular variables of the photon) we find:

$$d\sigma_B^T = \frac{2\alpha}{\pi} \frac{dq_0}{q_0} h(E, q_0)\, \sigma_0(4E(E-q_0)), \tag{9}$$

with

$$h(E, q_0) = \left(1 - \frac{q_0}{E} + \frac{q_0^2}{2E^2}\right)\left(-1 + 2\log\frac{2E}{m}\right).$$

So for the total cross section the factorization of formula (7) is valid with a very good approximation; there is no corrective term (coming from formula (6)).

2.4. *Radiative corrections due to the photon emitted by the initial electrons*

To the third order in α we have to take into account all the diagrams of figs. 2 and 4.

2.4.1. Renormalization. Using Feynmann techniques or analytic continuation of the results given by Chou and Dresden (dispersive methods) (ref. [4]), and by Schweber (ref. [9]), we find the contribution of the five diagrams of fig. 4. In the e^+e^- c.m. and in the ultrarelativistic limit, we find that this contribution is given by:

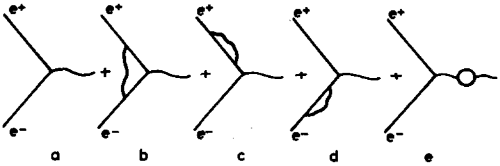

Fig. 4. Renormalization graphs.

$$d\sigma_1 = d\sigma_0(1 + A_V + A_{SE}) , \qquad (10)$$

where

$$A_V = \frac{\alpha}{\pi}\left[2\left(\log\frac{m}{\gamma} - 1\right)\left(1 - 2\log\frac{2E}{m}\right) - \log\frac{2E}{m} - 2\left(\log\frac{2E}{m}\right)^2 + \tfrac{2}{3}\pi^2\right],$$

$$A_{SE} = \frac{2\alpha}{3\pi}\left[-\tfrac{5}{3} + 2\log\frac{2E}{m}\right].$$

Note that being in the ultrarelativistic limit, we have neglected the term coming from the function $F_2(s)$ of ref. [2]. We notice in A_V the difference with refs. [5,6]: $(2\alpha/\pi)(-\frac{1}{6}\pi^2)$ in ref. [5] and $(2\alpha/\pi)(\frac{1}{12}\pi^2)$ in ref. [6] when our result is $(2\alpha/\pi)(\frac{1}{3}\pi^2)$. One can see some details of the analytic continuation in appendix 2.

2.4.2. Bremsstrahlung. To deal with the infrared divergence problem, we consider the cross section for $e^+e^- \to \gamma f$ where the photon γ has a mass λ. We integrate the photon variables in the volume $\mathcal{S}_A - \mathcal{S}_\lambda$ between the two spheres of centre 0 and radius $A(A/E \ll 1)$ and $\lambda(\lambda < A)$. Neglecting terms like q_0/E in front of 1 we obtain for the differential cross section

$$d\sigma_\lambda = d\sigma_0 \frac{e^2}{2(2\pi)^3} \int_{\varphi_A - \varphi_\lambda} \left(\frac{p_+}{qp_+ + \frac{1}{2}\lambda^2} - \frac{p_-}{qp_- + \frac{1}{2}\lambda^2}\right)^2 \sqrt{q_0^2 - \lambda^2}\, dq_0\, d\Omega_{\boldsymbol{q}} . \qquad (11)$$

In the e^+e^- c.m. and in the ultra-relativistic limit, formula (11) becomes:

$$d\sigma_\lambda = d\sigma_0 A_B ,$$

with

$$A_B = \frac{2\alpha}{\pi}\left\{\log\frac{2A}{\lambda}\left(-1 + 2\log\frac{2E}{m}\right) + \log\frac{2E}{m} - \log^2\frac{2E}{m} - \tfrac{1}{6}\pi^2\right\} . \qquad (12)$$

2.4.3. Results. Taking into account the previous radiative corrections the differential cross section for $e^+e^- \to f$ and $e^+e^- \to \gamma f$ (figs. 1 and 2) is:

$$d\sigma = d\sigma_0(1 + \delta) ,$$

with

$$\delta = A_V + A_{SE} + A_B + \int_{\mathcal{V} - \mathcal{S}_A} \frac{d\sigma_B}{d\sigma_0} . \qquad (13)$$

In $d\sigma_0$ the phase-space differential volume is given by $\delta_4(p_f - p_+ - p_-)\, d\rho_f$,

$$\delta_1 = A_V + A_{SE} + A_B = \frac{2\alpha}{\pi}\left\{\left(-1 + 2\log\frac{2E}{m}\right)\left(\log\frac{A}{E} + \tfrac{13}{12}\right) - \tfrac{17}{36} + \tfrac{1}{6}\pi^2\right\} . \qquad (14)$$

In making corrections to an experimental cross section the final lines are integrated in the volume $\mathcal{V}_{f\,\exp}$ and the photon variables are integrated in the volume $\mathcal{V}_{\exp} - \mathcal{S}_A$, so

$$\sigma_{\exp} = \sigma_{0\,\exp}\left(1+\delta_1+\frac{1}{\sigma_{0\,\exp}}\int_{\mathcal{V}_{\mathrm{f}\,\exp}}\int_{\mathcal{V}_{\exp}-\mathcal{S}_{\mathrm{A}}} \mathrm{d}\sigma_{\mathrm{B}}\right). \tag{15}$$

The corrections to the total cross section are obtained from (14) and (9):

$$\sigma = \sigma_0(4E^2)\left[1+\frac{2\alpha}{\pi}\left\{\left(-1+2\log\frac{2E}{m}\right)\left(\log\frac{A}{E}+\tfrac{13}{12}+\int_A^{q_{\max}}\frac{\mathrm{d}q_0}{q_0}\left(1-\frac{q_0}{E}+\frac{q_0^2}{2E^2}\right)\times\frac{\sigma_0[4E(E-q_0)]}{\sigma_0[4E^2]}\right)-\tfrac{17}{36}+\tfrac{1}{6}\pi^2\right\}\right]. \tag{16}$$

3. APPLICATION TO A PARTICULAR EXAMPLE. THE REACTION $e^+e^- \to \pi^+\pi^-$

We begin with a short description of the experimental set-up. Then we write the expression for radiative corrections in that case, using the Gounaris-Sakuraï model for the charged π-meson form factor. Finally we give our results, discuss the influence of angular cuts and form-factor parameters and compare them with those given in the "quasi-réel" process approximation in order to check the results of the first part of this paper (sect. 2).

3.1. *Experimental set-up*

This experiment has been performed at ACO [10].

Experimental constraints:

(i) The charged mesons are detected in two semi-cones (angle $\nu = \frac{1}{4}\pi$) of vertical axis normal to the beam plane (there should be a trace in each semi-cone).

(ii) Angular cuts on the tracks angle are made to discard the processes with more than two body final state (apart some radiative events $e^+e^- \to \pi^+\pi^-\gamma$).

We shall study the influence of a cut in the transverse plane (angle between the projections of the two tracks less than DELT), of a cut in the "diffusion plane", defined in ref. [10] (angle less than DELL), and of a cut in space (angle between the tracks less than η).

Note that as the π-meson charge is not measured, we have not to consider the two-photon-exchange graphs (fig. 5) due to a theorem proved by Putzolu [5].

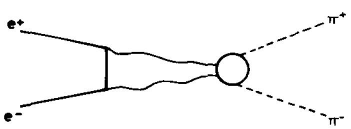

Fig. 5. Two-photon-exchange graph for $e^+e^- \to \pi^+\pi^-$.

3.2. *Expression of the radiative correction*

Notation: k_+, k_- are the moments of the π^+ and π^-, $k = k_+ - k_-$, $K = k_+ + k_-$, and ψ is the angle between the beam line and $\boldsymbol{k}$.

Here we have $\langle\pi^+\pi^-|J_\mu^{\rm em}(0)|0\rangle = (k_+ - k_-)_\mu F_\pi(s)$ and we use the Gounaris-Sakuraï model for $F_\pi(s)$ (ref. [7], formula (B.28)). So in every formula of the first part we replace F_μ by $k_\mu |F_\pi(s)|$ and $\delta_4(p_f + q - p_+ - p_-)\,d\rho_f$ by $\delta_4(K + q - p_+ - p_-)\,(d^3\boldsymbol{k}_+/E_+)\,(d^3\boldsymbol{k}_-/E_-)$. In the e$^+e^-$ c.m. system we find for $\delta_4(p_f + q - p_+ - p_-)\,d\rho_f$

$$\frac{|\boldsymbol{k}|(2E - q_0)}{2[(2E-q_0)^2 - q_0^2\cos^2\varphi]}\,d\Omega_k,$$

with

$$|\boldsymbol{k}|^2 = \frac{(2E - q_0)^2 k^2}{(2E - q_0)^2 - q_0^2\cos^2\varphi}, \qquad k^2 = 4(E^2 - E q_0 - M^2).$$

The formula (13) when integrated over the photon phase space becomes $d\sigma = d\sigma_0(1 + \delta)$, $\delta = \delta_1$ (independent of ψ) $+ g(E, \psi)$, $d\sigma_0 = h(E)[1 - (p^2/E^2)\cos^2\psi]\,d\Omega_f$.
Hence $\sigma = \sigma_0(1 + \delta_{\rm total})$ and, in the case of cones

$$\delta_{\rm total} = \delta_1 + \frac{2\int_\nu^{\frac{1}{2}\pi} (1 - (p^2/E^2)\cos^2\psi)\,{\rm Arc\,cos}(\cos\nu/\sin\psi)\,g(E,\psi)\sin\psi\,d\psi}{(1 - (p^2/3E^2))(1 - \cos\nu) + (p^2/6E^2)\cos\nu\sin^2\nu}.$$

Here $\nu = \frac{1}{4}\pi$. (All these integrations are done with a computer.)

3.3. *Results*

3.3.1. Relations between photon phase-space shape and angular cuts. One finds a surface with the axis $\boldsymbol{k}$ well approximated by the bissectrix of the tracks, the semi major axis equal to $2E(E - M)/(2E - M)$, where M is the pion mass, the cross section given in fig. 6 and with the longitudinal sections generally symmetrical, being well fitted by an arc of a conic.

3.3.2. Influence of angular cuts on the radiative correction. In this experiment, the photon phase space is the intersection of ⑤ and ④ (hachured part in fig. 6). It is equivalent to the intersection of ③ and ④ as can be seen on fig. 6: the difference is very small on the transverse section, and moreover the phase space is very long (430 MeV on the axis) so the volume contributing to this difference is very small. The radiative correction for ③ ∩ ④ is labelled as ② in the following table 1.

Comments

(i) One finds that ① and ④ are nearly the same. (See table 2.)

(ii) To go further, fig. 7 gives $(1 + \delta)$ integrated over photons lying in different parts of the photon phase space (more precisely, these numbers are written on the "transverse section averaged over ψ"). The results

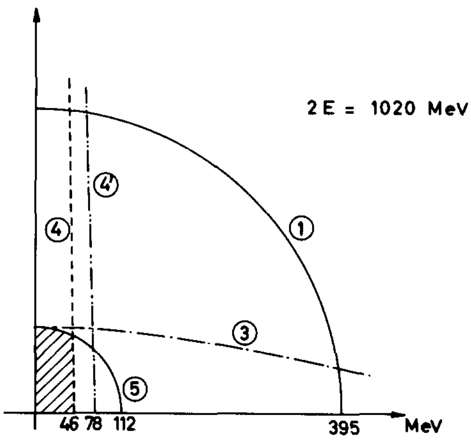

Fig. 6. Cross section of the photon phase space; $2E = 1020$ MeV; semi major axis $2E(E-M)/(2E-M) \approx 430$ MeV. ① One event in each semi cone (obtuse angle between the tracks). ③ Angle between the projections of the two tracks in the "diffusion plane" less than DELL = 15^o. ④ Angle between the projections of the two tracks on a plane perpendicular to the beam line less than DELT = 10^o ($\psi = \frac{1}{4}\pi$). ④' Same for $\psi' = \frac{1}{2}\pi$. ⑤ Angle between the tracks less than $\eta = 15^o$ in space.

Table 1
Radiative correction δ_{total}

E	322.2	352.3	382.0	412.4	442.9	510.0	
δ (obtuse) without cuts	- 0.0492	- 0.0726	- 0.0461	0.1519	0.4119	1.112	①
δ (10,15) DELT, DELL	- 0.0672	- 0.0835	- 0.0553	0.1073	0.1853	0.1104	②
δ (/,15) DELL only	not computed	not computed	- 0.0543	0.1126	0.2068	0.1654	③
δ (10,/) DELT only	- 0.053	- 0.075	- 0.048	0.143	0.371	0.92	④
δ ($\eta = 15^o$) $m_\rho = 760$	- 0.0689	- 0.0841	- 0.0338	0.133	0.181	0.108	
extrapolated to $m_\rho = 770$	- 0.067	- 0.082	- 0.055	0.109	0.185	0.112	⑤

of table 2 are to be compared to those given in subsect. 2.2.3; there it was found that for $E = 400$ MeV, 26% of the Bremsstrahlung events - that is to say 3% of the total number of events - occur with an angle θ greater than $\frac{1}{10}\pi$. To make that comparison, we need a relation between θ and $\Delta\varphi_{transverse}$.

On the transverse plane we have approximately $q_0 \times \theta \approx \frac{1}{2}|k|\Delta\varphi_{transverse}$ with $\frac{1}{2}|k| \approx E$ for a not too hard photon. So $\theta/\Delta\varphi_T \approx E/q_0 \approx 4$ mostly.

Table 2

E	322.2	352.3	382.0	412.4	442.9	510.0
① - ④	0.004	0.002	0.002	0.009	0.041	0.19
percentage of events with tracks making an angle of more than 10^{o} in a transverse plane	< 1%	< 1%	< 1%	< 1%	≈ 3%	≈ 9%

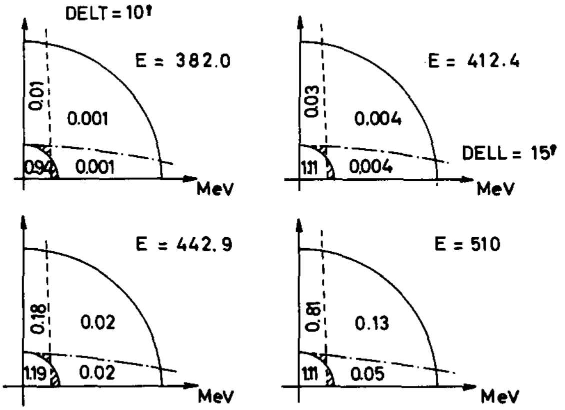

Fig. 7. $(1+\delta)$ integrated over photons lying on different parts of the photon phase space. (These numbers are written on "the transverse section averaged over ψ".)

This equality explains why for $\Delta\varphi_T = 10^o$, $\theta = 18^o$ we find 1% in table 2 and 3% in the previously mentioned subsection.

(iii) We compare now our results with those given in the "quasi-réel" process approximation in refs. [10, 2]. In these references the work of Rossi [6] is used and so there is an error as mentioned in subsect. 2.4.1 ($(2\alpha/\pi) \times \frac{1}{4}\pi^2$ is missing in δ).

The difference between the two last lines of table 3 arises from two reasons:

(a) Tavernier [2] uses a sphere for the Bremsstrahlung photon phase space when we have used the exact shape for it.

(b) We have seen in subsect. 2.3 that the "quasi-réel" process approximation is not an approximation, only when the final integration is over the whole space. In that experiment, it is not the case. The error mentioned in subsect. 2.2.1 was that of 10% on $\delta_{\text{Bremsstrahlung}}$ for $E = 400$ MeV - that is to say, 1% for $(1+\delta_\pi)$. The difference of the two last lines of table 3 is compatible with this error of 1%.

(iv) As an outlying result we have computed the contamination of $e^+e^- \to \varphi \to$ final state (especially $\pi^+\pi^-$ coming from K_S^0 decay) by $e^+e^- \to \rho \to \pi^+\pi^-$. The Orsay group, in their analysis of $e^+e^- \to \varphi \to$ final state, considers that by taking into account only the events in which the tracks an-

Table 3

E	322.2	352.3	382	412.4	442.9
$(1+\delta_\pi)$ (ref. [10])	0.925	0.911	0.935	1.081	1.144
$(1+\delta_\pi)$ (ref. [10]) plus $(2\alpha/\pi)\times\frac{1}{4}\pi^2 = 0.0115$	0.936	0.922	0.946	1.092	1.155
$(1+\delta_\pi)$ our work	0.933	0.916	0.945	1.107	1.185

gle is larger than DELT = 5° in transverse plane and DELL = 10° in space, they discard in the total number of events, almost all the one coming from the ρ-meson.

$$\text{contamination} = \frac{N}{N'}$$

$$= \frac{\sigma_0^{\text{total}}(e^+e^- \xrightarrow{\rho} \pi^+\pi^-) \times 0.24 \times \text{efficiency} \times \text{luminosity}}{\sigma^{\text{total}}(e^+e^- \to \Phi \to \text{all}) \times B_{\Phi\to K_0\bar{K}_0} \times B_{K_S^0\to\pi^+\pi^-} \times \text{efficiency} \times \text{luminosity}},$$

at $2E$ = 1020 MeV (0.24 comes from table 4).
Contamination = 1.7% which is under the experimentalists' estimation. Note that the variation of χ-function (see fig. 8) of the transverse cut is not very sharp, at least less than expected by the experimentalists of Orsay who thought that 5° was a crucial value.
(v) The radiative correction as a function of angular cuts (see figs. 9 and 10). The error made by experimentalists is $\pm 3^\circ$ for each angular cut. As a matter of fact, we can see on the curves that $|\delta(\text{DELL}+3^\circ) - \delta(\text{DELL})|$ and $|\delta(\text{DELL}-3^\circ) - \delta(\text{DELL})|$ are almost equal (and the same for DELL changed to DELT). Now statistically one accepts as many events with a too large angle (compared to the cut), as with a too small angle. Consequently the error coming from this error is nearly negligible.

3.3.3. The radiative correction as a function of ρ-meson parameters. With a first approximation for the radiative correction, the experimentalists have fitted the parameters of the ρ-meson to be $m_\rho = 770 \pm 4$ MeV, $\Gamma_\rho = 111 \pm 6$ MeV (ref. [10]).

Table 4

DELT	2°	5°	8°
χ	0.38	0.24	0.18

χ = contribution to cross section of events $\rho \to 2\pi$ with tracks in an angle greater than DELT in the transverse plane and $\eta = 10^\circ$ in space (in percentage of σ_0^{total} without radiative correction). E = 510 MeV.

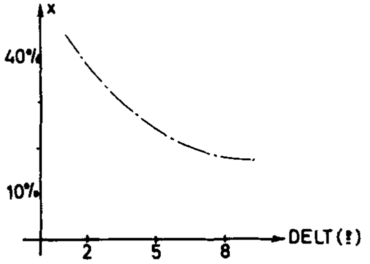

Fig. 8. χ (contribution to cross section of events $\rho \to 2\pi$ with tracks in an angle greater than DELT in the transverse plane and $\eta = 10^{\circ}$ in space) as a function of DELT.

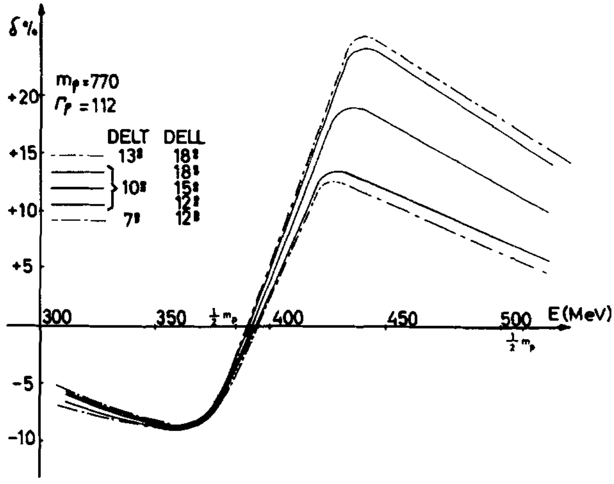

Fig. 9. Radiative correction as a function of longitudinal angular cut (DELL).

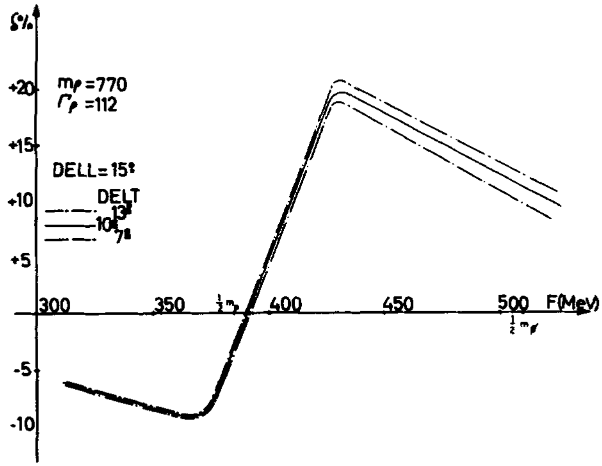

Fig. 10. Radiative correction as a function of transverse angular cut (DELT).

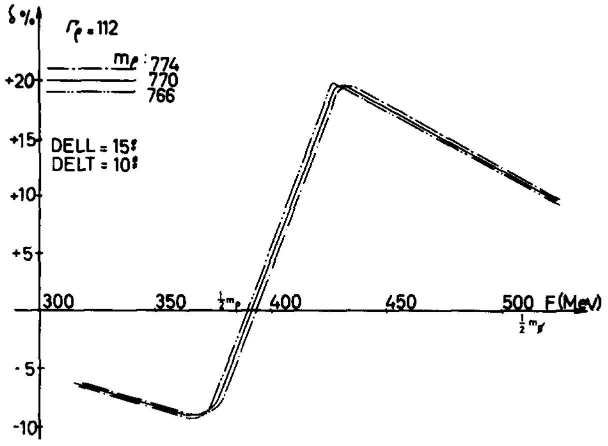

Fig. 11. Radiative correction as a function of m_ρ.

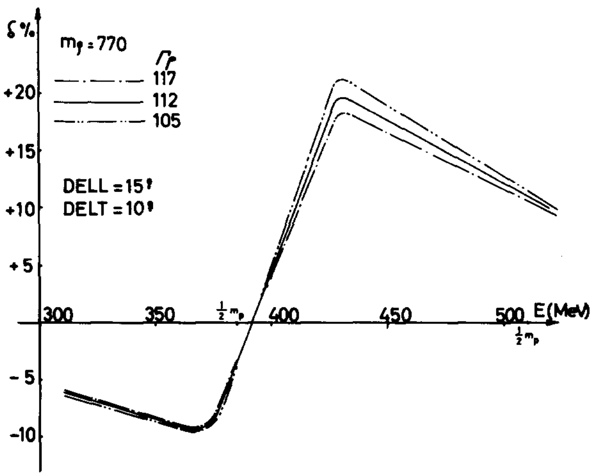

Fig. 12. Radiative correction as a function of Γ_ρ.

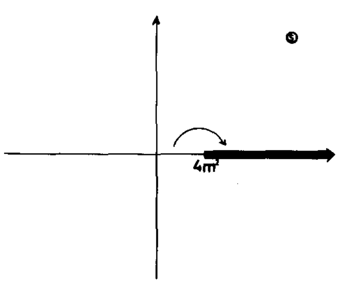

Fig. 13. Analytic domain for the function $F_1(s)$.

We have determined the error on our calculation coming from these imprecisions on m_ρ and Γ_ρ (see figs. 11 and 12).

As expected, when m_ρ goes from 766 to 774 MeV we have a translation of the curve. And when Γ_ρ goes from 105 to 117 MeV we have a dilatation centered on the point $E = 392$, $\delta = 0$. But these errors are correlated; the small discrepancy between our results and those of Tavernier does not seem to justify a new and very long fit for the ρ-meson parameters.

We thank Professor M. Gourdin and the experimentalists of ACO for useful discussions.

APPENDIX 1.

To understand formula (6) the following integrals are useful:

$$\int_0^1 \frac{\mathrm{d}(\cos\theta)}{(1-(p^2/E^2)\cos^2\theta)^2} = \frac{p^2}{2E^2}\left[\frac{p^2}{m^2}+\frac{p}{2E}\log\frac{E+p}{E-p}\right] \approx \tfrac{1}{2}\left[\frac{E^2}{m^2}+\log\frac{2E}{m}\right],$$

$$\int_0^1 \frac{\sin\theta\,\mathrm{d}(\cos\theta)}{(1-(p^2/E^2)\cos^2\theta)^2} = \tfrac{1}{4}\frac{E}{m}\pi,$$

$$\int_0^1 \frac{\sin^2\theta\,\mathrm{d}(\cos\theta)}{(1-(p^2/E^2)\cos^2\theta)^2} = \frac{E^2}{2p^2}\left[-1+\frac{E^2+p^2}{2Ep}\log\frac{E+p}{E-p}\right] \approx \tfrac{1}{2}\left[-1+2\log\frac{2E}{m}\right],$$

$$\int_0^1 \frac{\sin^n\theta\,\mathrm{d}(\cos\theta)}{(1-(p^2/E^2)\cos^2\theta)^2} \approx \int_0^1 \sin^{n-4}\theta\,\mathrm{d}(\cos\theta) \quad \text{for} \quad n \geqslant 3.$$

We see that only the three first integrals go to infinity when E goes to infinity.

APPENDIX 2.

The function $F_1(s)$ of refs. [4, 9] is given in a useful way for $0 < s < 4m^2$. For $s = 4E^2 > 4m^2$ we have to do an analytic continuation over the cut (fig. 13). The difficulty is in the analytic continuation of $\int_0^\theta x \operatorname{tg} x \mathrm{d}x$ (where $\sin^2\theta = s/4m^2$) because the function $x \operatorname{tg} x$ has a pole at $x = \frac{1}{2}\pi$. The result is:

$$\int_0^\theta x \operatorname{tg} x \, dx = -\tfrac{1}{2}\pi \log\left(\frac{2p}{m}\right) + i\left(\tfrac{1}{4}\pi^2 - \int_0^{\frac{1}{2}\log\frac{1+v}{1-v}} z \coth z \, dz\right),$$

where $v = p/E$ and

$$\int_0^{\frac{1}{2}\log\frac{1+v}{1-v}} z \coth z \, dz = \tfrac{1}{2}\left\{\tfrac{1}{2}\log\frac{4v^2}{1-v^2}\log\frac{1+v}{1-v} - \tfrac{1}{4}\left(\log\frac{1+v}{1-v}\right)^2 + \tfrac{1}{6}\pi^2 + \int_0^{\frac{1-v}{1+v}} \frac{\log(1-y)}{y}\, dy\right\}.$$

The analytic continuation gives for $A_V = 2 \operatorname{Re} F_1(s)$, when $s = 4E^2$

$$A_V = \frac{\alpha}{\pi}\left\{2\left(\log\frac{m}{\lambda} - 1\right)\left(1 - \frac{E}{p}\left(1 - \frac{m^2}{2E^2}\right)\log\frac{E+p}{E-p}\right) - \frac{E}{2p}\log\frac{E+p}{E-p} + \frac{4E}{p}\left(1 - \frac{m^2}{2E^2}\right)\left(\tfrac{1}{4}\pi^2 - \int_0^{\frac{1}{2}\log\frac{E+p}{E-p}} z \coth z \, dz\right)\right\}.$$

In the ultrarelativistic limit we find the A_V given in formula (10).

NOTE ADDED IN PROOF

This paper was finished when we came to hear of the work of Nguyen Ngoc Hoan (note interne du Laboratoire de l'Accélération linéaire, RI68, 18 Orsay). His work is on the same topic and is in part similar to ours.

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